Chapter - 7
MHD Flow with Heat Transfer Due to a Point Sink
7.1 INTRODUCTION

The analysis of laminar boundary layer and convective heat transfer has attracted considerable attention because of its increasing applications in many scientific and engineering fields such as oil recovery techniques, geophysics, heat storage beds, thermal insulation engineering etc. The study of heat transfer and flow field is necessary for determining the quality of the final products of such fields. Ackerberg and Robert (1965) studied the boundary layer flow inside a cone and obtained a series solution of the problem under some restricted conditions on the potential flow. Takhar et al. (1986) studied the MHD flow with heat and mass transfer due to a point sink. Pop et al. (1992), Eswara and Nath (1994), Khound and Hazarika (2000), Eswara and Bommaih (2004) and Hossain et al. (2004) investigated the different flow models under variable viscosity and point source/sink. Chamkha (2004) studied MHD convective heat and mass transfer past a semi-infinite vertical permeable moving plate with heat absorption. Molla et al. (2005) investigated the effects of MHD natural convection flow on a sphere in presence of heat source. More recently, Choudhary and Hazarika (2008) studied the effects of variable viscosity and thermal conductivity on MHD flow due to a point sink.

Based on the above mentioned investigations and applications, the present chapter is concerned with a steady, two-dimensional axisymmetric
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boundary layer flow of an electrically conducting fluid in a circular cone with a hole at the vertex in the presence of magnetic field. Numerical solutions are obtained for the governing equations by Runge-Kutta method of order four and shooting technique.

7.2 FORMULATION OF THE PROBLEM

Consider the steady laminar incompressible two-dimensional axisymmetric boundary layer flow of an electrically conducting fluid in a circular cone with a hole at the vertex in the presence of uniform magnetic field. The hole can be regarded as a three-dimensional point sink. The cone has been taken as semi-infinite in length so that it can be regarded as independent of length $r$. The electrical conductivity of the fluid is assumed to be small so that the induced magnetic field and hall-effect are neglected. The effects of surface mass transfer, viscous dissipation and Joule heating have been taken into consideration. The fluid properties are assumed to be isotropic and constant. The uniform magnetic field $B_0$ is fixed relative to the fluid is applied in $z$-direction (normal to $r$), $r$ and $z$ are the distances along and perpendicular to conical surface respectively (Figure 7.1).

Under the foregoing assumptions, the boundary layer equations governing the motion (Takhar et al. (1986) and Cramer and Pai (1973)) are:

$$\frac{\partial}{\partial r} (ru) + \frac{\partial}{\partial z} (rw) = 0$$

(7.1)
\[ u \frac{\partial u}{\partial r} + w \frac{\partial u}{\partial z} = - \frac{1}{\rho} \frac{\partial p}{\partial r} + \nu \frac{\partial^2 u}{\partial z^2} - \frac{\sigma B_0^2 u}{\rho} \]  
(7.2)

\[ u \frac{\partial T}{\partial r} + w \frac{\partial T}{\partial z} = \alpha \frac{\partial^2 T}{\partial z^2} + \nu \left( \frac{\partial u}{\partial z} \right)^2 + \frac{\sigma_e B_0^2 u^2}{\rho c_p} \]  
(7.3)

where \[- \frac{1}{\rho} \frac{\partial p}{\partial r} = U_\infty \frac{dU_\infty}{dr} + \frac{\sigma B_0^2 U_\infty}{\rho} \]  
(7.4)

**Figure 7.1:** Physical model and co-ordinate system

where \(u\) and \(w\) are the radial and axial velocities in the directions of \(r\) and \(z\) respectively, \(\rho\) is the density, \(\nu = \frac{\mu}{\rho}\) is the kinematic viscosity, \(\sigma_e\) is the electrical conductivity, \(\alpha\) is the thermal diffusivity, \(c_p\) is the specific heat at constant pressure, \(T\) is the temperature and \(U_\infty\) is the free stream velocity and given by \(U_\infty = -\frac{m}{r^2}\), \(m > 0, r > 0\).

The boundary conditions are:

\[ z = 0: \quad u = 0, \quad w = w_w; \quad T = T_w \]
\[ z \to \infty: \quad u \to U_\infty; \quad T \to T_\infty \]  
(7.5)
where \( w \) is the suction/injection velocity and the subscripts \( w \) and \( \infty \) denote conditions at the wall and in the free stream respectively.

