CHAPTER 5

SETTING THE CLOCK BACK TO ZERO
PROPERTY OF DISCRETE DISTRIBUTIONS

5.1 Introduction

In the last two chapters we were discussing the concept of SCBZ property in the continuous univariate as well as multivariate situations. The study of this property in the discrete set up is of more interest, since in actual practice the life of the components are measured in discrete time units, that is the time is measured discretely as the completed years of life or as number of cycles. The difficulties, in measuring the time continuously are discussed by authors like Xekalaki (1983), Cox (1972) and Kalbfleish and Prentice (1980). From this view point we try to define SCBZ property in the discrete cases- both for univariate and multivariate distributions. It can be noticed that, we will get almost all the results parallel to those we have obtained for continuous distributions with necessary modifications.
5.2 Univariate discrete SCBZ property. (Nair and Mini, 1999)

Let $X$ be a discrete random variable defined on the support of $I^+ = \{0, 1, 2, \ldots \}$ with a survival function $R(x, \theta) = P(X \geq x)$ and probability mass function $f(x, \theta)$. As in the case of continuous random variable the SCBZ property can be defined in the discrete case as follows.

**Definition**

A non-negative discrete random variable $X$ defined on $I^+$ or its family of survival functions $\{R(x, \theta) : x \in I^+, \theta \in \Theta\}$ is said to have the SCBZ property if for each $\theta \in \Theta$ and $x, t \in I^+$, the survival function $R(x, \theta)$ satisfies the condition

$$R(x + t, \theta) = R(t, \theta^*) R(x, \theta^*)$$

(5.1)

where $\theta^* = \theta^*(t) \in \Theta$, the parametric space.

By this property we mean that the conditional distribution of additional time of survival of a device or an organism given that it has already survived $t$ units of time remains in the same family. In the reliability context, this property ensures that the residual life distribution remains in the original family of distributions.
Examples

1. Consider the geometric distribution with survival function

\[ R(x, \theta) = q^x, \quad x = 0, 1, 2, \ldots; \quad 0 \leq q \leq 1. \]  

where \( \theta = q \) and \( \Theta = \{q; 0 \leq q \leq 1\} \). It can be noticed that

\[ \frac{R(x+t, \theta)}{R(t, \theta)} = \theta^x = R(x, \theta^*), \]

where \( \theta^* = \theta \).

2. Consider the Waring distribution specified in Irwin (1975) with survival function

\[ R(x, \theta) = \frac{(b)_x}{(a)_x}, \quad x = 0, 1, 2, \ldots; \quad a, b > 0 \]  

where \( (a)_x = a(a+1)\cdots(a+x-1) \); with \( \theta = (a, b) \) and \( \Theta = \{(a, b); \ a, b > 0; \ a > b+1\} \). In this case

\[ \frac{R(x+t, \theta)}{R(t, \theta)} = \frac{(b+t)_x}{(a+t)_x} = \frac{R(x, \theta^*)}{R(t, \theta^*)}, \]

where \( \theta^* = (a+t, b+t) \).

3. Consider the case of negative hypergeometric distribution with survival function
where \( \theta=(k,n) \) and \( \Theta = \{(k,n) : k,n>0\} \).

Here

\[
R(x+t, \theta) = \frac{(k+n-t-x)}{n-t} R(x, \theta), \quad x = 0, 1, 2, \ldots, n;
\]

(5.4)

and

\[
\frac{R(x+t, \theta)}{R(t, \theta)} = \frac{k+n-t-x}{n-t}.
\]

(5.5)

5.2.1 Reliability Characteristics (Nair and Mini, 1999)

In this section we establish the equivalent conditions of SCBZ property in terms of reliability characteristics especially the failure rate and mean residual life.

Theorem 5.1

The SCBZ property is equivalent to

\[
h(x+t, \theta) = h(x, \theta')
\]

where \( h(., \theta) \) is the failure rate defined in (2.50).
Proof

The SCBZ property of the random variable \(X\) implies (5.1).

