Chapter VI

Exact Fuzzy Modeling and Optimal Control of some
Well-known systems
Exact Fuzzy Modeling and Optimal Control of some Well-known systems

6.1 Launch Vehicle in the atmospheric phase

6.1.1 Introduction

There are several approaches for the flight control of launch vehicles and aircrafts. A typical and simple approach is the divide-and-conquer approach [130], where the flight is first partitioned into many separate operating regimes. For each of these regimes, a linearized dynamic model is approximated and the tools of linear control theory, is used to design individual compensators to satisfy closed loop specifications. Then the individual compensators are ‘stitched together’ with gain schedules to cover the full flight. The resulting scheduled control law is then verified with extensive nonlinear simulations. This raises questions on the legitimacy of the scheme, since there is no strict mathematical assurance that stability in the time-slice or individual regime sense is equivalent to stability in the time varying and nonlinear case.

Another evolving methodology for flight control is the dynamic inversion [131] or feedback linearization. This approach does not require the problem to be split in to large number of separate regimes. A nonlinear control law is fashioned which globally reduces the dynamics of selected control variables to integrators. Even though gain scheduling is avoided, this method requires considerable degree of sophistication and tends to result in rather complicated controllers. Adams et al [132] have applied $H_\infty$ robust control design methods to flight control. $H_\infty$ theory is based on linear time invariant systems where the nonlinearities and time variation of the plant are treated as uncertainties of the nominally invariant plant. This generally results in performance compromises for robustness. To cope with this problem, Adams et al [132] have designed $H_\infty$ control laws at four widely spaced operating points for a pitch axis autopilot and these are interpolated depending upon the actual operating condition.
In the present work, the *Theorem: 5.1 of Chapter: V* is applied for exact fuzzy modeling and optimal control of a launch vehicle in the atmospheric phase [117]. This is a new, rigorous and efficient approach for launch vehicle control.

### 6.1.2 Launch Vehicle Control and Guidance

The performance of a space launch vehicle during the launch phase of the flight is generally studied in two distinct, though related phases. The first deals with the trajectory of the vehicles with reference to some specified inertial frame and is concerned with such factors as payload capacity, dispersions from nominal and orbit capability. Dispersions of the actual, from nominal trajectory due to such factors as parameters uncertainty and random loads are generally referred to as “long period dynamics”. In this context, the vehicle is usually assumed to be a point mass and the oscillations about the nominal trajectory have a “long” period. However, the action of the control system in orienting the vehicle centre of mass, induce oscillations and these must be damped out if the mission is to be successful. These oscillations have a comparatively short period and the study of these motions is the “short period dynamics”.

Thus the control of an aerospace launch vehicle basically involves two control loops, the autopilot loop (inner loop) and the guidance loop (outer loop) as shown in the Fig: 6.1. The autopilot loop mainly deals with the short period dynamics of the vehicle to ensure stability and performance. The guidance loop mainly deals with the long period dynamics of the vehicle to ensure accurate target conditions.

A meaningful investigation of the stability and performance of the launch vehicle autopilot requires proper modeling of the short period dynamics. Perturbation models (perturbation from nominal) are used for this purpose. For deriving perturbation equations, factors such as center of gravity eccentricity, dynamic pressure, bending, propellant sloshing, engine inertia, control power plant nonlinearity etc. are to be considered.

Here, we consider the time varying rigid body short period dynamics of a typical launch vehicle with control power plant nonlinearity in the atmospheric phase for fuzzy modeling and optimal control of the same.
6.1.3 Short period modeling of a Launch Vehicle

The conventional type of Launch Vehicles has a high degree of symmetry about the longitudinal axis, which means that the inertial and aerodynamic cross coupling terms between pitch, yaw and roll are negligible. This affords a crucial simplification that pitch, yaw and roll control systems can be analyzed separately. For a symmetric Launch Vehicle and whose mass center coincides with geometric center, the short period perturbation model for the rigid body in the pitch plane is given as [130].

