Chapter 2

Not External Domination of Vertices (Ned) and Internally Stable Vertex Set (Int) of Fuzzy Graphs

2.1 Introduction

In crisp graph $G = (X, R)$, the well known concepts of not external domination Ned ($R$) and internally stable sets Int ($R$) are studied. For the fuzzy graphs those concepts have been extended under some valued logical operators by Kitainik[39] and [40].

Assia et al.[5] considered the logical operators $\wedge$, $\vee$ and composition max $\rightarrow L$ (composition $L$). By using these logical operators they extended the concepts of Int ($R$) and show that this set is characterized by solving a mathematical programming problem. Also, they discussed t- norm, t-co-norm and s-norm. Finally, they investigated the fuzzy counterpart of the set not external domination Ned ($R$).

In this chapter, we recall some basic concepts and known properties related to crisp graphs. We establish rigorous results describing the extension of Ned ($\rho$) and Int($\rho$) of fuzzy graphs. Here, we consider the logical operators $\bar{L}$, $L$ and the composition max-$\bar{L}$ (composition $\bar{L}$). We introduce new notions of the sets Ned ($\rho$), Int ($\rho$), weak lattices and sub weak lattices in fuzzy graphs. Further, we investigate the relation between the sets Ned ($\rho$), Int ($\rho$), weak lattices and sub weak lattices of fuzzy graphs.
with some useful illustrations. Finally, the set \( \text{Int}( \rho ) \) is characterized by solving a mathematical programming problem in fuzzy graph.

### 2.2 Preliminaries

This section contains the discussion of some basic definitions and results which are helpful for our main results.

**Definition 2.2.1**[5] A t-norm is a function \( T: [0, 1] \times [0, 1] \rightarrow [0, 1] \), associative, symmetric, monotonic and such that \( T(\alpha, 1) = \alpha \) for each \( \alpha \in [0, 1] \).

**Definition 2.2.2**[5] A t-conorm \( S \) can be constructed from a t-norm \( T \), by the following version of DeMorgan’s identity: for all \( \alpha, \beta \in [0, 1] \), \( S(\alpha, \beta) = T(\text{N}(\alpha), \text{N}(\beta)) \).

**Note.** The logical operators \(( \land, \lor, \neg, \bar{L}, \bar{L}, \bar{N} )\) on fuzzy subsets are defined as follows: \( \land(\alpha, \beta) = \min\{\alpha, \beta\} \), \( \lor(\alpha, \beta) = \max\{\alpha, \beta\} \), \( \bar{L}(\alpha, \beta) = \max\{\alpha + \beta - 1, 0\} \), \( \bar{L}(\alpha, \beta) = \min\{\alpha + \beta, 1\} \) and \( \text{N}(\alpha) = 1 - \alpha \).

**Definition 2.2.3**[5] Let \( G = (X, R) \) be graph, where \( X \) is an arbitrary finite non empty set, \( R \) a relation on \( X \). If \( A \subseteq X \), the set elements of \( X \) are dominated by \( A \) then the composition of \( A \) and \( R \) such that \( A \circ R = \{ y \in X \mid (\exists x \in A) x R y \} \).

**Definition 2.2.4**[5] Let \( G = (X, R) \) be graph, where \( X \) is an arbitrary finite non empty set, \( R \) a relation on \( X \). If \( A \subseteq X \) is said to be not externally dominated then “no element
in $A$ is dominated by an element in $\overline{A}$. That is, $(\forall y)[ y \in A \Rightarrow (\forall x \in \overline{A}) \text{ Not } (x R y)]$.

Which is denoted by Ned $(R)$. Here, $\overline{A}$ is the complement of $A$ and $\overline{A} = X - A$.

**Definition 2.2.5**[5] Let $G = (X, R)$ be graph, where $X$ is an arbitrary finite non empty set, $R$ a relation on $X$. If $A \subseteq X$ is said to be internally stable then “no element in $A$ is dominated by another element in $A$.” Then, $(\forall y)[ y \in A \Rightarrow (\forall x \in A) x \neq y, \text{ Not } (x R y)]$.

