CHAPTER 6

RETRIAL QUEUE SYSTEM
UNDER FUZZY ENVIRONMENT
CHAPTER-6

RETRIAL QUEUE SYSTEM UNDER FUZZY ENVIRONMENT

6.1 INTRODUCTION

In retrial queue system an arriving customer who finds the server busy is obliged to leave the service area and join a pool of unsatisfied customers called the “Orbit”. From the orbit, each customer further applies for the service after an uncertain amount of time called retrial time (J.R Artalejo 1999). The retrial queues have applications to model problems in telephone, computer, communication system, teletraffic theory. To understand the concept of retrial queue system suppose a telephone caller on dialling a number gets a busy signal in such cases, the caller repeats his calls after a random amount of time; other callers also do the same. These callers become sources of repeated calls and remain in what is termed as orbit, while a fresh caller (called a primary caller) who finds the facility empty immediately gets the service. This is the retrial queue system. Here we have assumed the capacity of orbit is infinite.

The successive retrial times are independent and identically distributed according to an exponential distribution with retrial rate $\theta$. Arriving customers form a single waiting line in front of server and get service in order of their arrivals. The service time provided by a single server is exponentially distributed with service rate $\mu$. Arriving customers at the service facility from outside at rate $\lambda$. This retrial model is denoted as M/M/1/1-R.

There are so many applications found for retrial queues in science and engineering streams. Recently Diamond and Alfa (1999) constructed a
method for approximating the stationary distribution and waiting time moments of M/PH/1 retrial queue with phase inter-retrial times. The BMAP/G/1 retrial system with search for customers immediately on termination of service was studied by Dudin et al. (2004) in which inter-retrial is followed by exponential distribution and duration of search is characterized by a generally distributed variable. Lopez-Herrero (2002) presented the explicit formulae for the probabilities of the number of customers being served in a busy period and an explicit expression for the second moment for M/G/1 retrial queuing system is also been given.

In the above described literature, the inter arrival times, inter retrial times, and service times of customers are required to follow certain probability distributions with fixed parameters. However in many real world applications, the parameters distributions may only be characterized subjectively, that is the arrival, retrial and service pattern are typically described in everyday language. Thus retrial queues in crisp nature can be extended to fuzzy environment. By extending the crisp retrial queues to fuzzy retrial queues in the context, these models become approximate for a wider range of applications. Chuen – Horng Lin, Jau-Chuan, Hsin-I Huang (2007) constructed the membership function for fuzzy retrial queuing system using non-linear programming approach with three fuzzy variables, fuzzified exponential arrival, retrial and service rate. Recently, W.Ritha & Robert (2009) applied the fuzzy set theory in order to construct the membership function of retrial queue by considering the arrival rate, retrial rate and service rate in fuzzy triangular numbers. Singh T.P & Kusum (2011) constructed a “Trapezoidal fuzzy network queue model with blocking”.

This chapter is an extended work of W.Ritha & Robert (2009) and Singh T.P & kusum (2011) in which the fuzzy set theory is applied to construct the membership function of a fuzzy retrial queues by using arrival rate, retrial rate, service rates in fuzzy trapezoidal numbers. $\alpha -$cut approach and fuzzy arithmetic operations are used to derive system characteristics. The model in notation form can be denoted by FM/FM/1/1-FR where F denotes the fuzzified exponential rate, with single server and single system capacity. The proposed model is illustrated with a numerical example. The fuzzy expected waiting time in the orbit for retrial system as well as expected number of customers in orbit has been presented through $\alpha -$cut and graphs. $\alpha -$cut approach and fuzzy arithmetic are used to construct system characteristic membership function.

6.2 FUZZY SET

In the universe of discourse $X$, a fuzzy subset $\tilde{A}$ on $X$ is defined by the membership function $\mu_{\tilde{A}}(X)$ which maps each element $x$ into $X$ to a real number in the interval [0,1]. $\mu_{\tilde{A}}(X)$ denotes the grade or degree of membership and it is usually denoted as $\mu_{\tilde{A}}(X) : X \rightarrow [0,1]$.