\[ \ddot{\alpha} = \frac{-g \cos \theta_e \theta}{U} - \frac{L_a}{mU} \alpha + \frac{T_c}{mU} \delta + \dot{\theta} \quad (6.1) \]

\[ \dot{\theta} = \mu_c \delta + \mu_a \alpha \quad (6.2) \]
\[ \mu_e = \frac{T_e l_e}{I_{yy}}, \quad \mu_a = \frac{L_a l_a}{I_{yy}} \]

g: acceleration due to gravity, \( U \): total forward velocity,
\( \theta_0 \): Attitude angle, \( m \): total mass, \( T_c \): control thrust \( L_a \): aerodynamic load, \( \delta \): engine deflection angle; \( \theta, \alpha \): perturbation attitude angle and angle of attack, \( l_c, l_a \): distance of the center of gravity from control point and center of pressure respectively, \( \mu_e, \mu_a \): coefficients of control and aerodynamic moments. \( I_{yy} \): Moment of inertia.

Equations (6.1) and (6.2) can be rewritten eliminating, \( \alpha \) as follows.

\[ \ddot{\theta} = -\frac{\mu_a g \cos \theta_0}{U} \theta + \mu_a \dot{\theta} - \frac{L_a}{mU} \dot{\theta} + \left\{ \mu_e \frac{L_a}{mU} \left( 1 + \frac{l_a}{l_c} \right) \right\} \delta + \mu_e \dot{\delta} \]  

\( 6.3 \)

The parameters \( g, \theta_0, U, L_a, m, T_c, \mu_e, \mu_a \) are all time varying in reality. But, in classical approach, they are assumed to be time invariant for the time slice of analysis.

### 6.1.4 Control Power Plant Non-linearity

Engine gymbal control and Secondary Injection Thrust Vector Control (SITVC) are commonly used for steering a Launch Vehicle. The former is used for liquid engines and the latter is used for solid booster stages. In SITVC, a nonflammable fluid is forced on the main flame to deflect it to achieve a side force. Electromechanical valves are used to control the port opening, which in turn controls the amount of fluid forced on the main flame. A typical SITVC port opening versus side force relation is given by

\[ F_s = k_1 \delta_p^2 + k_2 \delta_p \]  

\( 6.4 \)

where \( k_1, k_2 \) are constant and the slope is given by

\[ \frac{dF_s}{d\delta_p} = K_{a} (\delta_p) = k_2 + 2k_1 \delta_p \]

\( \delta_p \): is pintle opening of the port in meters and \( F_s \): is the side force in Newton. The side force is nonlinear with port opening. Hence, in the case of SITVC for solid stages, the control moment coefficient, \( \mu_e \) is represented as
\[ \mu_c(\delta_p, t) = \frac{K_{dc} (\delta_p) l_e(t)}{I_{yy}(t)} \]

which, shows that \( \mu_c \) is not only time varying but also nonlinear.

The control voltage to port opening is typically a linear second order system given as

\[ \delta_p = -2\xi\omega_n \delta_p - \omega_n^2 \delta_p + \omega_n^2 \delta_{vc} \quad (6.5) \]

\( \delta_{vc} \) : command voltage.

6.1.5 Rigid body with actuator dynamics

Defining the following scalar valued auxiliary functions [117] for the terms in (6.3), we have

\[ f_1(t) = -\mu_a(t) g(t) \cos(\theta_0(t)) \]
\[ f_2(t) = \mu_a(t) \]
\[ f_3(t) = \frac{-L_a(t)}{m(t)U(t)} \]
\[ f_4(t, \delta_p) = \mu_c(t, \delta_p) \frac{L_a(t)}{m(t)U(t)} \left[ 1 + \frac{l_a(t)}{l_e(t)} \right] \]
\[ f_5(t, \delta_p) = \mu_c(t, \delta_p) \]

Now the time varying nonlinear plant model for Rigid Body with actuator, in state space form can be written as

\[ \dot{X} = A(t, X) X + Bu \quad (6.6) \]

where \( u = \delta_{vc} \) and

\[ X = [\theta \ \dot{\theta} \ \ddot{\theta} \ \delta_p \ \dot{\delta}_p]^T \]
\[ B = \begin{bmatrix} 0 & 0 & 0 & 0 & \omega_n^2 \end{bmatrix}^T \]
\[ A(.) = \begin{bmatrix} f_1(t) & f_2(t) & f_3(t) & f_4(t, \delta_p) & f_5(t, \delta_p) \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\omega_n^2 & -2\xi\omega_n \end{bmatrix} \]
There are five scalar valued nonlinear auxiliary functions of states inside the system matrix $A(.)$. Typical profile of the auxiliary functions for a typical Launch Vehicle is given in Fig: 6.2 to 6.6. $f_4(t)$ & $f_5(t)$ are plotted for $\delta_p = 0$. 