Which is denoted by Int $(R)$.

**Proposition 2.2.6**[5] Let $G = (X, R)$ be a loop free graph, where $X$ is an arbitrary finite non empty set, $R$ a relation on $X$ and $A \subseteq X$. The following are equivalence

(i) $A$ is not externally dominated $\iff \overline{A} \circ R \subseteq A \iff A \circ R^{-1} \subseteq A$

(ii) $A$ is internally stable $\iff A \circ R \subseteq \overline{A} \iff A \circ R^{-1} \subseteq \overline{A}$.

**Definition 2.2.7**[94] Let $\mu: X \to [0, 1]$ be a fuzzy subset, if $\mu(x) = 1$ for some $x \in X$, then $\mu$ is called normalized.

**Note.** (i) The set of all fuzzy subsets defined on $X$, denoted by $\varphi(X)$, identified with points $(\mu(x_1), \mu(x_2), \ldots, \mu(x_n))$ of $[0, 1]^n$.

(ii) The zero of $\varphi(X)$ is denoted by $0 (0(x) = 0$ for all $x \in X)$.

(iii) The units by $1 (1(x) = 1$ for all $x \in X)$.

(iv) The constant $(\frac{1}{2}, \frac{1}{2}, \ldots, \frac{1}{2}) \in [0, 1]^n$ by $\frac{1}{2}$.
Definition 2.2.8[5] The set of all fuzzy relations defined on $X$ is denoted by $\varphi(X \times X)$.

Let $\rho, \sigma \in \varphi(X \times X)$, then the max-$L$ composition of $\rho$ and $\sigma$ is defined as follows

$$(\rho \circ \sigma)(x, y) = \max_{z \in X} \{ \rho(x, z) + \sigma(z, y) - 1, 0 \}$$

for all $x, y \in X$.

Note. If $R$ and $S$ are crisp relations then, the composition of $R$ and $S$ can be defined as

$$(R \circ S)(x, y) = \{ (x, y) \in X \times X / \exists z \in X, xRz, zSy \}$$

for each $x, y \in X$. The operation $L$ is associative; its zero elements are the zero relation and the unity is identity relation $I$. If $R \circ R \circ \ldots \circ R$ (m times) then it denoted by $R^m$.

Definition 2.2.9[76] A lattice is an algebraic system $(L, \land, \lor)$ with two binary operations $\land$ and $\lor$ on a non empty set $L$ where $\land$ and $\lor$ satisfy idempotent, commutative, associative and the absorption laws.

Example 2.2.10[5] The algebraic system $(\varphi(X), \land, \lor)$ is a lattice under Zadeh’s inclusion $\mu_1 \subseteq \mu_2 \iff \mu_1(x) \leq \mu_2(x)$ for all $x \in X$.

Definition 2.2.11[76] Let $(L, \land, \lor)$ be lattice then the non empty subset $S$ of the lattice $L$ is said to be sub lattice if it is closed under the operations $\land$ and $\lor$ of $L$ that is, if $(a \land b) \in S$ and $(a \lor b) \in S$ for all $a, b \in S$. 
2.3 Logical Operators in Fuzzy Graphs

In this section, we find some properties of logical operators and composition \( \mathcal{L} \) in fuzzy graphs with examples.

**Definition 2.3.1.** The logical operators \( \overline{L} \) and \( L \) are defined as follows; Let \( \mu_1 \) and \( \mu_2 \) be any two elements of \( \varphi(X) \), then for all \( x \in X \),

(i) \( (\mu_1 \overline{L} \mu_2)(x) = \max \{ \mu_1(x) + \mu_2(x) - 1, \ 0 \} \)

(ii) \( (\mu_1 L \mu_2)(x) = \min \{ \mu_1(x) + \mu_2(x), \ 1 \} \)

(iii) \( \overline{\mu_1}(x) = \overline{\mu_1(x)} = 1 - \mu_1(x). \)
Definition 2.3.2. Let $\mu$ be fuzzy subset and $\rho$ a fuzzy relation on a non empty finite set $X$ and the composition $L$, then the composition of $\mu$ and $\rho$, $(\mu L \rho)$ is defined as for each $x \in X$, $(\mu L \rho)(x) = \max \{ \mu(x) + \rho(x, y) - 1, 0 \}$.