$\alpha -$cut: if a fuzzy set $A$ is defined on $X$, for any $\alpha \in [0,1]$, then $\alpha -$cut $A(\alpha)$ is represented by the following crisp set,

Strong $\alpha -$cut: $A(\alpha^+) = \{x \in X | \mu_A(x) > \alpha \}; \alpha \in [0,1]$.

Weak $\alpha -$cut: $A(\alpha) = \{x \in X | \mu_A(x) \geq \alpha \}; \alpha \in [0,1]$.

Therefore, it is inferred that fuzzy set $A$ can be treated as a crisp set $A(\alpha)$ in which all the members have their membership values greater than or atleast equal to $\alpha$. the concept of $\alpha -$cut is one of the most important concepts in fuzzy set theory. And here, we define support and height of a fuzzy set in terms of $\alpha -$cut.
Definition 1: The support of a fuzzy set $A$ is a crisp set represented as $\text{supp}A(x)$ such that, $\forall \{x \in X|\mu(x) > 0\}$. Thus, support of a fuzzy set of all members with a strong $\alpha$ – cut where $\alpha=0$.

Definition 2: The height of fuzzy set $h\{A(x)|x \in X\}$ is the maximum value of its membership function $\mu(x)$ such that $A(\alpha) = \{x \in X|\mu_A(x) \geq \alpha\}$ and 0 does not belong to $\alpha$.

A fuzzy set where Max $\{\mu(x) = 1\}$ is called as a normal fuzzy set, otherwise, it is referred as sub-normal fuzzy set.

### 6.3 TRAPEZOIDAL FUZZY NUMBER:

The fuzzy number $\tilde{A}$ is said to be trapezoidal fuzzy number if it is fully determined by $(a_1, a_2, a_3, a_4)$ of crisp numbers such that $a_1 < a_2 < a_3 < a_4$ with membership function, representing a trapezoid of the form

$$
\mu_{\tilde{A}}(x) = \begin{cases} 
\frac{x-a_1}{a_2-a_1}, & a_1 \leq x \leq a_2, \\
1, & a_2 \leq x \leq a_3, \\
\frac{a_4-x}{a_4-a_3}, & a_3 \leq x \leq a_4, \\
0, & \text{otherwise}
\end{cases}
$$
Where \( a_1, a_2, a_3 \) and \( a_4 \) are the lower limit, lower mode, upper mode and upper limit respectively of the fuzzy number \( \tilde{A} \). When \( a_2 = a_3 \), the trapezoidal fuzzy number becomes a triangular fuzzy number. Here vertical line shows the membership function.

6.4 **TAPEZOIDAL FUZZY NUMBER OPERATION:**

Let \( \tilde{A} = (a_1, a_2, a_3, a_4) \) and \( \tilde{B} = (b_1, b_2, b_3, b_4) \) be two trapezoidal fuzzy numbers, then the arithmetic operation on \( \tilde{A} \) and \( \tilde{B} \) are given as follows:

(i) Addition of \( \tilde{A} \) and \( \tilde{B} \)

\[
\tilde{A} + \tilde{B} = (a_1 + b_1, a_2 + b_2, a_3 + b_3, a_4 + b_4)
\]

(ii) Subtraction of \( \tilde{A} \) and \( \tilde{B} \)

\[
\tilde{A} - \tilde{B} = (a_1 - b_4, a_2 - b_3, a_3 - b_2, a_4 - b_1)
\]

(iii) Multiplication of \( \tilde{A} \) and \( \tilde{B} \)

\[
\tilde{A} \times \tilde{B} = (a_1 b_1, a_2 b_2, a_3 b_3, a_4 b_4)
\]

(iv) Division of \( \tilde{A} \) and \( \tilde{B} \)

\[
\tilde{A} / \tilde{B} = \left( \frac{a_1}{b_4}, \frac{a_2}{b_3}, \frac{a_3}{b_2}, \frac{a_4}{b_1} \right)
\]