![Fig: 6.2, $f_1(t)$ Vs Time](image1)

![Fig: 6.3, $f_2(t)$ Vs Time](image2)
Fig: 6.4, $f_3(t)$ Vs Time

Fig: 6.5, $f_4(t)$ Vs Time ($\delta_p = 0$)
6.1.6 Premise variables for exact fuzzy modeling

In general, the TS fuzzy model uses the states of the system as the premise variables in the fuzzy rules. Here, we choose the transformed variables namely the scalar valued auxiliary functions as the premise variables for the fuzzy rules. In the present case, we have transformed variables in the system matrix $A(.)$ only. But this procedure is general and applicable for transformed variables if any in the input matrix, $B(.)$ also as proved in Theorem: 5.1.

Each of these scalar valued auxiliary functions requires 2 membership functions to represent it exactly in a predetermined domain, knowing their bounds in that domain. Accordingly, there are 10 membership functions required in total for the exact representation of (6.6) and they are obtained as

\[ M_{j1} = \frac{\beta_j - f_j(\cdot)}{\beta_j - \alpha_j}, M_{j2} = 1 - M_{j1} \text{ for } j = 1, 2, \ldots, 5. \]

where $\beta_j = \max (f_j)$ and $\alpha_j = \min (f_j)$

Since there are five auxiliary functions and each function has two membership functions, we need $2^5 = 32$ rules to exactly represent this nonlinear time-varying
system. Now, the exact fuzzy model of the system can be expressed using new premise variables and **Theorem: 5.1** as follows.

**Rule 1**

If \( f_1(.) \) is \( M_{11} \) and \( f_2(.) \) is \( M_{21} \) and \( f_3(.) \) is \( M_{31} \) and \( f_4(.) \) is \( M_{41} \) and \( f_5(.) \) is \( M_{51} \)

Then

\[
\dot{X} = A_1 X + B_1 u
\]

\[
A_1(.) = \begin{bmatrix}
\alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 & \alpha_5 \\
0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & -\omega_n^2 & -2\xi\omega_n
\end{bmatrix}
\]

**Rule 2:**

If \( f_1(.) \) is \( M_{11} \) and \( f_2(.) \) is \( M_{21} \) and \( f_3(.) \) is \( M_{31} \) and \( f_4(.) \) is \( M_{41} \) and \( f_5(.) \) is \( M_{52} \)

Then

\[
\dot{X} = A_2 X + B_2 u
\]

\[
A_2(.) = \begin{bmatrix}
\alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 & \beta_5 \\
0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & -\omega_n^2 & -2\xi\omega_n
\end{bmatrix}
\]

Proceeding in this way, we have in **Rule 1**

\[
A_{i}(3,1) = \beta_1 \text{ if } f_1(.) \text{ is } M_{12} \\
= \alpha_1 \text{ if } f_1(.) \text{ is } M_{11}
\]

\[
A_{i}(3,2) = \beta_2 \text{ if } f_2(.) \text{ is } M_{22} \\
= \alpha_2 \text{ if } f_2(.) \text{ is } M_{21}
\]
Launch Vehicle Control

\[ A_i(3,3) = \beta_3 \text{ if } f_3(.) \text{ is } M_{32} \]
\[ = \alpha_3 \text{ if } f_3(.) \text{ is } M_{31} \]

\[ A_i(3,4) = \beta_4 \text{ if } f_4(.) \text{ is } M_{42} \]
\[ = \alpha_4 \text{ if } f_4(.) \text{ is } M_{41} \]

\[ A_i(3,5) = \beta_5 \text{ if } f_5(.) \text{ is } M_{52} \]
\[ = \alpha_5 \text{ if } f_5(.) \text{ is } M_{51} \]

Thus all the 32 rules can be written.

\[ B_i = \begin{bmatrix} 0 & 0 & 0 & 0 & \omega_i \end{bmatrix}, \text{ for } 1, 2, 3, \ldots, 32. \]

