CHAPTER 5

SOME MINIMAX AND SADDLE POINT THEOREMS

5.1 INTRODUCTION

Fixed point theorems provide important tools in game theory which are used to prove the equilibrium and existence theorems. For instance, the fixed point theorems by Brouwer, Kakutani and Fan have been among the most used tools in economics and game theory, see [4], [13], [161]. These theorems are further applied to prove the minimax theorems and the existence of saddle points.

The first minimax theorem was given by von Neumann [157] in 1928. He established the following.

Theorem 5.1.1 [188]. Let \( A \) be an \( m \times n \) matrix and \( x \) and \( y \) be the sets of nonnegative row and column vectors with unit sum. Then

\[
\min_{x \in X} \max_{y \in Y} x^T Ay = \max_{y \in Y} \min_{x \in X} x^T Ay. \tag{5.1.1}
\]

After nine years of this result von Neumann [186], showed that the bilinear character of Theorem 5.1.1 was not needed. He extended his result by using Brouwer’s fixed point theorem as follows.
Theorem 5.1.2 [186]. Let $X$ and $Y$ be nonempty compact, convex subsets of Euclidean spaces and $f : X \times Y \to \mathbb{R}$ be jointly continuous. Suppose that $f$ is quasiconcave on $X$ and quasiconvex on $Y$. Then

$$\min_{y \in Y} \max_{x \in X} f(x, y) = \max_{x \in X} \min_{y \in Y} f(x, y).$$

Thereafter, many papers appeared in the literature verifying the Neumann inequality. It is noticed that majority of minimax theorems are proved by applying fixed point theorems under different conditions on the spaces or on the mappings.

Sion [159] generalized von Neumann’s result on the basis of Knaster, Kuratowski and Mazurkiewicz theorem [160], which was given in the year 1929 on the $n$-dimensional simplex in the following way.

Theorem 5.1.3 [160]. Let $D$ be the set of vertices of an $n$-simplex $\Delta_n$ and a multivalued map $G : D \to \Delta_n$ be a KKM map (that is, $co(A) \subseteq G(A)$ for each $A \subseteq D$) with closed [resp., open] values. Then $\bigcap_{z \in D} G(z) \neq \emptyset$.

This theorem is an equivalent formulation of the famous Brouwer fixed-point theorem. At the beginning, the results of the KKM theory were established for convex subsets of topological vector spaces mainly by Ky Fan [161]-[164]. A number of intersection theorems and applications were followed in the convex subsets of topological vector spaces. In 1961, Ky Fan generalized KKM theorem to infinite dimensional topological vector space, which is known as Fan–Knaster-Kuratowski-Mazurkiewicz theorem (i.e. Fan KKM theorem or KKM theorem). Fan established following minimax inequality from the KKM theorem.

Theorem 5.1.4 [162]. Let $X$ be a nonempty compact convex set in a topological vector space. If a function $\phi : X \times X \to (-\infty, \infty)$ satisfies the following conditions

(i) for each $x \in X$, $\phi(x, y)$ is lower semicontinuous function of $y$ on $X$,

(ii) for each $y \in X$, $\phi(x, y)$is quasiconcave function of $x$ on $X$.

Then the minimax inequality $\min_{y \in X} \sup_{x \in X} \phi(x, y) \leq \sup_{x \in X} \phi(x, x)$ holds.
Lassonde [184] extended the KKM theory to convex spaces by proving several variants of KKM theorems for convex spaces and proposed a systematic development of the method based on the KKM theorem. The KKM theorem was further extended to pseudo-convex spaces, contractible spaces and spaces with certain contractible subsets or H-spaces by Horvath [166]-[167]. In these papers, most of the Fan’s results in the KKM theory are extended to H-spaces by replacing the convexity condition by contractibility. With these terminology, Park [168] established new versions of KKM theorems, minimax inequalities, and others on H-spaces. Park and Kim [169]-[173], Park [174]-[178] and others extended these results to more general spaces such as G-convex, abstract convex and KKM spaces.

The purpose of this chapter is to present a generalized version of the KKM theorem by using the concept of Chang and Zhang [182]. As an application of it, a generalized minimax inequality is obtained and an existence result for the saddle-point problem under general settings is derived. Consequently, a saddle point theorem for two person zero sum parametric game is also proved. Several existing well known results are obtained as special cases.

5.2 PRELIMINARIES

First we give basic definitions used in our results. We follow [160], [180], [182],[185], [193], [195], [304] and [305] for notations and preliminaries.

**Definition 5.2.1** [304]. A two-person, zero-sum game $G$ is defined by $G = (A_1, A_2, f_1, f_2)$ where, for $i = 1, 2$, we have

(i) $A_i$ is a finite set of player $i$’s actions,

(ii) $f_i : A \rightarrow R$, where $A = A_1 \times A_2$, is player $i$’s payoff function,

the player’s payoff functions satisfy

$$f_1(a) + f_2(a) = 0, \text{ for all } a \in A.$$

If we consider two person zero sum game generated by function $f : X \times Y \rightarrow R$. This means that the first player selects a point $x$ from $X$ and the second player selects a point $y$ from $Y$. Because of this choice, the second player pays the first one the amount $f(x, y)$.

**Definition 5.2.2** [305]. A point $(x^*, y^*) \in X \times Y$ is said to be saddle point of $f$ if $X \times Y$ if
A point \((x^*, y^*) \in X \times Y\) is said to be a solution of the game if and only if it is a saddle point of \(f\) in \(X \times Y\).