Hence

\[ R(x+t+1, \theta) = R(x+1, \theta^*) \]  \hspace{1cm} (5.5)

On taking the difference between (5.1) and (5.5) and dividing by (5.1) we get

\[ \frac{R(x+t, \theta) - R(x+t+1, \theta)}{R(x+t, \theta)} = \frac{R(x, \theta^*) - R(x+1, \theta^*)}{R(x, \theta^*)} \] (say)

That is in terms of the failure rate \(h(., \theta)\), it can be written as

\[ h(x+t, \theta) = h(x, \theta^*). \] \hspace{1cm} (5.6)

For the converse part use the relation (2.52) connecting \(h(x, \theta)\) and \(R(x, \theta)\)

\[ \frac{R(x+t, \theta)}{R(t, \theta)} = \prod_{y=0}^{x+t-1} \frac{1 - h(y, \theta)}{1 - h(y+1, \theta)} \]

\[ = \prod_{y=t}^{x+t-1} \frac{1 - h(y, \theta)}{1 - h(y+t, \theta)} \]

\[ = \prod_{y=0}^{x-1} \frac{1 - h(y, \theta^*)}{1 - h(y+t, \theta^*)} \].
Assuming (5.6), we can write
\[
\frac{R(x+t, \theta)}{R(t, \theta)} = R(x, \theta^*),
\]
which establishes the desired result.

From the above theorem, it is noticed that the failure rate is a function of the transformed parameter. That is
\[
h(x, \theta) = h(0, \theta^*) = g(\theta^*) \text{ (say)}.
\]

The following theorems give the two important results concerning the functional form of the failure rate and SCBZ property.

**Theorem 5.2**

If the failure rate is linear, then the family of survival function possesses the SCBZ property.

**Proof**

The survival function of a discrete random variable defined on \( I^+ \) is uniquely determined by the equation (2.52). If \( h(x, \theta) = a+bx \), we have
Theorem 5.3

If $h(x, \theta) = \theta^*$ is a one to one function $g(.)$ of $\theta^*$, then $g(\theta^*)$ uniquely determines the distribution.

Proof

Let the one to one function of $h(x, \theta)$ is $g(\theta^*)$. Since the failure rate uniquely related to the survival function by (2.52)

$$R(x, \theta) = \prod_{t=0}^{x-1}[1-h(t, \theta)]$$

$$= \prod_{t=0}^{x-1}[1-g(\theta^*)]$$

which implies that $g(\theta^*)$ uniquely determines the distribution.
Now we can think about the mean residual life. The following result gives a characterization of SCBZ property in terms of the mean residual life.

**Theorem 5.4**

SCBZ property of a discrete random variable with $f(0)=0$ is equivalent to

$$r(x+t, \theta) = r(x, \theta^*) \tag{5.7}$$

where $r(., \theta)$ is the mean residual life defined in (2.51).

**Proof**

SCBZ property of the random variable implies (5.1) and (5.5).

Also

$$R(u+t+1, \theta) = R(t, \theta) R(u+1, \theta^*).$$

Therefore

$$\frac{R(u+t+1, \theta)}{R(x+t+1, \theta)} = \frac{R(u+1, \theta^*)}{R(x+1, \theta^*)}.$$}

That is

$$\frac{1}{R(x+t+1, \theta)} \sum_{u=x}^{\infty} R(u+t+1, \theta) = \frac{1}{R(x+1, \theta^*)} \sum_{u=x}^{\infty} R(u+1, \theta^*).$$

That is

$$r(x+t, \theta) = r(x, \theta^*)$$
For the converse part, let us assume the relation (5.7) with \( f(0)=0 \).

Using (2.53),

\[
\frac{R(x+t, \theta)}{R(t, \theta)} = \prod_{u=1}^{x+t-1} \left[ \frac{r(u-1, \theta) - 1}{r(u, \theta)} \right] = \prod_{u=0}^{x+t-1} \left[ \frac{r(u+t-1, \theta) - 1}{r(u+t, \theta)} \right]
\]

which shows the SCBZ property.