(v) Scalar multiplication

Let \( \alpha \) be any real number. Then

For \( \alpha \geq 0 \), \( \alpha \times \tilde{A} = (\alpha a_1, \alpha a_2, \alpha a_3, \alpha a_4) \)

For \( \alpha < 0 \), \( \alpha \times \tilde{A} = (\alpha a_4, \alpha a_3, \alpha a_2, \alpha a_1) \)

6.5 **FUZZY RETRIAL QUEUES:**

In this model, the fuzzy arrival rate \( \tilde{\lambda} \), fuzzy retrial rate \( \tilde{\theta} \) and fuzzy service rate \( \tilde{\mu} \) can be represented as convex fuzzy sets. Let \( \eta_{\tilde{\lambda}}(\tilde{\lambda}), \eta_{\tilde{\theta}}(\tilde{\theta}), \eta_{\tilde{\mu}}(\tilde{\mu}) \)
denoted the membership functions of $\tilde{\lambda}, \tilde{\theta}, \tilde{\mu}$ respectively. The fuzzy sets are described as:

$$\mu_{\tilde{\lambda}} (\tilde{\lambda}) : X \rightarrow [0,1]$$

$$\tilde{\lambda} = \{(\lambda, \eta_{\tilde{\lambda}}(\tilde{\lambda}))/\lambda \in X\}$$

$$\mu_{\tilde{\theta}} (\tilde{\theta}) : Y \rightarrow [0,1]$$

$$\tilde{\theta} = \{((\theta, \eta_{\tilde{\theta}}(\tilde{\theta}))/\theta \in Y\}$$

$$\mu_{\tilde{\mu}} (\tilde{\mu}) : Z \rightarrow [0,1]$$

$$\tilde{\mu} = \{(\mu, \eta_{\tilde{\mu}}(\tilde{\mu}))/\mu \in Z\}$$

Where $X, Y, Z$ are the crisp universal sets of arrival rate, retrial rate and service rate respectively.

Assume that the system characteristic of interest is the expected waiting time and expected number of customer in the orbit. From the traditional queuing theory if $\frac{\lambda}{\mu} < 1$, the expected waiting time and expected number of customer in the orbit for a crisp retrial queuing system respectively given by

$$E(W) = \frac{\lambda}{\mu - \lambda} \left(\frac{1}{\lambda} + \frac{1}{\theta}\right)$$

And

$$E(N) = \frac{\lambda^2}{\mu - \lambda} \left(\frac{1}{\lambda} + \frac{1}{\theta}\right)$$

The membership function for the expected waiting time $E(\tilde{W})$ in the orbit is
The system characteristics are described by membership function, this conserve fuzziness completely.

Theorem: (First Decomposition theorem) for every $A \in X$,

$$A = \bigcup_{\alpha \in (0,1]} \alpha A$$

Where $\alpha A(x)$

From first decomposition theorem, if $Z = (A * B)$ and

$$Z \in R, (A * B) = \bigcup_{\alpha \in (0,1]} \alpha (A * B)$$

Since $\alpha (A * B)$ is a closed interval for each $\alpha \in (0,1]$ with both $A$ and $B$ fuzzy, $(A * B)$ is also a fuzzy number.

6.6 Solution Methodology:

However $\bar{\lambda}, \bar{\theta}, \bar{\mu}$ are defined by trapezoidal fuzzy numbers

Let $\bar{\lambda} = (\bar{\lambda}_1, \bar{\lambda}_2, \bar{\lambda}_3, \bar{\lambda}_4)$

$$\bar{\theta} = (\bar{\theta}_1, \bar{\theta}_2, \bar{\theta}_3, \bar{\theta}_4)$$
\( \tilde{\mu} = (\tilde{\mu}_1, \tilde{\mu}_2, \tilde{\mu}_3, \tilde{\mu}_4) \)

Where \( \tilde{\lambda}_1 < \tilde{\lambda}_2 < \tilde{\lambda}_3 < \tilde{\lambda}_4, \tilde{\theta}_1 < \tilde{\theta}_2 < \tilde{\theta}_3 < \tilde{\theta}_4, \tilde{\mu}_1 < \tilde{\mu}_2 < \tilde{\mu}_3 < \tilde{\mu}_4 \) based on subjective judgement.