Note. In fuzzy graph $G = (\mu, \rho)$, the composition of $\mu$ and $\rho$ can be defined as for each $a \in X$, $(\mu(a) L \rho(a, b)) = \max \{ \mu(a) + \rho(a, b) - 1, 0 \}$.

Proposition 2.3.3. Let $\rho, \sigma \in \varphi(X \times X)$ and the fuzzy subsets $\mu_1, \mu_2 \in \varphi(X)$, under the composition $L$ the following axioms are valid

(i) $(\Theta L \rho) = \Theta$

(ii) $\mu_1 \subseteq \mu_2 \Rightarrow (\mu_1 L \rho) \subseteq (\mu_2 L \rho)$

(iii) $\rho \subseteq \sigma \Rightarrow (\mu_1 L \rho) \subseteq (\mu_2 L \sigma)$

(iv) $\mu_1 \mu_2 (\rho L \sigma) = (\mu_1 L \rho) (\mu_2 L \sigma)$

(v) $(\mu_1 \mu_2) L \rho \subseteq (\mu_1 L \rho) L (\mu_2 L \rho)$

(vi) $(\mu_1 \mu_2) L \rho \subseteq (\mu_1 L \rho) L (\mu_2 L \rho)$

(vii) $\mu_1 (\rho L \sigma) = (\mu_1 L \rho) L (\mu_1 L \sigma)$

(viii) $\mu_1 (\rho L \sigma) \subseteq (\mu_1 L \rho) L (\mu_1 L \sigma)$

Proof. (i) By the definition 2.3.2, $(\Theta L \rho)(x) = \max \{ 0 + \rho(x, y) - 1, 0 \}$, (since $0 \leq \rho(x, y) \leq 1$ for all $x \in X$)

(ii) We know that, $\mu_1 \subseteq \mu_2 \Rightarrow \mu_1(x) \leq \mu_2(x)$ for all $x \in X$. Then, there exist $y \in X$, for any relation $\rho(x, y)$ such that $\mu_1(x) \geq \rho(x, y)$ and $\mu_2(x) \geq \rho(x, y)$

$\Rightarrow [\mu_1(x) + \rho(x, y)] \leq [\mu_2(x) + \rho(x, y)]$

$\Rightarrow [\mu_1(x) + \rho(x, y) - 1] \leq [\mu_2(x) + \rho(x, y) - 1]$

$\Rightarrow \max \{ \mu_1(x) + \rho(x, y) - 1 \} \leq \max \{ \mu_2(x) + \rho(x, y) - 1, 0 \}$

$\Rightarrow (\mu_1 L \rho)(x) \leq (\mu_2 L \rho)(x) \Rightarrow (\mu_1 L \rho) \subseteq (\mu_2 L \rho)$
Therefore, we have \( \mu_1 \subseteq \mu_2 \Rightarrow (\mu_1 \ll L \rho) \subseteq (\mu_2 \ll L \rho) \).