Using the results in the proof of Theorem: 5.1 in Chapter: V, the corresponding convex weighting coefficients are obtained as follows.

\[ h_i = \frac{M_{1i_1} M_{2i_2} M_{3i_3} M_{4i_4} M_{5i_5}}{S}; \text{ where} \]
\[ i_1 = 1 \text{ if } A_i(3,1) = \alpha_1; \quad i_1 = 2 \text{ if } A_i(3,1) = \beta_1 \]
\[ i_2 = 1 \text{ if } A_i(3,2) = \alpha_2; \quad i_2 = 2 \text{ if } A_i(3,2) = \beta_2 \]
\[ i_3 = 1 \text{ if } A_i(3,3) = \alpha_3; \quad i_3 = 2 \text{ if } A_i(3,3) = \beta_3 \]
\[ i_4 = 1 \text{ if } A_i(3,4) = \alpha_4; \quad i_4 = 2 \text{ if } A_i(3,4) = \beta_4 \]
\[ i_5 = 1 \text{ if } A_i(3,5) = \alpha_5; \quad i_5 = 2 \text{ if } A_i(3,5) = \beta_5 \]

\[ S = \sum_{i_1=1}^{2} \sum_{i_2=1}^{2} \sum_{i_3=1}^{2} \sum_{i_4=1}^{2} \sum_{i_5=1}^{2} M_{1i_1} M_{2i_2} M_{3i_3} M_{4i_4} M_{5i_5} = 1 \]

and \( i = 1, 2, 3 \ldots 32 \)

The association of \( h_i \) in this way to \((A_i, B_i)\) follows from Theorem: 5.1

Thus the exact fuzzy model of the nonlinear system in (6.6) is

\[ \dot{X} = \sum_{i=1}^{32} h_i(X) [A_i(X) + B_i u] \quad (6.7) \]
**Remark**

Here, each subsystem is not local, but ‘boundary subsystems’ defined by the bounds of the auxiliary functions in the restricted domain. Each of these 32 boundary subsystems can be viewed as 32 vertices of a hypercube, according to the geometric view of fuzzy sets [80]. These 32-vertices correspond to conventional crisp systems (Linear Time Invariant Systems). The fuzzy system to be modeled lies inside this hypercube and models the system exactly.

### 6.1.7 Optimal control design using exact fuzzy model

For each of the subsystem (6.7), the optimal control result in Section: 4.4 of Chapter: IV is used to solve the respective Steady State Ricatti Equation, since Theorem: 4.4 of Section:4.4 is applicable because the 32 subsystems are linear time invariant. The corresponding $B_i R^{-1} B_i^T \pi_{\alpha}^i$ are computed. Thus the optimal boundary feedback fuzzy subsystems are

$$\dot{X}^* (t) = [A_i - B_i R^{-1} B_i^T \pi_{\alpha}^i] \dot{X}^* (t) \text{ for } i = 1,2, ..., 32$$

The Optimal Global feedback fuzzy system is

$$\dot{X}^* (t) = \sum_{i=1}^{i=32} h_i (X^*)[A_i - B_i R^{-1} B_i^T \pi_{\alpha}^i] \dot{X}^* (t)$$

This optimal global feedback fuzzy system is based on the exact fuzzy model of the true system under consideration in the domain of interest and is exponentially stable as noted in Section:4.4 and proved in [111].

### 6.1.8 Simulation Results

Optimal controller design method in Section: 6.1.7 above is used to design 32 fuzzy controllers for the 32 fuzzy subsystems in (6.7). Simulation was carried out in Matlab-Simulink with bounds of auxiliary functions computed for a reference trajectory. The $Q$ and $R$ matrices chosen for this design are
The overall optimal controller is obtained as the convex fuzzy blending of the 32 individual controllers as explained in Section: 6.1.7. A typical slow varying command input profile, in planar simulation is given in Fig: 6.7 and the tracking error performance of the controlled system are compared in Fig: 6.8 with that of a classical design with gain scheduling. Tracking error for the exact fuzzy model based optimal control is much lower than that of the controller based on classical design with gain scheduling. The performance improvement is obvious and validates the new design method.