5.2.2 Distribution of Partial Sums (Nair and Mini, 1999)

In this section we investigate certain characteristics of the distributions based on the partial sums. Let \( X \) be a discrete random
variable defined on the set of non-negative integers with probability mass function $f(x, \theta)$, survival function $R(x, \theta) = P(X \geq x, \theta)$ and a finite mean $\mu$. Then the random variable $Y$ specified by

$$g(x, \theta) = P(Y=x) = \mu^{-1} P(X>x) = \mu^{-1} R(x+1, \theta) \quad (5.8)$$

is said to have the distribution based on partial sums corresponding to $X$. Gupta (1979) has shown that $Y$ is the residual lifetime of a component in a system where a component of life length $X$ is replaced upon failure by another having the same distribution, so that it forms a renewal process. He also showed that the failure rate of $Y$ is the reciprocal of the mean residual life of $X$. Some other properties are studied by Johnson and Kotz (1969). Nair and Hitha (1989) obtained certain relations between the failure rate and mean residual life function to characterize certain discrete distributions by considering the relevance of these models in reliability analysis.

Let $h_y(x, \theta)$ denote the failure rate of $Y$ and $r(x, \theta)$ denote the mean residual life of $X$. Then we have

$$h_y(x, \theta) = [r(x, \theta)]^{-1}. \quad (5.9)$$
The following theorem shows that the SCBZ property of the parent distribution preserves in the distribution based on partial sums.

**Theorem 5.5**

The SCBZ property of $X$ implies the SCBZ property of $Y$.

**Proof**

Let $G(., \theta)$ denote the survival function of $Y$.

$$G(x+t, \theta) = \sum_{u=x}^{\infty} g(u, \theta)$$

$$= \mu^{-1} \sum_{u=x+t}^{\infty} R(u+1, \theta)$$

$$= \mu^{-1} \sum_{u=0}^{\infty} R(x+t+u+1, \theta)$$

$$= \mu^{-1} \sum_{u=0}^{\infty} R(t, \theta) R(x+u+1, \theta^*)$$

$$\mu^* R(t, \theta) = R(t, \theta) \sum_{u=1}^{\infty} R(u, \theta^*)$$

$$= R(t, \theta) \sum_{u=0}^{\infty} R(u+1, \theta^*)$$

$$= \sum_{u=0}^{\infty} R(t+u+1, \theta)$$

Therefore
\[ R(t, \theta) = (\mu^*)^{-1} \sum_{u=0}^{\infty} R(t+u+1, \theta) \]

and

\[ G(x+t, \theta) = (\mu \mu^*)^{-1} \sum_{u=0}^{\infty} R(t+u+1, \theta) \sum_{u=0}^{\infty} R(x+u+1, \theta') \]

\[ = G(t, \theta) G(x, \theta'), \quad (5.10) \]

which shows the SCBZ property of \( Y \).

The converse of the above theorem need not holds always. If \( f(0) = 0 \), then it holds. It is established in the following theorem.

**Theorem 5.6**

The SCBZ property of \( Y \) implies SCBZ property of \( X \) if \( f(0) = 0 \).

**Proof**

SCBZ property of \( Y \) implies (5.10). Then as in the line of proof of Theorem 5.1 we can have

\[ h_y(x+t, \theta) = h_y(x, \theta'). \]

By the relation (5.9), we have

\[ r(x+t, \theta) = r(x, \theta'). \]

Then by Theorem 5.4, we have the desired result that \( X \) has SCBZ property.
5.3 SCBZ Property of Bivariate Distributions

The analogous discrete situations of the Chapter IV is discussed in this section. In the previous section we define the SCBZ property of univariate case. The extension of this property to the bivariate case is not unique and hence we can define it in various ways. Let \((X_1, X_2)\) be a vector of random variables with support \(I_2^+ = \{(x_1, x_2): x_1, x_2 = 0, 1, 2, \ldots\}\) and family of survival functions \(\{R(x_1, x_2, \theta): (x_1, x_2) \in I_2^+, \theta \in \Theta\}\), where

\[ R(x_1, x_2, \theta) = P(X_1 \geq x_1, X_2 \geq x_2, \theta) \]

5.3.1 Bivariate SCBZ(1) Property

The natural extension of the definition of SCBZ property in univariate case leads to the following definition.

**Definition**

A family of survival distributions with support \(I_2^+\) or the random vector \((X_1, X_2)\) is said to have bivariate SCBZ(1) property if for each \(\theta \in \Theta\) and \(x_1, x_2, t = 0, 1, \ldots\), the following condition

\[ R(x_1 + t, x_2 + t, \theta) = R(t, t, \theta) R(x_1, x_2, \theta') \]  

(5.11)
where $\theta^* = \theta^*(t) \in \Theta$, the parametric space holds.

