CHAPTER 2

WAVE INTERACTION

2.1 INTRODUCTION

The study of wave-wave interactions has its origin in the fundamental paper by Peierls (1929) on heat conduction in solids. Since then it has been applied to other branches of physics, particularly in quantum field theory. Litvak (1960) has applied the theory to study plasma wave interactions. The same has found applications in scattering geophysical fields through the works of Phillips (1960), Hasselmann (1960, 1962), Benney (1962) and Longuet-Higgins (1962). The theory has also been applied to study exchange of energy in internal and surface waves (Ball, 1964), capillary waves (McGoldrick, 1965), waves in stratified fluids (Thorpe, 1966) and nonlinear interaction between gravity waves and turbulent atmospheric boundary layer (Hasselmann, 1967).

Nishikawa et al. (1974), Kawahara et al. (1975) and Benney (1976, 1977) have studied interactions between short and long waves by means of the coupled equations for a single monochromatic wave and a long wave. The interaction and the statistics of many localised waves have been investigated by
Zakharov (1972) in connection with Langmuir turbulence. Miles (1977a, b) has studied the general interaction of two oblique solitary waves and interaction associated with the parametric end points of the singular regime.

In the case of two solitary waves propagating in opposite directions, there are significant differences between experimental results (Maxworthy, 1976) and theory based on Boussinesq equation (Oikawa and Yajima, 1973). Su and Mirie (1980) recasted nonlinear surface boundary conditions into a pair of equations involving the free surface elevation and the velocity along the horizontal bottom boundary and determined a third-order perturbation solution to the head-on collision of two solitary waves. They have shown that although the waves emerged from the collision without any change in height, were symmetric and changed slowly in time.

Fenton and Rienecker (1982) have investigated the interaction of one solitary wave overtaking another, and the results supported experimental evidence for the applicability of the KdV equation. The phase-shift due to the interaction of large and small solitary waves has been studied by Johnson (1983).

Various authors have investigated solitary wave propagation at the interface of an inviscid two-fluid system (Miles, 1980; Koop and Butler, 1981; Segur and Hammack, 1982; Gear and Grimshaw, 1983). Mirie and Su (1984) have studied
internal solitary waves and their head-on collision by a perturbation method.

Strong interactions between solitary waves belonging to different wave modes have been studied by Gear (1985). Mirie and Su (1986) have investigated the head-on collision between two modified KdV solitary waves where cubic and quadratic nonlinearities balance dispersion. It is shown that the collision is elastic because of a dispersive wave train generated behind each emerging solitary wave.

Byatt-Smith (1988, 1989) has studied the reflection of a solitary wave by a vertical wall by considering the head-on collision of two equal solitary waves. He has found analytically that the amplitude of the solitary wave after reflection is reduced.

The resonant interaction between two internal gravity waves in a shallow stratified liquid can be modelled by a system of two KdV equations coupled by small linear and nonlinear terms. Kivshar and Boris (1989) have used this system. It is shown that two solitons belonging to different wave modes form an oscillatory bound state (bi-soliton). They have calculated the frequency of internal oscillations of a bi-soliton and the intensity of the radiation emitted by a weakly excited bi-soliton.

It has been pointed out by Kawahara (1973) that the derivative expansion method can be applied in a systematic way.
to the analysis of weak nonlinear dispersive waves in uniform media. He (1975a) has studied the weak nonlinear self-interactions of capillary gravity waves using this method. Derivative expansion method that avoids secularity incorporates partial sums in the sense that the solution thus obtained by a perturbation is not a simple power series solution (Jeffrey and Kawahara, 1981). Kakutani and Michihiro (1976) have applied this method to study the far-field modulation of stationary water waves and the same has been applied by Kawahara (1975b) to problems of wave propagation in nonhomogeneous medium. Using this method Kawahara and Jeffrey (1979) have derived several asymptotic kinematic equations for a wave system composed of an ensemble of many monochromatic waves having a continuous spectrum together with a long wave. Since the introduction of multiple scale concepts simplifies the order estimation necessary in a perturbation analysis, this method systematizes the wave-packet formalism.

Nirmala and Vedan (1990) have used derivative expansion method to study wave interaction on water of variable depth based on Johnson's (1973a) equation.