(iii) We know that, \( \rho \subseteq \sigma \Leftrightarrow \rho(x, y) \leq \sigma(x, y) \) for all \( x, y \in X \)

\[ \Rightarrow [ \mu_1(x) + \rho(x, y) ] \leq [ \mu_2(x) + \sigma(x, y) ] , \] since \( \mu_1(x) \geq \rho(x, y) \) and \( \mu_2(x) \geq \sigma(x, y) \)

\[ \Rightarrow [ \mu_1(x) + \rho(x, y) - 1 ] \leq [ \mu_2(x) + \sigma(x, y) - 1 ] \]

\[ \Rightarrow \max \max \{ \mu_1(x) + \rho(x, y) - 1, 0 \} \leq \max \max \{ \mu_2(x) + \sigma(x, y) - 1, 0 \} \]

\[ \Rightarrow (\mu_1 \ll L \rho)(x) \leq (\mu_2 \ll L \sigma)(x) \Rightarrow (\mu_1 \ll L \rho) \subseteq (\mu_2 \ll L \sigma) \]

Thus, we get \( \rho \subseteq \sigma \Rightarrow (\mu_1 \ll L \rho) \subseteq (\mu_2 \ll L \sigma) \).

(iv) Since the composition \( \ll L \) is associative, we have \( \mu, \ll L (\rho \ll L \sigma) = (\mu \ll L \rho) \ll L \sigma \).

By the definitions of \( \ll L \), logical operators \( \ll L \) and \( \ll \), we have (v), (vi), (vii) and (viii). ■

**Example 2.3.4.** Let \( G = (\mu, \rho) \) be a fuzzy graph where \( X = \{ a, b, c, d, e \} \),

\( \mu : X \rightarrow [0, 1] \) and \( \rho : X \times X \rightarrow [0, 1] \) with \( \mu(a) = 0.6, \mu(b) = 0.8, \mu(c) = 0.7, \)

\( \mu(d) = 0.9, \mu(e) = 0.5, \rho(a, b) = 0.4, \rho(b, c) = 0.7, \rho(c, d) = 0.5, \rho(b, d) = 0.6, \)

\( \rho(a, d) = 0.5, \rho(e, d) = 0.5 \) and \( \rho(a, e) = 0.3 \), defined as shown in the Figure 2.1.

![Figure 2.1: Logical operators in fuzzy graph](image)

(i) \( \Theta \ll L \rho = \Theta \) is trivial, see in page 21
(ii) If $\mu(a) \leq \mu(b) \Rightarrow 0.6 \leq 0.8$, now $\rho(a, b) = 0.4$

Then, $\overline{\mu(a)} \rho(a, b) = \max \max \{ \mu(a) + \rho(a, b) - 1, 0 \}$

$$= \max \max \{0.6 + 0.4 - 1, 0\} = 0$$ (2.1)

$\overline{\rho(b)} \rho(a, b) = \max \max \{ \mu(b) + \rho(a, b) - 1, 0 \}$

$$= \max \max \{0.8 + 0.4 - 1, 0\} = 0.2$$ (2.2)

From (2.1) and (2.2), we get $(\mu(a) \overline{\rho(a, b)}) \leq (\mu(b) \overline{\rho(a, b)})$.

Similarly, to show this result is satisfied for all the vertices in given fuzzy graph.

(iii) If $\rho(a, b) = 0.4$, $\rho(b, c) = \sigma(b, c) = 0.7$, $\mu(a) = 0.6$ and $\mu(b) = 0.8$

Now, $\rho(a, b) \leq \sigma(b, c)$

Then, we have $\overline{\mu(a)} \rho(a, b) = \max \max \{ \mu(a) + \rho(a, b) - 1, 0 \}$

$$= \max \max \{0.6 + 0.4 - 1, 0\} = 0$$ (2.3)

$\overline{\sigma(b, c)} \rho(a, b) = \max \max \{ \mu(b) + \sigma(b, c) - 1, 0 \}$

$$= \max \max \{0.8 + 0.7 - 1, 0\} = 0.5$$ (2.4)

From (2.3) and (2.4), we have $(\mu(a) \overline{\rho(a, b)}) \leq (\mu(b) \overline{\sigma(b, c)})$.

Similarly, to show this result is satisfied for all the edges in given fuzzy graph.