**Examples**

1. Bivariate Waring distribution with survival function

$$R(x_1, x_2, \theta) = \frac{B(\alpha, \beta + x_1 + x_2)}{B(\alpha, \beta)}, \quad \alpha, \beta > 0, \ x_1, x_2 = 0, 1, \ldots, \quad (5.12)$$

where $B(\alpha, \beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha + \beta)}$ with $\theta = (\alpha, \beta)$.

(This distribution can be regarded as the discrete analogue of bivariate Pareto distribution and it belongs to the Pearson’s system of discrete distributions (Ord, 1972). Here

$$R(x_1 + t, x_2 + t, \theta^*) = \frac{B(\alpha, \beta + x_1 + x_2 + 2t)}{B(\alpha, \beta + 2t)} = R(x_1, x_2, \theta^*)$$

where $\theta^* = (\alpha, \beta + 2t)$.

2. Bivariate negative hypergeometric distribution with survival function

$$R(x_1, x_2, \theta) = \binom{k + n - x_1 - x_2}{n - x_1 - x_2} \binom{n - x_1 - x_2}{k + n} \binom{k + n - x_1 - x_2}{n}, \quad (5.13)$$

$x_1, x_2 = 0, 1, \ldots, n; \ n, k > 0; \ x_1 + x_2 \leq n$ with $\theta = (k, n)$. 

...
(Xekalaki (1983), by the slope to mean ordinate ratio method has shown that the continuous analogue of negative hypergeometric distribution is finite range distribution).

\[
R(x_1 + t, x_2 + t, \theta) = \frac{\binom{k+n-x_1-x_2-2t}{k+n-2t}}{\binom{n-x_1-x_2-2t}{n-2t}} = R(x_1, x_2, \theta^*)
\]

where \( \theta^* = (k,n-2t) \).

3. A bivariate geometric distribution (discrete analogue of Gumbel's bivariate exponential) with survival function

\[
R(x_1, x_2, \theta) = p_1^{x_1} p_2^{x_2} \alpha^{x_1 x_2}, x_1, x_2 = 0, 1, \ldots
\]

where \( 0 < p_1, p_2 < 1; 0 \leq \alpha \leq 1; 1 - \alpha \leq (1 - p_1 \alpha)(1 - p_2 \alpha) \) where \( \theta = (p_1, p_2, \alpha) \).

It can be noticed that \( \theta^* = (p_1 \alpha^*, p_2 \alpha^*, \alpha) \).

5.3.2 Bivariate SCBZ(2) Property

The second definition to the SCBZ property of bivariate distributions is as follows
Definition

A random vector \((X_1,X_2)\) defined on \(I_2^+\) or its family of survival distributions \(\{R(x_1,x_2,\theta), \theta \in \Theta, (x_1,x_2) \in I_2^+\}\) is said to have bivariate SCBZ(2) property if for each \(\theta \in \Theta\) and all \((x_1,x_2)\) and \((t_1,t_2) \in I_2^+\), it should satisfy the condition

\[
R(x_1+t_1,x_2+t_2,\theta) = R(t_1,t_2,\theta) R(x_1,x_2,\theta^*)
\]

where \(\theta^* = \theta^*(t_1,t_2) \in \Theta\).

Examples

1. Bivariate Waring distribution specified in (5.12). Here \(\theta^*=(\alpha,\beta+t_1+t_2)\).

2. Bivariate negative hypergeometric distribution specified in (5.13) where \(\theta^*=(k,n-t_1-t_2)\).

3. Bivariate geometric distribution with survival (5.14). It can be seen that \(\theta^*=(p_1\alpha^k, p_2\alpha^h, \alpha)\).