### 7.3 ANALYSIS

The continuity equation (7.1) is identically satisfied by stream function \( \psi(r,z) \), defined as

\[
ru = \frac{\partial \psi}{\partial z}, \quad rw = -\frac{\partial \psi}{\partial r}
\]

(7.6)

For the solution of the momentum and energy equations (7.2) and (7.3), the following dimensionless variables are defined

\[
\eta = \frac{1}{r} \sqrt{\frac{m}{2w^2 z}}
\]

(7.8)

\[
\frac{T-T_w}{T-\infty} = \theta(\eta)
\]

(7.9)

The momentum and heat transfer equations reduce to

\[
f'' = 4\left(1 - f'^2\right) + M \left(1 - f'\right) = 0
\]

(7.10)

\[
\theta'' - Pr f\theta' + Pr Ecf^{-2} + M Pr Ecf^{-2} = 0
\]

(7.11)

The corresponding boundary conditions are:

\[
\eta = 0 : \quad f = f_w, \quad f' = 0; \quad \theta = 1
\]

\[
\eta \to \infty : \quad f' \to 1; \quad \theta \to 0
\]

(7.12)
where prime ( ) denotes the derivative with respect to $\eta$, $m$ is the strength of the point sink and dimensionless parameters are

$$M = \frac{2\sigma_e B_0^2 r^3}{m\rho} \quad \text{(Magnetic Parameter)}$$

$$Pr = \frac{\mu c_p}{\kappa} \quad \text{(Prandtl Number)}$$

$$Ec = \frac{U_\infty^2}{c_p(T_w - T_\infty)} \quad \text{(Eckert Number)}$$

and

$$f_w = rw_w \sqrt{\frac{2r}{vm}} \quad \text{(Mass Transfer Parameter)} \quad (7.13)$$

It may be noted that the boundary layer approximation is not valid in the immediate neighbourhood of the hole (Rosenhead (1963)). Also, mass transfer parameter $f_w$ will be a constant if the velocity is normal to the wall; suction/injection velocity $w_w$ varies as $\frac{1}{\sqrt{r^3}}$ as $vm$ is constant, further $f_w > 0$ or $< 0$ according as whether it is injection or suction.

The physical quantities of interest of the problem are the skin-friction coefficient $C_f$ and the Nusselt number $Nu$, can be expressed, respectively as

$$C_f = -\frac{\mu}{\rho U_\infty^2} \left( \frac{\partial u}{\partial z} \right)_{z=0} = \sqrt{\frac{2}{\text{Re}}} f'(0) \quad (7.14)$$

$$Nu = -\frac{r}{(T_w - T_\infty)} \left( \frac{\partial T}{\partial z} \right)_{z=0} = -\sqrt{\frac{\text{Re}}{2}} \theta'(0) \quad (7.15)$$

where $\text{Re} = \frac{m}{r\nu}$ is the local Reynolds number.
Numerical results for skin-friction coefficient $C_f$ which is proportional to the velocity gradient at the surface i.e. $f'(0)$ and the Nusselt number $Nu$ which is proportional to the temperature gradient at the surface i.e. $-\theta'(0)$ are tabulated in tables 7.1 and 7.2 respectively.

7.4 NUMERICAL METHOD FOR SOLUTION

The equations (7.10) and (7.11) along with boundary conditions (7.12) are solved by converting the boundary value problem into an initial value problem (Conte and Boor (1981)). We set

$$
\begin{align*}
    f' &= p, \quad p' = q, \quad q' = fq - 4\left(1 - p^2\right) - M(1 - p) \\
    \theta' &= s, \quad s' = Pr fs - Pr Ec q^2 - M Pr Ec p^2
\end{align*}
$$

(7.16)

with boundary conditions (7.12).

In order to integrate the system (7.16) as an initial value problem we require a value $q(0)$ (i.e. $f'(0)$) and $s(0)$ (i.e. $\theta'(0)$) but no such values are given at the boundary. Using shooting method with suitable given values for $q(0)$ and $s(0)$ are chosen and then integration is carried out by Runge–Kutta fourth order method. We compare the calculated values for $f'$ and $\theta$ at $\eta \to \infty$ with the given boundary condition $f(\infty) \to 4$ and $\theta(\infty) \to 0$ and adjust the estimated values $q(0)$ and $s(0)$, to give a better approximation for the solution.
7.5 RESULTS AND DISCUSSIONS

The set of non-linear differential equations (7.10) and (7.11) with boundary conditions (7.12) are solved numerically using Runge-Kutta fourth order algorithm with a systematic guessing of shooting technique until the boundary conditions at infinity are satisfied. The computations are done by a programme which uses a symbolic and computational computer language Matlab. Numerical computations are performed for various values of the physical parameters involved e.g. magnetic parameter $M$, mass transfer parameter $f_w$, Prandtl number $Pr$ and Eckert number $Ec$.