![Fig: 6.7, Typical Command profile](image)
Step responses at three representative time regions are also compared in Figures: 6.9 to 6.11. In all these three regions, the step responses are better for optimal controller based on exact fuzzy model than the classical design. Performance improvement for the exact fuzzy model based optimal design is obvious and hence validates the method.
6.1.9 Conclusion

A new, rigorous and optimal control approach for Launch Vehicle control is developed based on a single exact fuzzy dynamic model developed for the same in the entire atmospheric phase. The controlled global system is exponentially stable. It is to be noted that the time slice vice linear time invariant assumption in the conventional modeling and gain scheduling for control are avoided. The performance improvement obtained validates the methodology.
6.2 **Nonlinear second order systems**

A linear second order system is a standard control problem treated extensively in the literature. In practice, plants approximated as second order systems do have nonlinear properties like input saturation, output slew rate limit, nonlinear stiffness and hysteresis etc. Due to the practical and theoretical significance of second order systems, a second order system with three major nonlinear properties encountered in practice viz. input saturation, output slew rate limit and nonlinear stiffness is considered here for exact fuzzy modeling and its optimal control [118].

6.2.1 **Fuzzy modeling of nonlinear second order systems**

The nonlinear elements usually seen in a second order system are saturation, rate limit, stiffness as a function of states and hysteresis. Here, we consider the first three nonlinear properties in the fuzzy modeling. Consider the linear second order system in Fig: 6.12, \( \dot{X} = AX + Bu \) where,

\[
A = \begin{bmatrix} 0 & 1 \\ -\omega_n^2 & -2\xi\omega_n \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 \\ \omega_n^2 \end{bmatrix},
\]

\( X = [x_1, x_2]^T \) and \( u \) is the scalar input.

![Diagram](Fig:6.12)

**Fig:6.12 , Second order linear dynamic system**

Here, the effect of input saturation can be modeled as the effective reduction of the element \( B(2, 1) \) as input increases beyond the stipulated saturation limit. The effect of input saturation is modeled as
\[ B(2,1) = f_1(u) \quad \text{where} \]
\[ f_1(u) = \begin{cases} \omega_n^2 \text{ if } |u| \leq u_{sat} \\ \frac{\omega_n^2 u_{sat}}{|u|} \text{ if } |u| > u_{sat} \end{cases} \]

where, \( u_{sat} \) is the magnitude of input saturation limit for positive and negative inputs.

The output rate limit can be modeled by forcing the acceleration to zero whenever \( x_2 = \dot{x}_1 \) exceeds a specified limit. Now the combined effect of input saturation and rate limit can be modeled as follows. Let the rate limit is \( S_i = \max(x_2) \).

\[
\text{if} \quad (x_2 \geq S_i) \quad B(2,1) = \frac{\omega_n^2 x_1 + 2\xi \omega_n S_i}{u}
\]
\[
\text{elseif} \quad (|u| > u_{sat}) \quad B(2,1) = \frac{\omega_n^2 u_{sat}}{|u|}
\]
\[
\text{else} \quad B(2,1) = \omega_n^2
\]

Nonlinearity due to stiffness can be modeled as
\[ A(2,1) = f_2(x_1, x_2) \quad \text{where } f_2(\cdot) \text{ represents the dependency of the stiffness on } x_1, x_2. \]

For example, let
\[ f_2(\cdot) = \begin{cases} -2\omega_n^2 \text{ if } |x_1| \geq 0.01 \\ -\omega_n^2 \text{ otherwise} \end{cases} \]

this models a type of spring whose stiffness varies with displacement. In general, \( f_2(\cdot) \) can be any scalar valued function of states. Now the nonlinear second order system with input saturation, rate limit and nonlinear spring can be expressed as
\[ \dot{X} = A(X, u) X + B(X, u) u \quad (6.8) \]

where
\[ A(\cdot) = \begin{bmatrix} 0 & 1 \\ f_2(\cdot) & -2\xi \omega_n \end{bmatrix}, \quad B(\cdot) = \begin{bmatrix} 0 \\ f_1(\cdot) \end{bmatrix} \]