**Definition 5.2.3 [193].** A two-person zero-sum parametric game \((GP_\theta)\) is defined by the following form

\[(X, Y, f, g, \theta, F_\theta),\]

where,

1. \(X\) is a subset of a topological space \(E\), which is called the strategy set of player 1,
2. \(Y\) is a subset of a topological space \(E\), which is called the strategy set of player 2,
3. \(f : X \times Y \to R\) and \(g : X \times Y \to R_+\), where \(R_+ = (0, \infty)\),
4. \(\theta\) is a real number, which is called a parameter of the game,
5. \(F_\theta = f - \theta g : X \times Y \to R\), that is, for all \((x, y) \in X \times Y\), \(F_\theta(x, y) = f(x, y) - \theta g(x, y)\), is a loss function of player 1 and \(-F_\theta(x, y)\) is a loss function of player 2.

In general, \(F_\theta = \inf_{x \in X} \sup_{y \in Y} F_\theta(x, y)\) is called the minimal worst loss of player 1 and \(F_\theta = \sup_{y \in Y} \inf_{x \in X} F_\theta(x, y)\) is called the maximal worst gain of player 2.

We illustrate the above concept of two person zero sum parametric game by following example.

**Example 5.2.1.** Consider the sets

\[X = \{x_1, x_2, x_3, x_4\}, \ Y = \{y_1, y_2, y_3, y_4\}\] and parameter \(\theta = 2\).

A function \(f : X \times Y \to R\) defined by the following payoff matrix

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and $g : X \times Y \to R^+$ by

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When $\theta = 2$, then $F_\theta : X \times Y \to R$ becomes

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and $(x_2, y_3)$ is the saddle point of the function $F_\theta(x, y)$.

KKM mapping in Hausdorff topological vector space is defined as follows.

**Definition 5.2.4 [160].** Let $E$ be a Hausdorff topological vector space and $X$ be a nonempty subset of $E$. A multivalued mapping $G : X \to 2^E$, that is, mapping with the values $G(x) \subseteq E$, for each $x$ in $X$, is called a KKM mapping if $\text{co}\{x_1, x_2, \ldots, x_n\} \subseteq \bigcup_{i=1}^{n} G(x_i)$ for each finite subset $\{x_1, x_2, \ldots, x_n\} \subseteq X$, where $\text{co}\{x_1, x_2, \ldots, x_n\}$ denotes the convex hull of the set $\{x_1, x_2, \ldots, x_n\}$.

**Definition 5.2.5 [182].** Let $X$ be a nonempty subset of a topological vector space $E$. A multivalued mapping $G : X \to 2^E$ is called a generalized KKM mapping, if for any finite set
\{x_1, \ldots, x_n\} \subset X$, there exists a finite subset \{y_1, \ldots, y_n\} \subset E such that for any subset \{y_i, \ldots, y_k\} \subset \{y_1, \ldots, y_n\}, 1 \leq k \leq n, we have 
\[ \text{co}\{y_i, \ldots, y_k\} \subset \bigcup_{j=1}^{k} G(x_j). \]

We extend this definition for two maps in the following manner.

**Definition 5.2.6.** Let \( X \) be a nonempty subset of a topological vector space \( E \) and \( F, G : X \to 2^E \). Then \( G \) is said to be a generalized \( F \)-KKM mapping if for any finite set \( \{x_1, \ldots, x_n\} \subset X \) and each \( \{i_1, \ldots, i_k\} \subset \{1, 2, \ldots, n\} \), we have 
\[ \text{co}\{\bigcup_{j=1}^{k} F(x_{i_j})\} \subset \bigcup_{j=1}^{k} G(x_{i_j}). \]

It is remarked that definition 5.2.6 becomes definition 5.2.5 when \( E \to \mathbb{X} \) and \( x \notin G(x), \forall x \in [-2, -9/5) \cup (9/5, 2] \).

In the following example we have shown that concept of \( F \)-KKM mapping is general than KKM mapping.

**Example 5.2.2.** Let \( E = (-\infty, \infty), X = [-2, 2] \). Let \( F, G : X \to 2^E \) be defined as 
\[ F(x) = \left[ -\left(1 + \frac{x}{3}\right), 1 + \frac{x}{3} \right] \quad \text{and} \quad G(x) = \left[ -\left(1 + \frac{x^2}{5}\right), 1 + \frac{x^2}{5} \right]. \]

Since \( \bigcup_{x \in X} G(x) = [-9/5, 9/5], x \notin G(x), \forall x \in [-2, -9/5) \cup (9/5, 2] \).

So \( \text{co}\{x_1, \ldots, x_n\} \subset \bigcup_{i=1}^{n} G(x_i) \) and \( G \) is not a KKM map. But it is a \( F \)-KKM map, since 
\[ \bigcup_{x \in X} F(x) = \left[ -\frac{1}{3}, \frac{5}{3} \right] \quad \text{and} \quad \text{co}\{\bigcup_{j=1}^{k} F(x_{i_j})\} \subset \left[ -\frac{1}{3}, \frac{5}{3} \right] = \bigcap_{x \in X} G(x) \subset \bigcup_{j=1}^{k} G(x_{i_j}). \]

**Definition 5.2.7 [182].** Let \( E \) be a topological vector space and \( X \) be a nonempty convex subset of \( E \). A function \( \phi : X \times X \to (-\infty, \infty) \) is said to be \( \gamma \)-generalized quasiconvex in \( y \) for some \( \gamma \in (-\infty, \infty) \) if, for any finite subset \( \{y_1, y_2, \ldots, y_n\} \subset X \), there is a finite subset
Definition 5.2.8. Let $E$ be a topological vector space and $X$ be a nonempty convex subset of $E$. Let $\gamma \in (-\infty, \infty)$ and $F : X \to 2^Y$, where, $Y \in E$. Then the function $\phi : X \times X \to (-\infty, \infty)$ is said to be $F - \gamma$ generalized quasiconvex in $y$ if for any finite subset \{\gamma_1, \gamma_2, \ldots, \gamma_n\} \subset X$, each \{\gamma_i\} \subset Y, and any $x_0 \in \text{co}\{\bigcup_{j=1}^k F(x_j)\} \subset Y$ and any $y_{i_j} \in F(x_{j})$, we have $\gamma \leq \max_{i \in j} (\phi(x_0, y_{i_j})$.