In terms of the local lack of memory property, the SCBZ property can be defined as in the subsequent section.
5.3.3 Conditional SCBZ (1) Property

**Definition**

A class of life distributions \( \{R(x_1, x_2, \theta), \theta \in \Theta, (x_1, x_2) \in I_1^+ \} \) or a vector \((X_1, X_2)\) is said to have conditional SCBZ(1) property if for each \( \Theta \in \Theta \) and \( s_1, s_2, t_1, t_2 = 0, 1, 2, \ldots \)

\[
G_1(t_1 + s_1, t_2, \theta) = G_1(t_1, t_2, \theta) G_1(s_1, t_2, \theta^*)
\]

and

\[
G_2(t_1, t_2 + s_2, \theta) = G_2(t_1, t_2, \theta) G_2(t_1, s_2, \theta^{**})
\]

where \( \theta^* = \theta'(t_1) \) and \( \theta^{**} = \theta''(t_2) \) belong to the same parametric space and \( G_i(t_1, t_2, \theta) = P(X_i \geq t_i | X_j \geq t_j, \theta), i, j = 1, 2, i \neq j \).

Therefore

\[
G_1(t_1, t_2, \theta) = \frac{R(t_1, t_2, \theta)}{R(0, t_2, \theta)}
\]

and

\[
G_2(t_1, t_2, \theta) = \frac{R(t_1, t_2, \theta)}{R(t_1, 0, \theta)}.
\]

**Examples**

1. In the case of the bivariate Waring distribution (5.12), \( \theta^* = (\alpha, \beta + t_1) \) and \( \theta^{**} = (\alpha, \beta + t_2) \).
2. Bivariate negative hypergeometric distribution specified in (5.13). Here $\theta = (k, n)$. In this case $\theta^*=(k, n-t_1)$ and $\theta^{**}=(k, n-t_2)$.

3. Bivariate geometric distribution with survival function (5.14). It can be seen that $\theta^*=(p_1, p_2 \alpha^*, \alpha)$ and $\theta^{**}=(p_1 \alpha^*, p_2, \alpha)$.

### 5.3.4 Conditional SCBZ(2) Property

In view of the conditional lack of memory property defined by Nair and Nair (1991), here we investigate to study the SCBZ property of one component when the value of other component is preassigned.

**Definition**

A class of bivariate survival functions $\{R(x_1, x_2, \theta), \theta \in \Theta, (x_1, x_2) \in I^*_2 \}$ or a vector $(X_1, X_2)$ is said to have conditional SCBZ(2) property if the following set of conditions is satisfied for each $\theta \in \Theta$ and $s_1, s_2, t_1, t_2 = 0, 1, 2, ...$

$$S_1(t_1 + s_1, t_2, \theta) = S_1(t_1, t_2, \theta) S_1(s_1, t_2, \theta^*)$$

and

$$S_2(t_1, t_2 + s_2, \theta) = S_2(t_1, t_2, \theta) S_2(t_1, s_2, \theta^{**})$$

where $\theta^* = \theta^*(t_1)$ and $\theta^{**} = \theta^{**}(t_2) \in \Theta$. $S_i(t_1, t_2, \theta) = P(X_i \geq t_1 | X_j = t_j, \theta)$ for $i, j = 1, 2; i \neq j$. 
We have

\[
S_1(t_1, t_2, \theta) = \frac{P(X_1 \geq t_1 \mid X_2 = t_2, \theta)}{P(X_2 = t_2, \theta)}
\]

\[
= \frac{R(t_1, t_2, \theta) - R(t_1, t_2 + 1, \theta)}{R(0, t_2, \theta) - R(0, t_2 + 1, \theta)}
\]

(5.18)

and

\[
S_2(t_1, t_2, \theta) = \frac{R(t_1, t_2, \theta) - R(t_1 + 1, t_2, \theta)}{R(t_1, 0, \theta) - R(t_1 + 1, 0, \theta)}
\]

(5.19)

Examples

1. Consider the bivariate Waring distribution (5.12). Here

\[
S_1(t_1 + s_1, t_2, \theta) = B(\alpha, \beta + t_1 + t_2 + s_1) - B(\alpha, \beta + t_1 + t_2 + s_1 + 1)
\]

\[
S_1(t_1, t_2, \theta)
\]

\[
= S_1(s_1, t_2, \theta^*)
\]

where \( \theta^* = (\alpha, \beta + t_1) \).

Similarly

\[
S_2(t_1, t_2 + s_2, \theta) = S_2(t_1, s_2, \theta^{**})
\]

where \( \theta^{**} = (\alpha, \beta + t_2) \).