The membership functions of \( \eta_{\tilde{\lambda}}(\tilde{\lambda}), \eta_{\tilde{\theta}}(\tilde{\theta}), \eta_{\tilde{\mu}}(\tilde{\mu}) \) are defined as follows:

\[
\eta_{\tilde{\lambda}}(\tilde{\lambda}) = \begin{cases} 
\frac{\tilde{\lambda}_2 - \tilde{\lambda}_1}{\tilde{\lambda}_2 - \tilde{\lambda}_1}, & \tilde{\lambda}_1 \leq \lambda \leq \tilde{\lambda}_2 \\
1, & \tilde{\lambda}_2 \leq \lambda \leq \tilde{\lambda}_3 \\
\frac{\tilde{\lambda}_3 - \tilde{\lambda}}{\tilde{\lambda}_4 - \tilde{\lambda}_3}, & \tilde{\lambda}_3 \leq \lambda \leq \tilde{\lambda}_4 \\
0, & \text{otherwise}
\end{cases}
\]

\[
\eta_{\tilde{\mu}}(\tilde{\mu}) = \begin{cases} 
\frac{\tilde{\mu}_2 - \tilde{\mu}_1}{\tilde{\mu}_2 - \tilde{\mu}_1}, & \tilde{\mu}_1 \leq \mu \leq \tilde{\mu}_2 \\
1, & \tilde{\mu}_2 \leq \mu \leq \tilde{\mu}_3 \\
\frac{\tilde{\mu}_3 - \tilde{\mu}}{\tilde{\mu}_4 - \tilde{\mu}_3}, & \tilde{\mu}_3 \leq \mu \leq \tilde{\mu}_4 \\
0, & \text{otherwise}
\end{cases}
\]

\[
\eta_{\tilde{\theta}}(\tilde{\theta}) = \begin{cases} 
\frac{\tilde{\theta}_2 - \tilde{\theta}_1}{\tilde{\theta}_2 - \tilde{\theta}_1}, & \tilde{\theta}_1 \leq \theta \leq \tilde{\theta}_2 \\
1, & \tilde{\theta}_2 \leq \theta \leq \tilde{\theta}_3 \\
\frac{\tilde{\theta}_3 - \tilde{\theta}}{\tilde{\theta}_4 - \tilde{\theta}_3}, & \tilde{\theta}_3 \leq \theta \leq \tilde{\theta}_4 \\
0, & \text{otherwise}
\end{cases}
\]

Using the concept of \( \alpha \) -cut method we have

\[
\frac{\tilde{\lambda}_1 - \tilde{\lambda}}{\tilde{\lambda}_2 - \tilde{\lambda}_1} = \alpha \Rightarrow \tilde{\lambda} = \alpha(\tilde{\lambda}_2 - \tilde{\lambda}_1) + \tilde{\lambda}_1
\]

\[
\frac{\tilde{\lambda}_4 - \tilde{\lambda}}{\tilde{\lambda}_4 - \tilde{\lambda}_3} = \alpha \Rightarrow \tilde{\lambda} = \tilde{\lambda}_4 - \alpha(\tilde{\lambda}_4 - \tilde{\lambda}_3)
\]