(iv) If $\mu(a) = 0.6$, $\rho(a, e) = 0.3$ and $\rho(a, d) = \sigma(a, d) = 0.5$

Then, $\overline{\mu(a)} \rho(a, e) = \max \max \{ \mu(a) + \rho(a, e) - 1, 0 \}$

$$= \max \max \{0.6 + 0.5 - 1, 0\} = 0.1$$

$\overline{\sigma(a, d)} \rho(a, e) = \max \max \{ \rho(a, e) + \sigma(a, d) - 1, 0 \}$

$$= \max \max \{0.3 + 0.5 - 1, 0\} = 0$$

$\overline{\rho(a, e)} \sigma(a, d) = \max \max \{0.6 + 0 - 1, 0\} = 0$

$\overline{\overline{\mu(a)}} \rho(a, e) \overline{\sigma(a, d)} = \max \max \{0.1 + 0.5 - 1, 0\} = 0$
Thus, we have \((\mu(a) \lceil (\rho(a, e) \lceil \sigma(a, d))) = ((\mu(a) \lceil \rho(a, e)) \lceil \sigma(a, d))\)

Similarly, to show this result is satisfied for all the vertices and edges in given fuzzy graph.

(v) If \(\mu(b) = 0.8, \mu(c) = 0.7\) and \(\rho(c, b) = 0.7\)

Then, \((\mu(b) \lceil \mu(c)) = \max \{ \mu(b) + \mu(c) - 1, 0 \} = \max \{ 0.8 + 0.7 - 1, 0 \} = 0.5\)

\[(\mu(b) \lceil \rho(c, b)) = \max \{ 0.8 + 0.7 - 1, 0 \} = 0.5\] (2.5)

\[(\mu(c) \lceil \rho(c, b)) = \max \{ 0.7 + 0.7 - 1, 0 \} = 0.4\]

\[(\mu(b) \lceil \rho(c, b)) \lceil (\mu(c) \lceil \rho(c, b)) = \max \{ 0.5 + 0.4 - 1, 0 \} = 0\] (2.6)

From (2.5) and (2.6), we get

\[(\mu(b) \lceil \mu(c)) \lceil \rho(c, b)) = ((\mu(b) \lceil \rho(c, b)) \lceil (\mu(c) \lceil \rho(c, b)))\]

Similarly, to show this result is satisfied for remaining vertices in given fuzzy graph.

(vi) If \(\mu(b) = 0.8, \mu(c) = 0.7\) and \(\rho(c, b) = 0.7\)

\[(\mu(b) \lceil \mu(c)) = \min \{ \mu(b) + \mu(c) - 1, 0 \} = \min \{ 0.8 + 0.7, 1 \} = 1\]

\[(\mu(b) \lceil \rho(c, b)) = \max \{ 1 + 0.7 - 1, 0 \} = 0.7\] (2.7)

\[(\mu(c) \lceil \rho(c, b)) = \max \{ 0.8 + 0.7 - 1, 0 \} = 0.5\]

\[(\mu(b) \lceil \rho(c, b)) \lceil (\mu(c) \lceil \rho(c, b)) = \min \{ 0.5 + 0.4, 1 \} = 0.9\] (2.8)

From (2.7) and (2.8), we have

\[(\mu(b) \lceil \mu(c)) \lceil \rho(c, b)) \leq ((\mu(b) \lceil \rho(c, b)) \lceil (\mu(c) \lceil \rho(c, b)))\]

Similarly, to show this result is satisfied for remaining vertices in given fuzzy graph.
(vii) If $\mu(d)=0.9$, $\rho(e, d)=0.5$ and $\rho(d, b)=\sigma(d, b)=0.6$

Then, we get $(\mu(d)L(\rho(e, d)\widetilde{L}\sigma(d, b)))=0$

$(\mu(d)L\rho(e, d))=0.4$, $(\mu(d)L\sigma(d, b))=0.5$

$(\mu(d)L\rho(e, d)\widetilde{L}(\mu(d)L\sigma(d, b)))=0$

Therefore, $(\mu(d)L(\rho(e, d)\widetilde{L}\sigma(d, b)))=(\mu(d)L\rho(e, d))\widetilde{L}(\mu(d)L\sigma(d, b)))$.