\( f_i(\cdot) \) is defined as follows:
if \( x_2 \geq S_1 \)
\[
 f_1(\cdot) = B(2,1) = -f_2(\cdot) x_1 + 2\delta \omega_n S_1
\]
elseif \( |u| > u_{sat} \)
\[
 f_1(\cdot) = B(2,1) = \frac{\omega_n^2 u_{sat}}{|u|}
\]
else
\[
 f_1(\cdot) = B(2,1) = \omega_n^2
\]
end

and \( f_2(\cdot) \) is any given function of states. Fig: 6.13 represent the block diagram of the effective second order nonlinear system being exactly modeled by TS fuzzy dynamic model. There is one nonlinear auxiliary function of states \( f_2(\cdot) \) inside the system matrix \( A(\cdot) \) and another \( f_1(X, u) \) in \( B(\cdot) \).

![Block diagram of second order system](image)

Fig: 6.13 Second order system with input saturation, rate limit and nonlinear spring

### 6.2.2 Premise variables for exact fuzzy modeling

In general, the TSK fuzzy model uses the states of the system as the premise variables in the fuzzy rules. We had used transformed variables in [116,117] for the fuzzy modeling of inverted pendulum and a launch vehicle in the atmospheric phase. Here also we choose the two transformed variables \( f_1(\cdot), f_2(\cdot) \) as premise variables for the fuzzy modeling of nonlinear second order system defined in Section: 6.2.1. Each
of this function requires 2 membership functions to represent it exactly in a predetermined domain, knowing their bounds in that domain. Accordingly, 4 membership-functions are required in total for the exact representation of (6.8) and they are represented as

\[ M_{j1} = \frac{\beta_j - f_j(\cdot)}{\beta_j - \alpha_j}, \quad M_{j2} = 1 - M_{j1} \text{ for } j = 1, 2. \]

Since there are two auxiliary functions and each function has two membership functions, we need \(2^2 = 4\) rules to exactly represent this nonlinear system.

**Rule 1**  
If \( f_1(\cdot) \) is \( M_{11} \) and \( f_2(\cdot) \) is \( M_{21} \)  
Then  
\[
\dot{X} = A_1X + B_1u
\]

**Rule 2**  
If \( f_1(\cdot) \) is \( M_{11} \) and \( f_2(\cdot) \) is \( M_{22} \)  
Then  
\[
\dot{X} = A_2X + B_1u
\]

**Rule 3**  
If \( f_1(\cdot) \) is \( M_{12} \) and \( f_2(\cdot) \) is \( M_{21} \)  
Then  
\[
\dot{X} = A_1X + B_2u
\]

**Rule 4**  
If \( f_1(\cdot) \) is \( M_{12} \) and \( f_2(\cdot) \) is \( M_{22} \)  
Then  
\[
\dot{X} = A_2X + B_2u
\]

where 
\[
A_1(\cdot) = \begin{bmatrix} 0 & 1 \\ \alpha_2 & -2\xi\omega_n \end{bmatrix}, \quad A_2(\cdot) = \begin{bmatrix} 0 & 1 \\ \beta_2 & -2\xi\omega_n \end{bmatrix}
\]

\[
B_1(\cdot) = \begin{bmatrix} 0 \\ \alpha_1 \end{bmatrix}, \quad B_2(\cdot) = \begin{bmatrix} 0 \\ \beta_1 \end{bmatrix}
\]

using the results in the proof of Theorem 5.1, the corresponding convex weighting coefficients are obtained as 

\[
h_1(\cdot) = \frac{M_{11}M_{21}}{S}, \quad h_2(\cdot) = M_{11}M_{22}
\]

\[
h_3(\cdot) = M_{12}M_{21}, \quad h_4(\cdot) = M_{12}M_{22}
\]

where 
\[
S = M_{11}M_{21} + M_{11}M_{22} + M_{12}M_{21} + M_{12}M_{22} = 1
\]
Thus the exact fuzzy model of the nonlinear system above is

\[
\dot{X} = \sum_{i=1}^{4} h_i(X, u) [A_i X + B_i u]
\]  

(6.9)

**Remark**

Here, each subsystem is not local, but "boundary subsystems" defined by the bounds of the auxiliary functions in the restricted domain. Each of these 4 boundary subsystems can be viewed as 4 vertices of a square, according to the geometric view of fuzzy sets [80]. These 4 vertices correspond to conventional crisp systems (Linear Time Invariant Systems). The fuzzy system to be modeled lies inside this hypercube.