\[
\tilde{\lambda}(\alpha) = [ \alpha(\tilde{\lambda}_2 - \tilde{\lambda}_1) + \tilde{\lambda}_1, \tilde{\lambda}_4 - \alpha(\tilde{\lambda}_4 - \tilde{\lambda}_3) ]
\]
\[
\frac{\bar{\mu} - \mu_1}{\bar{\mu}_2 - \mu_1} = \alpha \Rightarrow \bar{\mu} = \alpha(\bar{\mu}_2 - \mu_1) + \mu_1
\]
\[
\frac{\bar{\mu}_4 - \bar{\mu}}{\bar{\mu}_4 - \mu_3} = \alpha \Rightarrow \bar{\mu} = \mu_4 - \alpha(\bar{\mu}_4 - \mu_3)
\]
\[
\bar{\mu}(\alpha) = [\alpha(\bar{\mu}_2 - \mu_1) + \mu_1, \bar{\mu}_4 - \alpha(\bar{\mu}_4 - \mu_3)]
\]
\[
\bar{\theta}_{2} - \bar{\theta}_{1} = \alpha \Rightarrow \bar{\theta} = \alpha(\bar{\theta}_2 - \bar{\theta}_1) + \bar{\theta}_1
\]
\[
\bar{\theta}_{4} - \bar{\theta}_{3} = \alpha \Rightarrow \bar{\theta} = \frac{\bar{\theta}_4 - \alpha(\bar{\theta}_4 - \bar{\theta}_3)}{\bar{\theta}_4 - \bar{\theta}_3}
\]
\[
\bar{\theta}(\alpha) = [\alpha(\bar{\theta}_2 - \bar{\theta}_1) + \bar{\theta}_1, \bar{\theta}_4 - \alpha(\bar{\theta}_4 - \bar{\theta}_3)]
\]

Now
\[
\tilde{\lambda}(\alpha) = [\alpha(\tilde{\lambda}_2 - \tilde{\lambda}_1) + \tilde{\lambda}_1, \tilde{\lambda}_4 - \alpha(\tilde{\lambda}_4 - \tilde{\lambda}_3)]
\]
\[
\tilde{\mu}(\alpha) = [\alpha(\tilde{\mu}_2 - \tilde{\mu}_1) + \tilde{\mu}_1, \tilde{\mu}_4 - \alpha(\tilde{\mu}_4 - \tilde{\mu}_3)]
\]
\[
\tilde{\theta}(\alpha) = [\alpha(\tilde{\theta}_2 - \tilde{\theta}_1) + \tilde{\theta}_1, \tilde{\theta}_4 - \alpha(\tilde{\theta}_4 - \tilde{\theta}_3)]
\]