Notice that the function $\phi$ becomes $\gamma$-generalized quasi convex in $y$ when $F : X \to Y$ and $F(x_i) = y_i$ for each $i \in \{1, 2, \ldots, n\}$.

Definition 5.2.9 [195]. Let $X$ and $Y$ be two topological spaces. A multivalued mapping $F : X \to 2^Y$ is said to be transfer closed valued on $X$ if,
$$\bigcap_{x \in X} \overline{F(x)} = \bigcap_{x \in X} F(x).$$

Definition 5.2.10 [185]. Let $X$ and $Y$ be two topological spaces. A multivalued mapping $F : X \to 2^Y$ is said to be intersectionally closed valued on $Y$ if,
$$\bigcap_{x \in X} \overline{F(x)} = \bigcap_{x \in X} F(x).$$

Definition 5.2.11 [180]. Let $X$, $Y$ be two topological spaces. Then a function $f : X \times Y \to R = R \cup \{\pm \infty\}$ is said to be

(i) quasiconcave in $x$ if for each $y \in Y$ and $\lambda \in R$, the set $\{x \in X : f(x, y) \geq \lambda\}$ is convex,

(ii) quasiconvex in $y$ if for each $x \in X$ and $\lambda \in R$, the set $\{y \in Y : f(x, y) \leq \lambda\}$ is convex,
(iii) upper semicontinuous (resp. generally upper semicontinuous) in $x$ if for each $y \in Y$ and $\lambda \in R$, the set $\{x \in X : f(x, y) \geq \lambda\}$ is closed (resp. intersectionally closed),

(iv) lower semicontinuous (resp. generally lower semicontinuous) in $y$ if for each $x \in X$ and $\lambda \in R$, the set $\{y \in Y : f(x, y) \leq \lambda\}$ is closed (resp. intersectionally closed).

**Definition 5.2.12** [180]. Let $X$, $Y$ be two topological spaces. Then a function $f : X \times Y \to \overline{R} = R \cup \{\pm \infty\}$ is said to be

(i) transfer upper semicontinuous in $x$ if for each $\lambda \in R$, and all $x \in X$, $y \in Y$ with $f(x, y) < \lambda$ there exists a neighbourhood $V(x)$ of $x$ and a point $y' \in Y$ such that $f(z, y') < \lambda$ for all $z \in V(x)$.

(ii) transfer lower semicontinuous in $y$ if for each $\lambda \in R$, and all $x \in X$, $y \in Y$ with $f(x, y) > \lambda$ there exists a neighbourhood $V(y)$ of $y$ and a point $x' \in X$ such that $f(x', u) > \lambda$ for all $u \in V(y)$.

**Definition 5.2.13** [305]. Let $X$ and $Y$ be two topological spaces. A function $\phi : X \times Y \to (-\infty, \infty)$ is said to be

(i) $\gamma$-transfer lower semicontinuous function in $x$ for some $\gamma \in (-\infty, \infty)$ if, for all $x \in X$ and $y \in Y$ with $\phi(x, y) > \gamma$, there exists some $y' \in Y$ and some neighborhood $N(x)$ of $x$ such that $\phi(z, y') > \gamma$ for all $z \in N(x)$.

(ii) $\gamma$-transfer upper semicontinuous function in $y$ for some $\gamma \in (-\infty, \infty)$ if, for all $x \in X$ and $y \in Y$ with $\phi(x, y) < \gamma$, there exists some $x' \in Y$ and some neighborhood $N(y)$ of $y$ such that $\phi(x', u) < \gamma$ for all $u \in N(y)$.

**Definition 5.2.14.** Let $X$, $Y$ be two nonempty sets. The mapping $F : X \to 2^Y$ is said to be surjective if $F(X) = \bigcup_{x \in X} F(x) = Y$. 

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5.3 GENERALIZED KKM THEOREMS AND THEIR APPLICATIONS

First we prove the main result of this section which will be used in the sequel.

**Theorem 5.3.1.** Let $X$ be a nonempty convex subset of a Hausdorff topological vector space $E$. Let $G : X \to 2^E$ be a multivalued mapping such that for each $x \in X$, $G(x)$ is finitely closed. Then the family of sets $\{G(x) : x \in X\}$ has a finite intersection property if and only if the mapping $G$ is a generalized $F$-KKM mapping for some mapping $F : X \to 2^E$.

**Proof.** Let $\{G(x)\}_{x \in X}$ has finite intersection property. Then for any finite subset $\{x_1, x_2, \ldots, x_n\} \subset X$, $\bigcap_{i=1}^n G(x_i) \neq \emptyset$. Taking $x_i \in \bigcap_{i=1}^n G(x_i)$, and define $F : X \to 2^E$ by $F(x) = \{x_i\}$ for each $x \in X$. Then for each $\{i_1, \ldots, i_k\} \subset \{1, 2, \ldots, n\}$, we have

$$\operatorname{co}\left\{\bigcup_{j=1}^k F(x_{i_j})\right\} = \operatorname{co}\{x_i\} = \{x_i\} \subset \bigcup_{j=1}^k G(x_{i_j}).$$

This implies that $G : X \to 2^E$ is a generalized $F$-KKM mapping.

Now consider $G : X \to 2^E$ to be a generalized $F$-KKM mapping, $F : X \to 2^E$ and suppose the family $\{G(x) : x \in X\}$ does not have the finite intersection property, i.e., there exists some finite subset $\{x_1, x_2, \ldots, x_n\} \subset X$ such that $\bigcap_{i=1}^n G(x_i) = \emptyset$. Since $G$ is a generalized $F$-KKM mapping, then for each $\{i_1, \ldots, i_k\} \subset \{1, 2, \ldots, n\}$, we have $\operatorname{co}\left\{\bigcup_{j=1}^k F(x_{i_j})\right\} \subset \bigcup_{j=1}^k G(x_{i_j})$. In particular, $\operatorname{co}\{F(x_{i_j})\} \subset G(x_i)$ for each $i \in \{1, 2, \ldots, n\}$.