Similarly, to show this result is satisfied for all the vertices and edges in given fuzzy graph.

(viii) If $\mu(d)=0.9$, $\rho(e, d)=0.5$ and $\rho(d, b)=\sigma(d, b)=0.6$

$(\rho(e, d)L\sigma(d, b))=\min\{0.5+0.6, 1\}=1$

$((\mu(d)L(\rho(e, d)L\sigma(d, b)))=\max\max\{0.9+1, 0\}=0.9$ (2.9)

$(\mu(d)L\rho(e, d))=0.4$, $(\mu(d)L\sigma(d, b))=0.5$

$(\mu(d)L\rho(e, d)\widetilde{L}(\mu(d)L\sigma(d, b)))=\min\{0.5+0.4, 1\}=0.9$ (2.10)

From (2.9) and (2.10), we have

$((\mu(d)L(\rho(e, d)L\sigma(d, b)))=(\mu(d)L\rho(e, d))\widetilde{L}(\mu(d)L\sigma(d, b)))$

But, from $\mu(a)=0.6$, $\rho(a, e)=0.3$ and $\rho(a, d)=\sigma(a, d)=0.5$

$(\rho(a, e)L\sigma(a, d))=\min\{0.3+0.5, 1\}=0.8$

$((\mu(a)L(\rho(a, e)L\sigma(a, d)))=\max\max\{0.6+0.8, 0\}=0.4$ (2.11)

$(\mu(a)L\rho(a, e))=0$, $(\mu(a)L\sigma(a, d))=0.1$

$(((\mu(a)L\rho(a, e))L(\mu(a)L\sigma(a, d)))=0.1$ (2.12)

From (2.11) and (2.12), we have

$((\mu(d)L(\rho(e, d)L\sigma(d, b)))\geq(\mu(d)L\rho(e, d))\widetilde{L}(\mu(d)L\sigma(d, b)))$.

Similarly, to show this result is satisfied for all the vertices and edges in given fuzzy graph.
Definition 2.3.5. Let $W$ be a non-empty collection of fuzzy subsets of $X$. The triplet $(W, \overline{L}, L)$, is called a weak lattice, if the following axioms hold:

(i) **Commutative laws**

\[
(\mu_1 L \mu_2) = (\mu_2 L \mu_1) \quad (\mu_1 \overline{L} \mu_2) \overline{L} \mu_3 = \mu_1 \overline{L} (\mu_2 \overline{L} \mu_3)
\]

(ii) **Associative laws**

\[
(\mu_1 L \mu_2) L \mu_3 = \mu_1 L (\mu_2 L \mu_3) \text{ for all } \mu_1, \mu_2 \text{ and } \mu_3 \in W.
\]

Remark 2.3.6. An algebraic system $(\wp(X), \overline{L}, L)$ is a weak lattice under Zadeh’s inclusion $\mu_1 \subseteq \mu_2 \iff (\forall x, (\mu_1(x) \leq \mu_2(x)))$. Let $G = (\mu, \rho)$ be a fuzzy graph, then $(\mu (X), \overline{L}, L)$ is a weak lattice under the condition for each $a, b \in X \Rightarrow \mu(a) \leq \mu(b)$ and the composition $L$.

Example 2.3.7. Let $\mu_1, \mu_2$ and $\mu_3$ be any fuzzy subsets of $\wp(X)$ where $\mu_1(x) = 0.4$, $\mu_2(x) = 0.7$ and $\mu_3(x) = 0.6$ for each $x \in X$ then, we have

(1) **Idempotent laws**

(i) $(\mu_1 \overline{L} \mu_1) = 0 \neq \mu_1$

(ii) $(\mu_1 L \mu_1) = 0.8 \neq \mu_1$, therefore, idempotent laws are not satisfied.