Now
\[
[\tilde{\mu}(\alpha) - \tilde{\lambda}(\alpha)] = [\alpha(\tilde{\mu}_2 - \tilde{\mu}_1) + \tilde{\mu}_1 - \alpha(\tilde{\lambda}_2 - \tilde{\lambda}_1) - \tilde{\lambda}_1, \tilde{\mu}_4 - \alpha(\tilde{\mu}_4 - \tilde{\mu}_3) - \tilde{\lambda}_4 + \alpha(\tilde{\lambda}_4 - \tilde{\lambda}_3)]
\]
\[
\left[ \frac{\bar{\lambda}(\alpha)}{(\bar{\mu}(\alpha) - \bar{\lambda}(\alpha))} \right] = \frac{\alpha(\bar{\lambda}_2 - \bar{\lambda}_1) + \bar{\lambda}_1}{\bar{\mu}_4 - \alpha(\bar{\mu}_4 - \bar{\lambda}_3) - \bar{\lambda}_4 + \alpha(\bar{\lambda}_4 - \bar{\lambda}_3)} \cdot \frac{\bar{\lambda}_4 - \alpha(\bar{\lambda}_4 - \bar{\lambda}_3)}{\alpha(\bar{\mu}_2 - \bar{\mu}_1) + \bar{\mu}_1 - \alpha(\bar{\mu}_4 - \bar{\mu}_3) - \bar{\lambda}_4 + \alpha(\bar{\lambda}_4 - \bar{\lambda}_3)}
\]
\[
\left[ \frac{1}{\bar{\lambda}(\alpha)} \right] + \frac{1}{\bar{\theta}(\alpha)} = \left[ \frac{1}{\bar{\lambda}_4 - \alpha(\bar{\lambda}_4 - \bar{\lambda}_3)} + \frac{1}{\bar{\theta}_4 - \alpha(\bar{\theta}_4 - \bar{\theta}_3)} \cdot \frac{1}{\bar{\lambda}_4 - \alpha(\bar{\lambda}_4 - \bar{\lambda}_3)} + \frac{1}{\bar{\theta}_4 - \alpha(\bar{\theta}_4 - \bar{\theta}_3)} \right]
\]
\[
E(\tilde{W}(\alpha)) = \left( \frac{\bar{\lambda}(\alpha)}{(\bar{\mu}(\alpha) - \bar{\lambda}(\alpha))} \right) \left[ \frac{1}{\bar{\lambda}(\alpha)} + \frac{1}{\bar{\theta}(\alpha)} \right]
\]
\[
= \left[ \frac{\alpha(\bar{\lambda}_2 - \bar{\lambda}_1) + \bar{\lambda}_1}{\bar{\mu}_4 - \alpha(\bar{\mu}_4 - \bar{\lambda}_3) - \bar{\lambda}_4 + \alpha(\bar{\lambda}_4 - \bar{\lambda}_3)} \right] \left( \frac{1}{\bar{\lambda}_4 - \alpha(\bar{\lambda}_4 - \bar{\lambda}_3)} \right) + \left[ \frac{1}{\bar{\theta}_4 - \alpha(\bar{\theta}_4 - \bar{\theta}_3)} \cdot \frac{\bar{\lambda}_4 - \alpha(\bar{\lambda}_4 - \bar{\lambda}_3)}{\alpha(\bar{\mu}_2 - \bar{\mu}_1) + \bar{\mu}_1 - \alpha(\bar{\mu}_4 - \bar{\mu}_3) - \bar{\lambda}_4 + \alpha(\bar{\lambda}_4 - \bar{\lambda}_3)} \right] \left( \frac{1}{\bar{\theta}_4 - \alpha(\bar{\theta}_4 - \bar{\theta}_3)} \right) + \frac{1}{\bar{\theta}_4 - \alpha(\bar{\theta}_4 - \bar{\theta}_3)}
\]
\[
E(\tilde{N}(\alpha)) = \left( \frac{\bar{\lambda}(\alpha)}{(\bar{\mu}(\alpha) - \bar{\lambda}(\alpha))} \right) \left[ \frac{1}{\bar{\lambda}(\alpha)} + \frac{1}{\bar{\theta}(\alpha)} \right]
\]
We put $\alpha=0$ and $\alpha=1$ in above and obtain an approximate trapezoidal fuzzy number.

Thus

$$
E(\bar{W}) = \left[ \frac{\bar{\lambda}_1}{\mu_4 - \bar{\lambda}_4} \left( \frac{1}{\bar{\lambda}_4} + \frac{1}{\bar{\theta}_4} \right), \frac{\bar{\lambda}_2}{\mu_3 - \bar{\lambda}_3} \left( \frac{1}{\bar{\lambda}_3} + \frac{1}{\bar{\theta}_3} \right), \frac{\bar{\lambda}_3}{\mu_2 - \bar{\lambda}_2} \left( \frac{1}{\bar{\lambda}_2} + \frac{1}{\bar{\theta}_2} \right), \frac{\bar{\lambda}_4}{\mu_1 - \bar{\lambda}_1} \left( \frac{1}{\bar{\lambda}_1} + \frac{1}{\bar{\theta}_1} \right) \right]
$$

$$
E(\bar{N}) = \left[ \frac{\bar{\lambda}_1^2}{\mu_4 - \bar{\lambda}_4} \left( \frac{1}{\bar{\lambda}_4} + \frac{1}{\bar{\theta}_4} \right), \frac{\bar{\lambda}_2^2}{\mu_3 - \bar{\lambda}_3} \left( \frac{1}{\bar{\lambda}_3} + \frac{1}{\bar{\theta}_3} \right), \frac{\bar{\lambda}_3^2}{\mu_2 - \bar{\lambda}_2} \left( \frac{1}{\bar{\lambda}_2} + \frac{1}{\bar{\theta}_2} \right), \frac{\bar{\lambda}_4^2}{\mu_1 - \bar{\lambda}_1} \left( \frac{1}{\bar{\lambda}_1} + \frac{1}{\bar{\theta}_1} \right) \right]
$$