Let $S = \operatorname{co}\{y_1, y_2, \ldots, y_n\}$, $L = \operatorname{span}\{y_1, y_2, \ldots, y_n\}$. Then $S \subset L$. Since $G$ is finitely closed, $G(x_i) \cap L$ is a closed set. Let $d$ be the Euclidean metric on $L$. It is easy to see that

$$d(x, L \cap G(x_i)) > 0 \iff x \notin L \cap G(x_i) \quad (5.3.1)$$

Now we define a function $f : S \to [0, \infty)$ as

$$f(c) = \sum_{i=1}^n d(c, L \cap G(x_i)) y_i$$

It follows from (5.3.1) and $\bigcap_{i=1}^n G(x_i) = \emptyset$ that for each $c \in S$, $f(c) > 0$. Let
\[ R(c) = \sum_{i=1}^{n} \frac{1}{f(c)} d(c, L \cap G(x_i)) y_i. \] (5.3.2)

Thus \( R \) is a continuous function from \( S \) into \( S \). By the Brouwer fixed point theorem, there exists an element \( c_\ast \in S \) such that

\[ c_\ast = F(c_\ast) = \sum_{i=1}^{n} \frac{1}{f(c)} d(c_\ast, L \cap G(x_i)) y_i \] (5.3.3)

Denote

\[ I = \{i \in \{1, 2, ..., n\} : d(c_\ast, G(x_i)) \cap L > 0\}. \] (5.3.4)

Then for each \( i \in I \), \( c_\ast \notin G(x_i) \cap L \). Since \( c_\ast \in L \), \( c_\ast \notin G(x_i) \), \( \forall i \in I \), so we have

\[ c_\ast \notin \bigcup_{i \in I} G(x_i) \] (5.3.5)

From (5.3.3) and (5.3.4), we have

\[ c_\ast = \sum_{i \in I} \frac{1}{f(c_\ast)} d(c_\ast, L \cap G(x_i)) y_i \in \text{co}(y_i : i \in I). \]

However, since \( G \) is a generalized \( F\)-\( KKM \) mapping from \( X \) into \( 2^E \), which follows that

\[ \text{co}\left( \bigcup_{j=1}^{k} F(x_j) \right) \subset \bigcup_{j=1}^{k} G(x_j) \].

Therefore we have

\[ c_\ast \in \text{co}(y_i : i \in I) \subset \bigcup_{i \in I} G(x_i). \] (5.3.6)

This contradicts (5.3.5). Hence \( \{G(x) : x \in X\} \) has the finite intersection property. This completes the proof.

It is remarked that if \( F : X \to E \) is defined as a single valued map with \( F(x_i) = y_i \) for each \( i \in \{1, 2, ..., n\} \), then above result implies the following results of Chang and Zhang [182, Theorem 3.1].

**Corollary 5.3.1** [182]. Let \( X \) be a nonempty convex subset of a Hausdorff topological vector space \( E \). Let \( G : X \to 2^E \) be a multivalued mapping such that for each \( x \in X \), \( G(x) \) is finitely closed. Then the family of sets \( \{G(x) : x \in X\} \) has the finite intersection property if and only if the mapping \( G \) is a generalized \( KKM \) mapping.

Following is a generalized version of the \( KKM \) mapping theorem.
**Theorem 5.3.2.** Let $Y$ be a nonempty convex subset of a Hausdorff topological vector space $E$, $\phi \neq X \subset Y$ and maps $F, G : X \to 2^E$ with $G$ an intersectionally closed valued map of $Y$. Furthermore, assume that there exists a nonempty subset $X_0 \subset X$, contained in some precompact convex subset $Y_0$ of $Y$, such that $G(x_0)$ is compact subset of $Y$ for at least one $x_0 \in X_0$ and let $\overline{G} : X \to 2^E$ be a generalized $F$-$KKM$ mapping. Then $\bigcap_{x \in X} G(x) \neq \phi$.

**Proof.** For any finite subset $\{x_1, x_2, \ldots, x_n\}$ of $X$, let $X_1 = X_0 \cup \{x_1, x_2, \ldots, x_n\}$. Since $Y_0$ is a precompact convex subset of $Y$, the convex hull of $Y_0 \cup \{x_1, x_2, \ldots, x_n\}$ is also a compact convex subset of $Y$, and denote it by $K$. For each $y \in X_1$, let $T(y) = G(y) \cap K$. Since $G(y)$ is intersectionally closed valued map of $Y$ and $K$ is a compact subset of $Y$, each $T(y)$ is also compact. Further, since $\overline{G} : X \to 2^E$ is defined by $\overline{G}(x) = G(x)$ for each $x \in X$ and $\overline{G}$ is a generalized $F$-$KKM$ mapping with closed values. So, we can easily show that $T$ is also a $F$-$KKM$ map. Therefore, by Theorem 5.3.1, we have $\bigcap_{x \in X_1} T(x) \neq \phi$. Hence we have

$\phi \neq \bigcap_{x \in X_1} T(x) = \bigcap_{x \in X_1} G(x) \cap K \subseteq \bigcap_{x \in X_0} G(x_0) \cap G(x_1) \cap \ldots \cap G(x_n) \cap \ldots \cap G(x_n) = \bigcap_{x \in X_0} G(x_0) \cap G(x_1) \cap \ldots \cap G(x_n)$

Let $C$ denotes the compact set $\bigcap_{x \in X_0} G(x_0)$. Then we have $\bigcap_{i=1}^n G(x_i) \cap C \neq \phi$ for every finite subset $\{x_1, x_2, \ldots, x_n\} \subset X$. Since each $G(x)$ is an intersectionally closed valued map of $Y$ and $C$ is a compact subset of $Y$, each $G(x) \cap C$ is also a compact subset of $Y$. Since the family $\{G(x) \cap C \mid x \in X\}$ has the finite intersection property, we have

$\bigcap_{x \in X} G(x) \cap C = \bigcap_{x \in X} G(x) \neq \phi$.