### 6.2.3 Optimal control design using exact fuzzy model

For each of the subsystem in (6.9), the optimal control result in Section: 4.4 of Chapter: IV is used to solve the respective Steady State Ricatti Equations, since the four subsystems are linear time invariant. The corresponding \(B, R^{-1} B^T \pi_{\alpha_i} \) are computed. Thus the optimal feedback boundary fuzzy subsystems are

\[
\dot{X}^*(t) = [A_i - B_i R^{-1} B_i^T \pi_{\alpha_i}^t] X^*(t) \quad \text{for } i=1,2,3,4.
\]

The optimal Global feedback fuzzy system is

\[
\dot{X}^*(t) = \sum_{i=1}^{4} h_i(X^*) [A_i - B_i R^{-1} B_i^T \pi_{\alpha_i}^t] X^*(t)
\]

This optimal global feedback fuzzy system is based on the exact fuzzy model of the true system under consideration in the domain of interest. This global controlled system is exponentially stable.

### 6.2.4 Simulation Results

Optimal controller design method in Section: 6.2.3 is used to design 4 fuzzy controllers for the 4 fuzzy subsystems in (6.9). Simulation was carried out in Matlab-Simulink with S-functions defined for \(f_1(.)\) and \(f_2(.)\) with

\[
u_{\text{sat}} = 0.5, \quad S_i = 5.0, \quad \omega_n = 12\pi, \quad \xi = 0.7 \quad \text{and} \quad f_2(.) \text{as}
\]

\[
f_2(.) = \begin{cases} 
-2\omega_n^2 & \text{if } |x_1| \geq 0.01 \\
-\omega_n^2 & \text{otherwise}
\end{cases}
\]

The bounds of \(f_1(.)\) and \(f_2(.)\) were determined for a regular trajectory with non-zero initial condition by trial simulation, The \(Q\) and \(R\) matrices chosen are
The overall optimal controller is obtained as the convex fuzzy blending of the 4 individual controllers as in Section: 6.2.3. Simulation was carried out with initial condition $x_1(0) = 1.0$ with a sinusoidal disturbance at the output. The state trajectory plots for the optimal Fuzzy Quadratic Regulator (FQR) are given in Figures: 6.14 and 6.15. The corresponding control input for the Fuzzy Quadratic Regulator is given in Fig 6.16.

![Graph showing state trajectory plots for the optimal Fuzzy Quadratic Regulator (FQR), $x_1$ Vs Time](image_url)

**Fig: 6.14, Fuzzy Quadratic Regulator (FQR), $x_1$ Vs Time**
Fig: 6.15, Fuzzy Quadratic Regulator, $x_2$ Vs Time

Fig: 6.16, Fuzzy Quadratic Regulator, input: $u$

For comparison purpose, a linear quadratic regulator (LQR) was also designed with the same $Q$ and $R$. The state trajectory, $x_1$ of the conventional LQR is given in figure: 6.17. The corresponding control input is given in Fig 6.18.
Second Order nonlinear

In order to compare the disturbance rejection of the optimal FQR with conventional LQR, the following sinusoidal disturbance in Fig 6.19 was applied at the output of the regulator. The disturbance rejection of optimal FQR is shown in Fig 6.20.
and that of the conventional LQR is shown in Fig 6.21. The disturbance rejection for the optimal FQR is better than the conventional LQR, which again validates the exact fuzzy dynamic modeling and optimal control methodology.
The controller switching due to the nonlinear spring is evident from the membership function plots below in Figures: 6.22 and 6.23 for the FQR.

**Fig: 6.21, Disturbance Rejection, Classical LQR**

**Fig: 6.22, Fuzzy Quadratic Regulator, membership-function \( M_{11} \) Vs Time**
For the optimal FQR, the membership function plots in Figures: 6.24 and 6.25 below indicate the varying controller weights due to input saturation and slew rate limit.
6.2.5 Conclusion

The effectiveness of optimal fuzzy control based on the exact fuzzy model is validated for a generic second order nonlinear dynamic system with input saturation, slew rate limit and nonlinear spring.