Here we use derivative expansion method considering equation (1.17) as a perturbation of system (1.16) due to the parameter $c$. 

\begin{equation}
3c \frac{\partial^2 \eta}{\partial x^2 \partial x} = 0
\end{equation}
2.2 STUDY OF WAVE INTERACTION USING DERIVATIVE EXPANSION METHOD

We consider the asymptotic series expansion

\[ \eta = \eta_1 + \varepsilon \eta_2 + \varepsilon^2 \eta_3 + \ldots \ldots \]  

(2.1)

regarding \( \eta \) as a function of multiple scales of the parameter \( \varepsilon \). The partial derivatives with respect to \( t \) and \( x \) are also expanded as

\[ \frac{\partial}{\partial t} = \frac{\partial}{\partial t_0} + \varepsilon \frac{\partial}{\partial t_1} + \varepsilon^2 \frac{\partial}{\partial t_2} + \ldots \ldots \]  

(2.2)

\[ \frac{\partial}{\partial x} = \frac{\partial}{\partial x_0} + \varepsilon \frac{\partial}{\partial x_1} + \varepsilon^2 \frac{\partial}{\partial x_2} + \ldots \ldots \]  

(2.3)

Then substituting for (1.17) we get,

\[ \frac{\partial \eta_1}{\partial t_0} + \varepsilon \frac{\partial \eta_2}{\partial t_0} + \varepsilon \frac{\partial \eta_1}{\partial t_1} + \left[ 1 - \frac{1}{2} \varepsilon \right] \left[ \frac{\partial \eta_1}{\partial x_0} + \varepsilon \frac{\partial \eta_2}{\partial x_0} + \varepsilon \frac{\partial \eta_1}{\partial x_1} \right] \]  

\[ + \left[ \frac{3}{2} \alpha + \frac{5}{4} \varepsilon \alpha \right] \left[ \eta_1 \frac{\partial \eta_1}{\partial x_0} + \varepsilon \eta_1 \frac{\partial \eta_2}{\partial x_1} + \varepsilon \eta_1 \frac{\partial \eta_1}{\partial x_1} \right] \]  

\[ + \varepsilon \eta_2 \frac{\partial ^3 \eta_1}{\partial x_0^2 \partial x_1} \]  

\[ + \left[ \frac{1}{6} \beta - \frac{1}{3} \varepsilon \beta \right] \left[ \frac{\partial ^3 \eta_2}{\partial x_0^3} + \varepsilon \frac{\partial ^3 \eta_2}{\partial x_0^3} \right] \]  

\[ + 3 \varepsilon \frac{\partial ^3 \eta_1}{\partial x_0^2 \partial x_1} \]  

= 0  

(2.4)

27
Collecting $O(\epsilon^0), O(\epsilon^1)$ terms in equation (2.4) we get,
\[ L_0 \eta_1 = 0, \quad \text{(2.5)} \]
where
\[ L_0 = \frac{\partial}{\partial t_0} + \frac{\partial}{\partial x_0} + \frac{1}{2} \alpha \eta_1 \frac{\partial}{\partial x_0} + \frac{1}{6} \beta \frac{\partial^3}{\partial x_0^3}, \quad \text{(2.7)} \]
\[ L_1 = \frac{\partial}{\partial t_1} + \frac{\partial}{\partial x_1} + \frac{1}{2} \left[ 3 \alpha \eta_1 \frac{\partial}{\partial x_1} - \frac{\partial}{\partial x_0} \right] + \frac{1}{6} \beta \left[ 3 \frac{\partial^3}{\partial x_0^3} x_1 - 2 \frac{\partial^3}{\partial x_0^3} \right], \quad \text{(2.8)} \]
\[ L_2 = \frac{3}{2} \alpha \frac{\partial \eta_1}{\partial x_0}, \quad \text{(2.9)} \]
and
\[ N_0 = -\frac{5}{8} \alpha \frac{\partial}{\partial x_0}. \quad \text{(2.10)} \]

We note that equation (2.5) is a nonlinear homogeneous equation in $\eta_1$. Solving this and substituting in equation (2.6) we get a nonlinear nonhomogeneous equation in $\eta_2$.