This completes the proof.

Following Corollary is the special case of Theorem 5.3.1.

**Corollary 5.3.2.** Let $X$ be a nonempty convex subset of a Hausdorff topological vector space $E$ and maps $F, G : X \to 2^E$ with $G$ an intersectionally closed valued map such that
\( \overline{G(x_0)} = K \) is compact for at least one \( x_0 \in X \) and let \( \overline{G} : X \to 2^E \) be a generalized \( F\)-KKM mapping. Then \( \bigcap_{x \in X} G(x) \neq \phi \).

**Proof.** Since \( \overline{G} : X \to 2^E \) is defined by \( \overline{G}(x) = G(x) \) for each \( x \in X \), we have that \( \overline{G} \) is a generalized \( F\)-KKM mapping with closed values. By Theorem 5.3.1 the family of sets \( \{G(x) : x \in X\} \) has the finite intersection property. Since \( G(x_0) \) is compact, we have \( \bigcap_{x \in X} G(x) \neq \phi \).

Since \( G \) is intersectionally closed valued mapping
\[
\bigcap_{x \in X} G(x) = \bigcap_{x \in X} G(x) \neq \phi.
\]

Notice that if \( F : X \to E \) be a single valued map and \( F(x_i) = y_i \) for each \( i \in \{1, 2, ..., n\} \), then Corollary 5.3.1 contains the result of Ansari et al [179, Th. 2.1].

**Corollary 5.3.3.** [179]. Let \( X \) be a nonempty convex subset of a Hausdorff topological vector space \( E \) and map \( G : X \to 2^E \) with \( G \) a transfer closed valued map such that \( \overline{G(x_0)} = K \) is compact for at least one \( x_0 \in X \) and let \( \overline{G} : X \to 2^E \) be a generalized KKM mapping. Then \( \bigcap_{x \in X} G(x) \neq \phi \).

The relation between generalized \( F\)-KKM mappings and \( S - \gamma - \) generalized quasi-convexity (quasi-concavity) is the following.

**Theorem 5.3.3.** Let \( X \) be a nonempty convex subset of a topological vector space \( E \). Let \( \phi : X \times X \to (-\infty, \infty), \gamma \in (-\infty, \infty) \) and \( F : X \to 2^X \). Then the following are equivalent

(i) The mapping \( G : X \to 2^X \) defined by
\[
G(y) = \{x \in X : \phi(x, z) \leq \gamma, \text{ for all } z \in F(x)\}
\]
are \( G(y) = \{x \in X : \phi(x, z) \geq \gamma, \text{ for all } z \in F(x)\} \)
for each \( y \in X \) is a generalized \( F\)-KKM mapping.

(ii) \( \phi(x, y) \) is \( S - \gamma - \) generalized quasi-concave, [resp. \( S - \gamma - \) generalized quasiconvex] in \( y \).
Proof. For the sake of simplicity, we prove the conclusion only for the first case in (i) and (ii). The other case can be proved similarly.

(i) ⇒ (ii). Since \( G : X \to 2^X \) is a generalized \( F-KKM \) mapping, for any finite set \( \{x_1, x_2, ..., x_n\} \subset X \), and each \( \{i_1, ..., i_k\} \subset \{1, 2, ..., n\} \), we have

\[
\co\left( \bigcup_{j=1}^{k} F(x_{i_j}) \right) \subset \bigcup_{j=1}^{k} G(x_{i_j})
\]

and hence for any \( y_0 \in \co\left( \bigcup_{j=1}^{k} F(x_{i_j}) \right) \), \( y_0 \in \bigcup_{j=1}^{k} G(x_{i_j}) \). So there exists some \( m \in \{1, 2, ..., k\} \), such that \( y_0 \in G(y_{i_m}) \), so we have \( \phi(y_0, z_{i_m}) \leq \gamma \) for all \( z_{i_m} \in F(x_{i_m}) \). Therefore, we have

\[
\min_{1 \leq j \leq k} (x_0, z_{i_j}) \leq \gamma \text{ for all } z_{i_m} \in F(x_{i_m}).
\]
i.e. \( \phi \) is \( F-\gamma \)− generalized quasi-concave in \( y \).

(ii) ⇒ (i). Since \( \phi \) is \( F-\gamma \)− generalized quasi-concave in \( y \), for any finite set \( \{x_1, x_2, ..., x_n\} \subset X \), each \( \{i_1, ..., i_k\} \subset \{1, 2, ..., n\} \), and any \( y_0 \in \co\left( \bigcup_{j=1}^{k} F(x_{i_j}) \right) \),

\[
\min_{1 \leq j \leq k} \phi(y_0, z_{i_j}) \leq \gamma \text{ for all } z_{i_m} \in F(x_{i_m}).
\]
Hence there exists some \( m : 1 \leq m \leq k \), such that \( \phi(y_0, z_{i_m}) \leq \gamma \) for all \( z_{i_m} \in F(x_{i_m}) \). This shows that \( y_0 \in G(x_{i_m}) \) and hence \( y_0 \in \bigcup_{j=1}^{k} G(x_{i_j}) \).

By the arbitrariness of \( y_0 \in \co\left( \bigcup_{j=1}^{k} F(x_{i_j}) \right) \), we have

\[
\co\left( \bigcup_{j=1}^{k} F(x_{i_j}) \right) \subset \bigcup_{j=1}^{k} G(x_{i_j}).
\]
This implies that \( G : X \to 2^X \) is a generalized \( F-KKM \) mapping.