Tables 7.1 and 7.2 present the variations of skin-friction coefficient $C_f$ which is proportional to the velocity gradient at the surface i.e. $f'(0)$ and Nusselt number $Nu$ which is proportional to the temperature gradient at the surface i.e. $-\theta(0)$, for various values of the parameters respectively. It is found that skin-friction coefficient $C_f$ and Nusselt number $Nu$ increase with suction ($f_w<0$) and the effect of injection ($f_w>0$) is just opposite. However, the effect of the magnetic parameter $M$ on skin-friction coefficient $C_f$ is slightly different. The skin-friction coefficient $C_f$ increases with increasing values of magnetic parameter $M$. The reason for such a behaviour is that the suction ($f_w<0$) and magnetic parameter $M$ reduce the thickness of the momentum boundary layer which results in increase in skin-friction. The effect of injection ($f_w>0$) is just opposite. The Nusselt number $Nu$ decreases with
increasing values of Eckert number $Ec$ and magnetic parameter $M$. Also, Nusselt number $Nu$ decreases with increasing values of Prandtl number $Pr$ at positive values of mass transfer parameter $f_w$ (i.e. for injection or $f_w > 0$).

The variations of velocity profile against similarity variable $\eta$ for various values of parameters are shown in the figures 7.2 to 7.4. It is observed from figure 7.2 that velocity decreases as mass transfer parameter $f_w$ increases. Figure 7.3 and 7.4 show the velocity distribution for various values of magnetic parameter $M$ when $f_w > 0$ (injection) and $f_w < 0$ (suction) respectively. It is noted from both the figures that velocity increases with increasing values of magnetic parameter $M$ both for injection as well as suction. The effects of the magnetic parameter $M$, Prandtl number $Pr$, Eckert number $Ec$ and mass transfer parameter $f_w$ on temperature profile against similarity variable $\eta$ have been shown in the figures 7.5 to 7.10. Temperature increases with increasing values of magnetic parameter $M$, Prandtl number $Pr$, Eckert number $Ec$ and mass transfer parameter $f_w$ for any particular values of the other parameters.

7.6 CONCLUSION

The steady two–dimensional, laminar boundary layer flow of a viscous incompressible electrically conducting fluid in a circular cone with hole at the vertex (point sink) in the presence of a uniform transverse magnetic field is studied. Taking suitable similarity variables, the governing boundary layer
equations are transformed into ordinary differential equations and solved them numerically by standard technique. The study shows that the magnetic field increases the skin-friction, but reduces the heat transfer. The rate of heat transfer increases with suction \((f_w < 0)\) and decreases with injection \((f_w > 0)\). The effect of mass transfer on the skin-friction and heat transfer is more pronounced than the magnetic parameter \(M\). Suction \((f_w < 0)\) reduces the thermal boundary layer whereas the injection \((f_w > 0)\) and the magnetic parameter \(M\) increase it.

**Table 7.1:** Variations of the velocity gradient at the surface i.e. \(f'(0)\) for various values of \(f_w\) and \(M\)

<table>
<thead>
<tr>
<th>(f_w)</th>
<th>(M=0.0)</th>
<th>(M=0.25)</th>
<th>(M=0.5)</th>
<th>(M=1.0)</th>
</tr>
</thead>
<tbody>
<tr>
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</tr>
<tr>
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<td>1.679</td>
<td>1.729</td>
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</tr>
<tr>
<td>2.0</td>
<td>1.449</td>
<td>1.500</td>
<td>1.549</td>
<td>1.644</td>
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</table>
Table 7.2: Variations of the temperature gradient at the surface i.e. $-\theta(0)$ for various values of $f_w$, $M$, $Pr$ and $Ec$

<table>
<thead>
<tr>
<th>$f_w$</th>
<th>$M$</th>
<th>$Pr=0.023$</th>
<th>$Pr=0.72$</th>
<th>$Pr=1.0$</th>
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<td></td>
<td>$Ec=0.1$</td>
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<td>1.002</td>
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<td>1.001</td>
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</tr>
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</tr>
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</tr>
<tr>
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</tr>
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Figure 7.2: Velocity distribution against $\eta$ for various values of $f_w$ when $M = 1.0$.

Figure 7.3: Velocity distribution against $\eta$ for various values of $M$ when $f_w = 1.0$. 
Figure 7.4: Velocity distribution against $\eta$ for various values of $M$ when $f_w = -1.0$

Figure 7.5: Temperature profile against $\eta$ for various values of $M$ when $f_w = 1.5$, $Pr = 1.0$ and $Ec = 0.5$
Figure 7.6: Temperature profile against $\eta$ for various values of $M$ when $f_w=0.0$, $Pr=1.0$ and $Ec=0.5$.

Figure 7.7: Temperature profile against $\eta$ for various values of $M$ when $f_w=-1.5$, $Pr=1.0$ and $Ec=0.5$. 
Figure 7.8: Temperature profile against $\eta$ for various values of $Pr$ when $f_w = 1.5$, $M = 0.2$ and $Ec = 0.5$

Figure 7.9: Temperature profile against $\eta$ for various values of $Ec$ when $f_w = 1.5$, $M = 1.0$ and $Pr = 0.7$
Figure 7.10: Temperature profile against $\eta$ for various values of $f_w$ when $M=0.2$, $Pr=1.0$ and $Ec=0.5$