2. For the bivariate negative hypergeometric distribution with survival function (5.13)
\[
\frac{S_1(t_1 + s_1, t_2, \theta)}{S_1(t_1, t_2, \theta)} = \begin{pmatrix}
{k+n-t_1-s_1} \\
{n-t_1-t_2-s_1} \\
{k+n-t_1-t_2} \\
{n-t_1-t_2}
\end{pmatrix} - \begin{pmatrix}
{k+n-t_1-s_1} \\
{n-t_1-t_2-s_1} \\
{k+n-t_1-t_2} \\
{n-t_1-t_2}
\end{pmatrix}
\]

\[= S_1(s_1, t_2, \theta^*)\]

where \(\theta^* = (k, n-t_1)\) and

\[
\frac{S_2(t_1, t_2 + s_2, \theta)}{S_2(t_1, t_2, \theta)} = S_2(t_1, s_2, \theta^{**})
\]

where \(\theta^{**} = (k, n-t_2)\).

3. In the case of bivariate geometric distribution with survival function (5.14),

\[
\frac{S_1(t_1 + s_1, t_2, \theta)}{S_1(t_1, t_2, \theta)} = \frac{p_1^{s_1} \alpha^{s_1} (1 - p_2 \alpha^{s_1})}{(1 - p_2 \alpha^h)}
\]

\[= S_1(s_1, t_2, \theta^*)\]

where \(\theta^* = (p_1, p_2 \alpha^h, \alpha)\) and

\[
\frac{S_2(t_1, t_2 + s_2, \theta)}{S_2(t_1, t_2, \theta)} = S_2(t_1, s_2, \theta^{**})
\]

where \(\theta^{**} = (p_1 \alpha^h, p_2, \alpha)\).

### 5.3.5 Extended Bivariate SCBZ Property

Rao et al. (1993) has extended the notion of SCBZ property to the bivariate continuous case as the one discussed in section 2.7. By
applying a similar approach we can define the SCBZ property of the
discrete case to that in bivariate case as follows.

**Definition**

A class of bivariate survival functions \( R(x_1, x_2, \theta), \theta \in \Theta, \)
\( (x_1, x_2) \in I_2^+ \) or a vector \( (X_1, X_2) \) is said to have the extended bivariate
setting the clock back to zero property if for each \( \theta \in \Theta \) and \( x_1, x_2 = 0, 1, 2, \ldots \) the survival function satisfies the equations

\[
\begin{align*}
R(x_1 + t, t, \theta) &= R(t, t, \theta) R(x_1, t, \theta') \\
R(t, x_2 + t, \theta) &= R(t, t, \theta) R(t, x_2, \theta^{**})
\end{align*}
\]

with \( \theta' = \theta(t) \in \Theta_0 \) and \( \theta^{**} = \theta^{**}(t) \in \Theta_0 \) where \( \Theta_0 \) denotes the boundary
of \( \Theta \).

**Example**

1. Consider the geometric distribution specified in (5.14)

\[
\frac{R(x_1 + t, t, \theta)}{R(t, t, \theta)} = \frac{p_1^{x_1 + t} p_2 p_2^{x_1 + t + 2}}{p_1^t p_2^t} = p_1^{x_1} \alpha^{\theta_t} = R(x_1, t, \theta')
\]

where \( \theta' = (p_1, p_2', \alpha) \) with \( p_2' = 1 \).
Also

\[
\frac{R(t, x_2 + t, \theta)}{R(t, t, \theta)} = p_2^* \alpha^{x_2}
\]

\[
= R(x_1, t, \theta^{**})
\]

where \( \theta^{**} = (p_1^*, p_2^*, \alpha) \) with \( p_1^* = 1 \). Here \( \theta^* \) and \( \theta^{**} \) belong to the boundary \( \Theta_0 \), which includes \( 0 \leq p_1, p_2 \leq 1 \).

5.4 Properties of SCBZ property

In this section we establish certain implications between various definitions and some of their properties.

Theorem 5.7

Bivariate SCBZ(2) property implies bivariate SCBZ(1) property, but the converse is not true.