Let \( X = E \) be a topological vector space, \( G : X \to 2^E \) defined by \( G(y) = \{x \in E : \phi(x, y) \leq \gamma\} \) (resp. \( G(y) = \{x \in E : \phi(x, y) \geq \gamma\} \)) for each \( y \in E \) be a generalized \( KKM \) mapping, \( \psi(x, y) \) is a \( \gamma \)− generalized quasiconcave (resp., convex) function in \( y \) and \( F : X \to E \) be a single valued map, \( F(x_i) = y_i \) for each \( i \in \{1, 2, ..., n\} \), then Theorem 5.3.2 implies Proposition 3.1 of Ansari et al [179].
Corollary 5.3.4 [179]. Let $X$ be a nonempty convex subset of a topological vector space $E$. Let $\phi : X \times X \to (\infty, \infty)$ and $\gamma \in (\infty, \infty)$. Then the following are equivalent.

(ii) The mapping $G : X \to 2^X$ defined by

$$G(y) = \{x \in X : \phi(x, y) \leq \gamma\} \quad [\text{resp. } G(y) = \{x \in X : \phi(x, y) \geq \gamma\}]$$

is a generalized KKM mapping.

(iii) $\phi(x, y)$ is $\gamma$–generalized quasi-concave, [resp. $\gamma$–generalized quasiconvex] in $y$.

Some Applications
As applications of the above results we present minimax and saddle point results.

First we state and prove the following general minimax inequality.

Theorem 5.3.4. Let $X$ be a nonempty closed convex subset of a Hausdorff topological vector space $E$ and $F : X \to 2^X$. Let $\gamma \in (\infty, \infty)$ be given number and maps $\phi, \psi : X \times X \to (\infty, \infty)$ satisfy the following conditions.

(i) For any fixed $y \in X$, $\phi(x, y)$ is a $\gamma$–generally lower semicontinuous function in $x$.

(ii) For any fixed $x \in X$, $\psi(x, y)$ is a $F$–$\gamma$–generalized quasiconcave function in $y$.

(iii) $\phi(x, y) \leq \psi(x, y)$ for all $(x, y) \in X \times X$.

(iv) The set $\{x \in X : \phi(x, y_0) \leq \gamma\}$ is precompact for at least one $y_0 \in X$. Then there exists $\bar{x} \in X$ such that $\inf_{x \in X} \sup_{z \in F(x)} \phi(\bar{x}, z) \leq \gamma$.

Proof. Define maps $T, G : X \to 2^X$ by

$$T(y) = \{x \in X : \psi(x, z) \leq \gamma \text{ for all } z \in F(x)\} \text{ and } G(y) = \{x \in X : \phi(x, z) \leq \gamma \text{ for all } z \in F(x)\}.$$ 

Condition (i) implies that $G$ is an intersectionally closed-valued mapping on $X$. Indeed, if $x \not\in G(y)$, then $\phi(x, z) > \gamma$. Since $\phi(x, y)$ is $\gamma$–generally lower semicontinuous function in $x$, there exists a $y' \in X$ and a neighborhood $N(x)$ of $x$ such that $\phi(r, y') > \gamma$, for all $r \in N(x)$.

Then $G(y') \subset X \setminus N(x)$. Hence $x \not\in G(y')$. Thus $G$ is intersectionally closed-valued. From condition (ii) and Theorem 5.3.1, $T$ is a generalized $F$-KKM mapping. From condition (iii), we
have \( T(y) \subseteq G(y) \), and hence \( G \) is also generalized \( F\)-KKM mapping. So \( \overline{G} \) is also generalized \( F\)-KKM mapping. By condition (iv) \( G(y_0) \) is precompact. Hence, \( G(y_0) \) is compact. 

Now by Theorem 5.3.2., it follows that \( \bigcap_{x \in X} G(x) \neq \emptyset \). Therefore, there exists \( \bar{x} \in X \) such that \( \phi(\bar{x}, z) \leq \gamma \), for all \( z \in F(X) \). In particular, we have \( \inf_{x \in X} \sup_{z \in F(x)} \phi(\bar{x}, z) \leq \gamma \).

This completes the proof.

It is remarked that If \( F : X \to X, \ F(x_i) = y_i \) for each \( i \in \{1, 2, ..., n\} \) and \( \psi(x, y) \) is a \( \gamma \)-generalized quasiconcave function in \( y \). Then Theorem 5.3.3 implies Theorem 3.1 of Ansari et al [179].

**Corollary 5.3.5** [179]. Let \( X \) be a nonempty closed convex subset of a Hausdorff topological vector space \( E \). Let \( \gamma \in (-\infty, \infty) \) be given number, and maps \( \phi, \psi : X \times X \to (-\infty, \infty) \) satisfy the following conditions.

1. For any fixed \( y \in X, \ \phi(x, y) \) is a \( \gamma \)-transfer lower semicontinuous function in \( x \).
2. For any fixed \( x \in X, \ \psi(x, y) \) is a \( \gamma \)-generalized quasiconcave function in \( y \).
3. \( \phi(x, y) \leq \psi(x, y) \), for all \( (x, y) \in X \times X \).
4. The set \( \{x \in X : \phi(x, y_0) \leq \gamma\} \) is precompact for at least one \( y_0 \in X \). Then there exists \( \bar{x} \in X \) such that \( \inf_{x \in X} \sup_{z \in F(x)} \phi(\bar{x}, y) \leq \gamma \), for all \( y \in X \).

We now present the following saddle point results. First, we present a saddle point theorem for \( \gamma \)-generally lower semicontinuous (upper) function in \( x \) and \( F\)-\( \gamma \)-generalized quasiconcave (convex) function in a Hausdorff topological vector space \( E \) using Theorem 5.3.4.