Then the membership functions for $E(\bar{W}), E(\bar{N})$ are

$$
\eta_{E(\bar{W})} = \begin{cases} 
\frac{\bar{W} - \bar{W}_1}{\bar{W}_2 - \bar{W}_1}, & \bar{W}_1 \leq \bar{W} \leq \bar{W}_2 \\
1, & \bar{W}_2 \leq \bar{W} \leq \bar{W}_3 \\
\frac{\bar{W}_4 - \bar{W}}{\bar{W}_4 - \bar{W}_3}, & \bar{W}_3 \leq \bar{W} \leq \bar{W}_4 \\
0, & \bar{W}_4 \leq \bar{W} \leq \bar{W}_5 
\end{cases}
$$
\[
\eta_{E(\bar{N})} = \begin{cases} 
\frac{N-N_1}{N_2-N_1}, & \bar{N}_1 \leq \bar{N} \leq \bar{N}_2 \\
1, & \bar{N}_2 \leq \bar{N} \leq \bar{N}_3 \\
\frac{N_3-N}{N_4-N_3}, & \bar{N}_3 \leq \bar{N} \leq \bar{N}_4 \\
0, & \bar{N}_4 \leq \bar{N} \leq \bar{N}_5 
\end{cases}
\]

Since the system characteristics are described by membership function, it preserves the fuzziness of input information. However the designer would prefer one crisp value for one of system characteristic rather than fuzzy set. In order to overcome this problem we defuzzify the fuzzy values of system characteristic by using formula,

\[
\text{Crisp } A = \frac{a_1 + 2a_2 + 2a_3 + a_4}{6} \quad (A)
\]

\[
\tilde{A} = (a_1, a_2, a_3, a_4)
\]

**6.7 Numerical illustration:**

In a packet switching network, considered a computer network, there are a group of host computers, connected to interface message processors. Arriving of the messages to host computer follows the poisson process. If the host computer wants to transmit the message to another host computer, it must send the message and the final address to the interface message processor to which it is associated. If the processor is vacant then message is accepted. Otherwise the message come back to the host computer and is stored in buffer to be transmitted later. The buffer in the host computer, the interface processor and the transmission policy correspond to the orbit, the server and the retrial discipline respectively in the queuing terminology. Concerned to the system efficiency, the management wants to obtain the system characteristics, including the expected waiting time and the number of customer in the orbit.
Suppose the arrival of the message to computer system is about 15 per hour, the message arrive to the buffer is around 3 per hour and the process of message is about 20 per hour. Let us define arrival, retrial and service rates are in fuzzy trapezoidal numbers represented by

\[ \lambda = [14, 15, 16, 18] \quad \tilde{\mu} = [19, 20, 21, 22] \]
\[ \tilde{\theta} = [2, 3, 4, 6] \]

\[ \eta_{\lambda}(\lambda) = \begin{cases} \frac{\lambda - 14}{15 - 14} & 14 \leq \lambda \leq 15 \\ 1, & 15 \leq \lambda \leq 16 \\ \frac{18 - \lambda}{18 - 16} & 16 \leq \lambda \leq 18 \\ 0, & \text{otherwise} \end{cases} \]

For \( \alpha - \text{cut} \)

\[ \lambda(\alpha) = [\alpha + 14, 18 - 2\alpha] \]
\[ \eta_{\tilde{\lambda}}(\tilde{\lambda}) = \begin{cases} \frac{\tilde{\lambda} - 19}{20 - 19} & 19 \leq \mu \leq 20 \\ 1, & 20 \leq \mu \leq 21 \\ \frac{22 - \tilde{\mu}}{22 - 21} & 21 \leq \mu \leq 22 \\ 0, & \text{otherwise} \end{cases} \]