Now we study the nonlinear interaction between a long wave and an ensemble of short waves (Jeffrey and Kawahara, 1982), i.e., a superposition of a number of monochromatic waves with
different wave numbers, or with a continuous spectrum. For this purpose, to the lowest order of approximation, we consider a solution of equation (2.5) in the form

\[ \eta_1 = \int_{-\infty}^{\infty} A_1(k; x_1, t_1, \ldots) \exp \left[ i(kx_0 - \omega t_0) \right] dk + B_1(x_1, t_1, \ldots), \quad (2.11) \]

where \( A_1(k) \) is a slowly varying complex amplitude with the wave number \( k \) and \( B_1 \) is a slowly varying real function representing the long-wave component. The reality of \( \eta_1 \) requires that \( A_1^*(k) = A_1(k) \) where the asterisk denote complex conjugate. The dispersion relation of the linear equation (2.5) is

\[ D(k, \omega) = -i\omega + ik - \frac{1}{6} \beta k^3 = 0, \quad (2.12a) \]

Then we have

\[ \omega(k) = k - \frac{1}{6} \beta k^3, \quad (2.12b) \]

and the group velocity is

\[ V(g) = \frac{\partial \omega}{\partial k} = 1 - \frac{1}{2} \beta k^2. \quad (2.13) \]

Substituting equation (2.11) in equation (2.6) we get,

\[ \left[ L_0 + L_2 \right] \eta_2 = \int_{-\infty}^{\infty} \left[ -\left( \frac{\partial}{\partial t_1} + Vg \frac{\partial}{\partial x_1} \right) + \frac{ik}{2} + \frac{i\beta k^3}{3} \right] A_1(k) \exp \left[ i(kx_0 - \omega t_0) \right] dk - \frac{3}{2} \alpha \left( \int_{-\infty}^{\infty} A_1(k') \right). \]
\[
\frac{\partial A_1(k'')}{\partial x_1} \exp i \left[ (k'+k'')x_0 - (\omega'+\omega'')t_0 \right] dk'dk''
\]

\[
+ B_1(x_1, t_1) \int_{-\infty}^{\infty} \frac{\partial A_1(k'')}{\partial x_1} \exp i \left[ k''x_0 - \omega''t_0 \right] dk''
\]

\[
+ \frac{\partial B_1}{\partial x_1} \int_{-\infty}^{\infty} A_1(k', x_1, t_1) \exp i \left[ (k'x_0 - \omega't_0) \right] dk'
\]

\[
+ B_1(x_1, t_1) \frac{\partial B_1}{\partial x_1}
\]

\[
- \frac{5}{8} \alpha \left\{ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \exp i \left[ (k' + k'')A_1(k')A_1(k'') \right] \exp i \left[ (k'x_0 - \omega't_0) \right] dk' \right\}
\]

where \( V_g \) denotes the group velocity and \( \omega' \) and \( \omega'' \) denote \( \omega(k') \) and \( \omega(k'') \) respectively.

2.3 THREE-WAVE INTERACTION

We now consider the resonant wave interaction between different wave modes. Two primary components of wave numbers

\[
\frac{\partial B_1}{\partial t_1} + V_g \frac{\partial B_1}{\partial x_1} + \frac{1}{2} \beta \frac{\partial^2 B_1}{\partial x_0^2 \partial x_1}, \quad \text{(2.14)}
\]

where \( V_g \) denotes the group velocity and \( \omega' \) and \( \omega'' \) denote \( \omega(k') \) and \( \omega(k'') \) respectively.
$k_1$ and $k_2$ and frequencies $\omega_1$ and $\omega_2$ give rise to an interaction term with the magnitudes of the wave number $k_3$ and corresponding frequency $\omega_3$ lying within the limits $|k_1+k_2|$ and $|k_1-k_2|$. Phillips (1960, 1977) has pointed out that a resonance is possible if the interaction frequencies $\omega_1+\omega_2$ and $\omega_1-\omega_2$ corresponds to wave numbers lying within that range and exchange of energy among wave modes is analogous to resonance in a forced linear oscillator. He has further shown that for three-wave interactions, the energy exchange is significant only when the conditions

$$k_1 + k_2 \pm k_3 = 0,$$

and

$$\omega_1 \pm \omega_2 \pm \omega_3 = 0.$$

are satisfied or nearly satisfied simultaneously.