**Theorem 5.3.5.** Let \( X \) be a nonempty closed convex subset of a Hausdorff topological vector space \( E \) and \( F : X \to 2^X \). Let \( \gamma \in (-\infty, \infty) \) be given number, \( F : X \to 2^X \) a surjective mapping and let \( \phi : X \times X \to (-\infty, \infty) \) satisfy the following conditions.
(i) $\phi(x, y)$ is a $\gamma -$ generally lower semicontinuous function in $x$ and $F - \gamma -$ generalized quasiconcave function in $y$.

(ii) $\phi(x, y)$ is a $\gamma -$ generally upper semicontinuous function in $y$ and $F - \gamma -$ generalized quasiconvex function in $x$.

(iii) There exists $x_1, y_1 \in X$ such that the sets
\[ \{ x \in X : \phi(x, y_i) \leq \gamma \} \text{ and } \{ y \in X : \phi(x_i, y) \geq \gamma \} \]
are precompact.

Then, there exists a saddle point of $\phi(x, y)$; that is, there exists $(\bar{x}, \bar{y}) \in X \times X$ such that
\[ \phi(\bar{x}, y) \leq \phi(\bar{x}, \bar{y}) \leq \phi(x, \bar{y}), \text{ for all } x, y \in X. \]

Moreover, we also have
\[ \inf_{x \in X} \sup_{y \in X} \phi(x, y) = \sup_{y \in X} \inf_{x \in X} \phi(x, y) = \gamma. \]

**Proof.** By Theorem 5.3.4 with $\phi = \psi$, there exists $\bar{x} \in X$ such that
\[ \phi(\bar{x}, z) \leq \gamma, \text{ for all } z \in F(y). \]

Since $F$ is surjective, then
\[ \sup_{y \in X} \inf_{x \in X} \phi(\bar{x}, z) = \sup_{y \in X} \inf_{x \in X} \phi(\bar{x}, y) \text{ and hence } \]
\[ \phi(\bar{x}, y) \leq \gamma, \text{ for all } y \in X. \] (5.3.7)

Let $f : X \times X \to (-\infty, \infty)$ be defined as $f(y, x) = -\phi(x, y)$. By assumption (ii) $f(x, y)$ is $\gamma -$ generally lower semicontinuous function in $x$ and $F - \gamma$ generalized quasiconcave function in $y$. Therefore, again by Theorem 5.3.4, there exists $\bar{y} \in X$ such that
\[ \sup_{y \in X} \inf_{x \in X} \phi(z, \bar{y}) = -\inf_{z \in F(x)} \sup_{y \in X} f(z, y) \]

Note that $F$ is surjective. Then
\[ \inf_{z \in F(x)} \sup_{y \in X} \phi(z, \bar{y}) = \inf_{x \in X} \sup_{y \in X} \phi(x, \bar{y}) \]
and hence
\[ \phi(x, \bar{y}) \geq \gamma, \text{ for all } x \in X. \] (5.3.8)

Combining (5.3.7) and (5.3.8) we have $\phi(\bar{x}, \bar{y}) = \gamma$ and
\[ \phi(\bar{x}, y) \leq \phi(\bar{x}, \bar{y}) \leq \phi(x, \bar{y}), \text{ for all } x, y \in X. \] (5.3.9)

Finally, again by (5.3.7) and (5.3.8), we get
\[ \sup_{y \in X} \inf_{x \in X} \phi(x, y) \leq \inf_{x \in X} \sup_{y \in X} \phi(x, y). \]
\[ \leq \sup_{y \in X} \phi(x, y) \leq \phi(x, y) \leq \inf_{x \in X} \phi(x, y) \quad \text{by (5.3.9)} \]
\[ \leq \sup_{y \in X} \inf_{x \in X} \phi(x, y). \]

Consequently,
\[ \inf_{x \in X} \sup_{y \in X} \phi(x, y) = \sup_{y \in X} \inf_{x \in X} \phi(x, y) = \gamma. \]

This completes the proof.

Let \( \phi(x, y) \) be a quasiconcave function in \( y \) and \( \gamma \) – generalized quasiconvex function in \( x \), \( F : X \to X, F(x_i) = y_i \) for each \( i \in \{1, 2, \ldots, n\} \). Then Theorem 5.3.4 implies Theorem 3.2 of Ansari et al [179].

**Corollary 5.3.6 [179]**. Let \( X \) be a nonempty closed convex subset of a Hausdorff topological vector space \( E \). Let \( \gamma \in (-\infty, \infty) \) be given number. Suppose that \( \phi : X \times X \to (-\infty, \infty) \) satisfy the following conditions.

(i) \( \phi(x, y) \) is a \( \gamma \) – transfer lower semicontinuous function in \( x \) and \( \gamma \) – generalized quasiconcave function in \( y \).

(ii) \( \phi(x, y) \) is a \( \gamma \) – transfer upper semicontinuous function in \( y \) and \( \gamma \) – generalized quasiconvex function in \( x \).

(iii) There exists \( x_1, y_1 \in X \) such that the sets
\[ \{x \in X : \phi(x, y_1) \leq \gamma\} \text{ and } \{y \in X : \phi(x_1, y) \geq \gamma\} \text{ are precompact.} \]

Then, there exists a saddle point of \( \phi(x, y) \); that is, there exists \( (\bar{x}, \bar{y}) \in X \times X \) such that
\[ \phi(\bar{x}, y) \leq \phi(\bar{x}, \bar{y}) \leq \phi(x, \bar{y}), \text{ for all } x, y \in X. \]

Moreover, we also have
\[ \inf_{x \in X} \sup_{y \in X} \phi(x, y) = \sup_{y \in X} \inf_{x \in X} \phi(x, y) = \gamma. \]

Now we present the saddle point theorem for parametric games in topological vector spaces.