Proof

When \( t_1 = t_2 = t \) in the equation (5.15) of bivariate SCBZ(2) property, it reduces to (5.11), which is the condition for bivariate SCBZ(1) property. For proving the converse part, let us consider a bivariate geometric distribution (the discrete analogue of Marshall-Olkin exponential distribution) with survival function (2.32). (2.32) can also be written as
\[ R(x_1, x_2, \theta) = p_{\min(x_1, x_2)} p_{1}^{\max(0, x_1 - x_2)} p_{2}^{\max(0, x_2 - x_1)}, \quad (5.21) \]

\[ 1+p \geq p_1 + p_2, \quad 0 < p \leq p_j < 1, \quad p = p_1 + p_2 - c_3 - 1, \quad \text{where} \quad c_3 = \frac{P(X_1 = x_1, X_2 = x_2)}{P(X_1 \geq x_1, X_2 \geq x_2)} . \]

Here

\[ \frac{R(x_1 + t, x_2 + t, \theta)}{R(t, t, \theta)} = p_{\min(x_1, x_2)} p_{1}^{\max(0, x_1 - x_2)} p_{2}^{\max(0, x_2 - x_1)} \]

\[ = R(x_1, x_2, \theta^*) \]

with \( \theta^* = \theta \). But

\[ \frac{R(x_1 + t, x_2 + t, \theta)}{R(t, t, \theta)} = p_{\min(x_1 + t_1, x_2 + t_2)} p_{1}^{\max(0, x_1 - x_2 + t_2 - t_1)} p_{2}^{\max(0, x_2 - x_1 + t_2 - t_1)} \]

\[ \neq R(x_1, x_2, \theta^*) \]

with \( \theta^* \in \Theta \). Then, it can be noticed that the bivariate geometric distribution with survival function (5.21) holds bivariate SCBZ(1) property but does not holds bivariate SCBZ(2) property. Hence we can conclude that bivariate SCBZ(1) need not imply bivariate SCBZ(2) property.

The next theorem shows that the bivariate SCBZ(2) property implies the SCBZ properties of the marginal distributions.

**Theorem 5.8**

Bivariate SCBZ(2) property implies the SCBZ property of marginal distributions.
Proof

Bivariate SCBZ(2) property implies (5.15). On setting $x_2 = t_2 = 0$, (5.15) becomes

$$R(x_1 + t_1, 0, \theta) = R(t_1, 0, \theta) R(x_1, 0, \theta').$$

That is, if $R_i(x_i, \theta) = P(X_i \geq x_i, \theta)$, we have

$$R(x_1 + t_1, \theta) = R(t_1, \theta) R(x_1, \theta')$$

with $\theta^* = \theta'(t_1) \in \Theta$, which indicates the SCBZ property of the component $X_1$. Similarly we can show the SCBZ property of $X_2$ also.

As in the case of continuous variables a parallel result of Theorem 4.13 holds for the bivariate discrete case also.

Theorem 5.9

The bivariate SCBZ(2) property of $(X_1, X_2)$ implies the univariate SCBZ property of $Z = \min(X_1, X_2)$.

Proof

Let $R(., \theta)$ denote the survival function of $Z$. Then

$$R(x+t, \theta) = P(Z \geq x+t, \theta)$$

with $\theta^* = \theta'(t_1) \in \Theta$, which implies the SCBZ property of the component $X_1$. In a similar way we can show that $X_2$ has SCBZ property with $\theta^* = \theta'(t) \in \Theta$. That is
\[ R(x+t, \theta) = R(t, \theta) R(x, \theta^*), \]

which shows the desired result.

The property of bivariate SCBZ(2) property described in Theorem 5.8 holds for the conditional SCBZ(1) property also and is established in the following theorem.

**Theorem 5.10**

Conditional SCBZ(1) property of \((X_1,X_2)\) implies SCBZ properties of \(X_1\) and \(X_2\).

**Proof**

Conditional SCBZ(1) property implies (5.16). When \(t_2 = 0\) in the first equation of (5.16), we get

\[ G_1(t_1 + s_1, 0, \theta) = G_1(t_1, 0, \theta) G_1(s_1, 0, \theta^*). \]

That is

\[ R(t_1 + s_1, 0, \theta) = R(t_1, 0, \theta) R(s_1, 0, \theta^*). \]

That is

\[ R_1(t_1 + s_1, \theta) = R_1(t_1, \theta) R_1(s_1, \theta^*) \]

with \(\theta^* = \theta^*(t_1) \in \Theta\), where \(R_1(t_1, \theta)\) is the survival function of \(X_1\) which implies the SCBZ property of the component \(X_1\). In a similar way we can show that \(X_2\) has SCBZ property.
5.5 Distribution based on Partial Sums in Bivariate case