Linear dispersion relation (2.12) admits the three-wave interaction process if $\omega'+\omega'' = \omega$ for $k'+k'' = k$. Here the three-wave interaction process does not occur since this condition is not satisfied.

In this case we obtain from equation (2.14) the following condition for the nonsecularity of the $O(\epsilon^3)$ solution,

$$\left[ \frac{\partial}{\partial t_1} + Vg \frac{\partial}{\partial x_1} - \frac{ik}{2} - \frac{ibk^3}{3} \right] A_1(k)$$
\[ \frac{3}{2} \alpha \left[ \frac{\partial}{\partial x_1} B_1(x_1, t_1) \frac{\partial A_1(k)}{\partial x_1} + A_1(k) \frac{\partial B_1}{\partial x_1} \right] + \frac{5}{4} \alpha \left[ A_1(k) i k B_1(x_1, t_1) \right] = 0, \quad (2.15) \]

with

\[ \frac{\partial B_1}{\partial t_1} = 0, \quad \frac{\partial B_1}{\partial x_1} = 0. \]

There is a corresponding equation for the complex conjugate \( A_1^*(k) \) also.

Equation (2.15) can be written as

\[ N = \left[ n(k) \right] dk, \quad (2.20) \]

\[ \left[ \frac{\partial}{\partial t_1} + V_g \frac{\partial}{\partial x_1} \right] A_1(k) + \frac{3}{2} \alpha \frac{\partial}{\partial x_1} \left[ A_1(k) B_1 \right] = \text{Im} \left\{ \left[ \frac{k}{2} + \frac{\beta k^3}{3} \right] A_1(k) - \frac{5}{4} \alpha k A_1(k) B_1(x_1, t_1) \right\}, \quad (2.16) \]

Multiplying by \( A_1^*(k) \) we get,

\[ N = \left[ n(k) \right] dk, \quad (2.20a) \]

\[ = \text{Im} \left\{ \left[ \frac{k}{2} + \frac{\beta k^3}{3} \right] A_1(k) - \frac{5}{4} \alpha k A_1(k) B_1(x_1, t_1) \right\}. \quad (2.17) \]

2.4 DISCUSSION

Perturbation method can be applied in the study of

Equating the real parts in equation (2.17) we get,
Equation (2.18) can be written as

$$\frac{\partial N}{\partial t} + \frac{\partial}{\partial x_1} \left[ N(V+R) \right] = 0,$$

(2.19)

where

$$n(k) = |A_1(k)|^2,$$

(2.20a)

$$N = \int_{-\infty}^{\infty} n(k) dk,$$

(2.20b)

$$V = \frac{1}{N} \int_{-\infty}^{\infty} Vg \ n(k) dk,$$

(2.20c)

and

$$R = \frac{1}{N} \int_{-\infty}^{\infty} \frac{3}{2} \alpha B_1 \ n(k) dk.$$

(2.20d)

Here $n(k)$ and $N$ represent, energy density and total energy density of the short waves respectively.

### 2.4 DISCUSSION

Perturbation method can be applied in the study of a wide range of physical phenomena. The guiding principle for obtaining asymptotic equations is merely the nonsecularity of
Equation (2.19) is a conservation law. It is found that the dispersion relation (2.12) does not admit three-wave interaction process. Thus there is no transfer of energy between different wave numbers. But the total energy of the short wave components is conserved by transfer of energy between the short wave components and the interacting long wave.

In this chapter we consider IST analysis and numerical study of equations (1.16) and (1.17).

As has been pointed out earlier IST method provides a procedure for obtaining pure soliton solutions and quantitative information about the general solutions of the KdV equation. Johnson (1973a) has briefly discussed development of solitary wave moving over an uneven bottom using IST method. He has obtained what he calls the eigendepths relating number of solitons formed to the depth of the shelves. Soliton solutions for various depths have also been examined by numerically integrating the relevant KdV equation.

The IST method for one-dimensional Schrödinger operator on a straight line has been used by Mel'nikov (1990) to derive solutions for the KdV equation with self consistent source which describe creation and annihilation of solitons. Meinhold (1991) has used IST method to find solutions for KdV