**Theorem 5.3.6**. Let \( E \) and \( F \) be two Hausdorff topological vector space and let \( X \subset E \) and \( Y \subset F \) be two nonempty closed convex subsets. Let \( \gamma \in (-\infty, \infty) \) be given number, \( F \) be a surjective mapping and let \( G_\theta = f - \theta g : X \times Y \to R \) satisfy the following conditions.
(i) \( f(x, y) \) is a \( \gamma \)-generally lower semicontinuous function in \( x \) and \( F \)-\( \gamma \)-generalized quasiconvex function in \( y \).

(ii) \( f(x, y) \) is a \( \gamma \)-generally upper semicontinuous function in \( y \) and \( F \)-\( \gamma \)-generalized quasiconcave function in \( x \).

(iii) \( g(x, y) \) is a \( \gamma \)-generally lower semicontinuous function in \( y \) and \( F \)-\( \gamma \)-generalized quasiconvex function in \( x \).

(iv) \( g(x, y) \) is a \( \gamma \)-generally upper semicontinuous function in \( x \) and \( F \)-\( \gamma \)-generalized quasiconcave function in \( y \).

(v) There exists \( x_1, y_1 \in X \) such that the sets
\[ \{x \in X : G_\theta(x, y) \leq \gamma \} \text{ and } \{y \in X : G_\theta(x_1, y) \geq \gamma \} \]
are precompact.

Then, there exists a saddle point of \( G_\theta(x, y) \); that is, there exists \((\bar{x}, \bar{y}) \in X \times X\) such that
\[ G_\theta(\bar{x}, y) \leq G_\theta(\bar{x}, \bar{y}) \leq G_\theta(x, \bar{y}), \text{ for all } x, y \in X. \]

Moreover, we also have
\[ \inf_{x \in X} \sup_{y \in X} G_\theta(x, y) = \sup_{y \in X} \inf_{x \in X} G_\theta(x, y) = \gamma. \]

**Proof.** Since \( \theta \geq 0 \), from (ii) and (iii) in the theorem, it follows that the function \( G_\theta(x, y) \) is \( \gamma \)-generally upper semicontinuous function in \( y \) and \( F \)-\( \gamma \)-generalized quasiconcave function in \( x \). Similarly \( G_\theta(x, y) \) is \( \gamma \)-generalized lower semicontinuous function in \( x \) and \( F \)-\( \gamma \)-generalized quasiconvex function in \( y \). Then from Theorem 5.4.5, \( G_\theta(x, y) \) has a saddle point \((\bar{x}, \bar{y}) \in X \times Y\).

The following result is derived from Theorem 5.3.5.

**Corollary 5.3.7.** Let \( E \) and \( F \) be two Hausdorff topological vector spaces and let \( X \subset E \) and \( Y \subset F \) be two nonempty closed convex subsets. Let \( \gamma \in (-\infty, \infty) \) be given number and let \( G_\theta = f - \theta g : X \times Y \to R \) satisfy the following conditions.

(i) \( f(x, y) \) is a \( \gamma \)-transfer lower semicontinuous function in \( x \) and \( \gamma \)-generalized quasiconvex function in \( y \).

(ii) \( f(x, y) \) is a \( \gamma \)-transfer upper semicontinuous function in \( y \) and \( \gamma \)-generalized quasiconcave function in \( x \).

(iii) \( g(x, y) \) is a \( \gamma \)-transfer lower semicontinuous function in \( y \) and \( \gamma \)-generalized quasiconvex function in \( x \).
(iv) \( g(x, y) \) is a \( \gamma \) – transfer upper semicontinuous function in \( x \) and \( \gamma \) – generalized quasiconcave function in \( y \).

(v) There exists \( x_1, y_1 \in X \) such that the sets
\[
\{ x \in X : G_\theta(x, y_1) \leq \gamma \} \quad \text{and} \quad \{ y \in X : G_\theta(x_1, y) \geq \gamma \}
\]
are precompact.

Then, there exists a saddle point of \( G_\theta(x, y) \); that is, there exists \( (\bar{x}, \bar{y}) \in X \times X \) such that
\[
G_\theta(\bar{x}, y) \leq G_\theta(\bar{x}, \bar{y}) \leq G_\theta(x, \bar{y}), \quad \text{for all} \quad x, y \in X.
\]

Moreover, we also have
\[
\inf_{x \in X} \sup_{y \in X} G_\theta(x, y) = \sup_{y \in X} \inf_{x \in X} G_\theta(x, y) = \gamma.
\]

As an illustration we give the following example.

**Example 5.3.1.** Consider the following payoff matrix

\[
F = \begin{pmatrix}
3 & 1 \\
2 & 4
\end{pmatrix}
\]

Let the strategies \( p \) and \( q \) of the players ‘1’ and ‘2’ be given respectively by \( p = (x, 1-x) \) and \( q = (y, 1-y) \), where \( x, y \in [0, 1] \). Then we have

\[
F(p, q) = (x, 1-x) \begin{pmatrix}
3 & 1 \\
2 & 4
\end{pmatrix} \begin{pmatrix}
y \\
1-y
\end{pmatrix}
= 3xy + x(1-y) + 2(1-x)y + 4(1-x)(1-y)
= 4xy - 3x - 2y + 4 = f(x, y).
\]

Now we have function \( f(x, y) = 4xy - 3x - 2y + 4 \), \( x, y \in [0, 1] \), whose graph is given like this:
This function is continuous so both upper and lower semicontinuous and convex so both quasiconvex and quasiconcave.

Now the slope of this function is \( \frac{\partial}{\partial x} f(x, y), \frac{\partial}{\partial y} f(x, y) \) \( = (4y - 3, 4x - 2) \), which is 0 for \( x = \frac{1}{2} \) and \( y = \frac{3}{4} \). Now \( f(1/2 + \varepsilon, 3/4 + \delta) = 5/2 + 4\varepsilon\delta \) yields that the point \((1/2, 3/4)\) is a saddle point and the value of the game is \( F(p, q) = 5/2 \).