An appropriate bivariate extension of partial sum distributions are firstly given by Kotz and Johnson (1991). Let $X_1$ and $X_2$ are the original random variables with survival function $R(x_1, x_2, \theta)$. Then the random vector $(Y_1, Y_2)$ corresponding to the partial sums has a probability density function of the form

$$g(x_1, x_2, \theta) = P(Y_1 = x_1, Y_2 = x_2, \theta)$$

$$= \frac{P(X_1 > x_1, X_2 > x_2, \theta)}{E(X_1 X_2)}$$

$$= \frac{R(x_1 + 1, x_2 + 1, \theta)}{\mu}, \quad (5.22)$$

where $\mu = E_\theta(X_1 X_2)$.

Stipulated along the lines of the univariate case, we can obtain the analogous result of Theorem 5.5.

Theorem 5.11

The bivariate SCBZ(2) property of $(X_1, X_2)$ implies the bivariate SCBZ(2) property of $(Y_1, Y_2)$.

Proof

Let $G(\ldots, \theta)$ denote the survival function of $(Y_1, Y_2)$. Then
\[ G(x_1, x_2, \theta) = \sum_{u=x_1}^{\infty} \sum_{v=x_2}^{\infty} g(u, v, \theta). \]

That is
\[ G(x_1, x_2, \theta) = \sum_{u=x_1}^{\infty} \sum_{v=x_2}^{\infty} \frac{R(u+1, v+1, \theta)}{\mu}. \]

Therefore
\[ G(x_1+t_1, x_2+t_2, \theta) = \mu^{-1} \sum_{u=x_1+t_1}^{\infty} \sum_{v=x_2+t_2}^{\infty} R(u+1, v+1, \theta) \]
\[ = \mu^{-1} \sum_{u=0}^{\infty} \sum_{v=0}^{\infty} R(t_1+t_2, \theta) R(u+x_1+1, v+x_2+1, \theta^*), \tag{5.23} \]

since \((X_1, X_2)\) has bivariate SCBZ(2) property. We have
\[ \mu^* R(t_1, t_2, \theta) = \sum_{u=0}^{\infty} \sum_{v=0}^{\infty} R(t_1, t_2, \theta) R(u+1, v+1, \theta^*). \]
\[ = \sum_{u=0}^{\infty} \sum_{v=0}^{\infty} R(u+t_1+1, v+t_2+1, \theta). \]

Therefore
\[ R(t_1, t_2, \theta) = (\mu^*)^{-1} \sum_{u=0}^{\infty} \sum_{v=0}^{\infty} R(u+t_1+1, v+t_2+1, \theta) \tag{5.24} \]

Hence on substituting (5.24) in (5.23), we get
\[ G(x_1 + t_1, x_2 + t_2, \theta) = (\mu \mu')^{-1} \sum_{u=0}^{\infty} \sum_{v=0}^{\infty} R(u + t_1 + 1, v + t_2 + 1, \theta) \]

That is
\[ G(x_1 + t_1, x_2 + t_2, \theta) = G(t_1, t_2, \theta) \cdot G(x_1, x_2, \theta'), \]

which shows the desired result.

5.6 Conclusion

In the present study we formed a class of univariate continuous distributions that admit a partial differential equation. The general solution of that equation also can be derived. But the formulation of the PDE in the case of all the bivariate models cannot be possible. Eventhough a PDE in the case of bivariate SCBZ (1) property is obtained, we are not able to show that all the distributions admitting that PDE should hold that property.

It can be proved that the SCBZ property preserves in the equilibrium distributions of univariate continuous and discrete cases. But the converse of that result can be proved only in the case of continuous distributions. In chapter 4 we have obtained that bivariate SCBZ (2) property preserves in the equilibrium distribution in
continuous distributions and vice versa. But the converse of the parallel result in the bivariate discrete case cannot be proved. The study of the preservation of conditional SCBZ (1) property also requires some interest. These problems are set for future work. Also the study of the measures for maintenance policies in the distributions with SCBZ properties are meant for our future work.