\[ \tilde{\mu}(\alpha) = [\alpha + 19, 22 - 2\alpha] \]
\[ \eta_{\tilde{\mu}}(\tilde{\mu}) = \begin{cases} \frac{\tilde{\mu} - 2}{3 - 2} & 2 \leq \theta \leq 3 \\ 1, & 3 \leq \theta \leq 4 \\ \frac{6 - \tilde{\theta}}{6 - 4} & 4 \leq \theta \leq 6 \\ 0, & \text{otherwise} \end{cases} \]

\[ \tilde{\theta}(\alpha) = [\alpha + 2, 6 - 2\alpha] \]
\[ E \left( \bar{W}(\alpha) \right) = \left( \frac{\bar{\lambda}(\alpha)}{\bar{\mu}(\alpha) - \bar{\lambda}(\alpha)} \right) \left[ \frac{1}{\bar{\lambda}(\alpha)} + \frac{1}{\bar{\theta}(\alpha)} \right] \]

\[ E \left( \bar{W}(\alpha) \right) = \left( \frac{\alpha + 14}{(\alpha + 19 - \alpha - 14)} \right) \left[ \frac{1}{18 - 2\alpha} + \frac{1}{6 - 2\alpha} \right], \left( \frac{18 - 2\alpha}{(2\alpha + 22 - \alpha - 18)} \right) \left[ \frac{1}{14 + \alpha} + \frac{1}{2 + \alpha} \right] \]

\[ E \left( \bar{W}(\alpha) \right) = \left( \frac{\alpha + 14}{5} \right) \left[ \frac{1}{18 - 2\alpha} + \frac{1}{6 - 2\alpha} \right], \left( \frac{18 - 2\alpha}{(\alpha + 4)} \right) \left[ \frac{1}{14 + \alpha} + \frac{1}{2 + \alpha} \right] \]

At \( \alpha = 0 \) and \( \alpha = 1 \) we have
\[ E(\bar{W}) = [.62, .9375, 1.28, 2.57] \]

\[ E(\bar{N}(\alpha)) = \left( \frac{(\alpha + 14)(\alpha + 14)}{5} \right) \left[ \frac{1}{18 - 2\alpha} + \frac{1}{6 - 2\alpha} \right], \left( \frac{18 - 2\alpha}{(\alpha + 4)} \right) \left( \frac{1}{14 + \alpha} + \frac{1}{2 + \alpha} \right) \]

\[ E(\bar{N}) = [8.71, 14.06, 20.48, 46.28] \]

Defuzzify the function using formula (A), we get
\[ E(W) = 1.2708 \quad \text{and} \quad E(N) = 20.678 \]
Fuzzy expected number of customer in the orbit $E(\bar{N})$

In above graphs, vertical line shows the membership function in fuzzy that lies in the interval $[0, 1]$ and horizontal lines shows the Fuzzy expected waiting time in the orbit for retrial system $E(\bar{W})$ and Fuzzy expected number of customer in the orbit $E(\bar{N})$.

From the graph it is clear that the support of $E(\bar{W})$ ranges from 0.62 to 2.57, this indicates that the expected waiting time is fuzzy and it is not possible that the values to fall below 0.62 or exceed 2.57. The $\alpha$-cut at $\alpha=1$ is exactly .9375 and 1.28, which are the most possible value for the expected waiting time in the orbit.

In the same manner the numbers of customers in the system fall between 8.71 and 46.28. It says the number of customers in the system will never exceed 46.28 or fall below 8.71. The most possible number of customers in the orbit is 14.06 and 20.48 at $\alpha=1$.

6.8 CONCLUSION:

In this chapter, we have studied a system characteristic of retrial queuing system under fuzzy environment. The fuzzy trapezoidal membership function is being used to derive the system characteristic of fuzzy retrial queue model. The waiting time as well as expected number of customers in
the orbit has been computed by using fuzzy arithmetic operators. The proposed model is more realistic and more suitable for designer and practioners since in practical situation the input information are almost uncertain, in precise incomplete. α-cut approach and fuzzy arithmetic operators are used to construct system characteristic membership function for preserving the fuzziness.