(2) **Commutative laws**

(i) $(\mu_1 \overline{L} \mu_2) = 0.1 = (\mu_2 \overline{L} \mu_1)$

(ii) $(\mu_1 L \mu_2) = 1 = (\mu_2 L \mu_1)$, therefore, commutative laws are satisfied.

(3) **Associative laws**

(i) $(\mu_1 \overline{L} \mu_2) \overline{L} \mu_3 = 0 = \mu_1 \overline{L} (\mu_2 \overline{L} \mu_3)$

(ii) $(\mu_1 L \mu_2) L \mu_3 = 1 = \mu_1 L (\mu_2 L \mu_3)$, therefore, associative laws are satisfied.
(4) Absorption laws

(i) \( \mu_1 \overline{L}(\mu_1 \overline{L} \mu_2) = 0.4 \neq \mu_1 \)

(ii) \( \mu_1 \overline{L}(\mu_1 \overline{L} \mu_2) = 0.5 \neq \mu_1 \)

If \( \mu_1(x) = 0.2 \), \( \mu_2(x) = 0.6 \), then we have \( \mu_1 \overline{L}(\mu_1 \overline{L} \mu_2) = 0 \neq \mu_1 \).

Therefore, absorption laws are not satisfied. Hence, \((\varphi(X), \overline{L}, \overline{L})\) is a weak lattice.

**Note.** In any algebraic system by using of the logical operators \( \overline{L} \) and \( L \), we have weak lattice conditions only, there is no strong lattice axioms.

**Definition 2.3.8.** Let \( \phi \neq S \subseteq W \subseteq \varphi(X) \). Then \((S, \overline{L}, \overline{L})\) is called a sub weak lattice of a weak lattice \((W, \overline{L}, \overline{L})\), if the following hold: \( (\mu_1 \overline{L} \mu_2) \in S \) and \( (\mu_1 \overline{L} \mu_2) \in S \) for all \( \mu_1, \mu_2 \in S \).

### 2.4 Not External Domination Set Ned \((\rho, \overline{L})\)

**Definition 2.4.1.** Let \( G = (\mu, \rho) \) be a fuzzy graph without loops and with underlying set \( X \) where \( \mu : X \rightarrow [0, 1], \rho : X \times X \rightarrow [0, 1] \), vertex \( a \in X \) is not externally dominated under the composition \( \overline{L} \) if \( ((\mu(a) \overline{L} \rho(a, b)) \leq \mu(a)), ((\mu(a) \overline{L} \rho^{-1}(a, b)) \leq \mu(a)) \) for some \( b \in X \). We denote it by Ned \((\rho, \overline{L})\).

**Example 2.4.2.** Let \( G = (\mu, \rho) \) be a fuzzy graph where \( X = \{a, b, c, d\} \), \( \mu : X \rightarrow [0, 1] \), \( \rho : X \times X \rightarrow [0, 1] \) with \( \mu(a) = 0.4, \mu(b) = 0.6, \mu(c) = 0.8, \mu(d) = 0.5, \rho(a, b) = 0.3, \rho(b, c) = 0.5, \rho(c, d) = 0.5, \rho(d, a) = 0.2 \) and \( \rho(d, b) = 0.4 \), defined as shown in the Figure 2.2.
The edge $ab$, we get $(\mu(a) \ L \ \rho(a, b)) = \max \max \{0.6 + 0.3 - 1, 0\} = 0 \leq 0.6 = \mu(a)$

Similarly, $(\mu(a) \ L \ \rho^{-1}(a, b)) = \max \max \{0.4 + 0.3 - 1, 0\} = 0 \leq 0.4 = \mu(a)$. Thus, $a$ is not external domination under the composition $L$ and $a \in \text{Ned}(\rho, L)$. The edge $bc$, we get $(\mu(b) \ L \ \rho(b, c)) = \max \max \{0.4 + 0.5 - 1, 0\} = 0 \leq 0.4 = \mu(b)$. Similarly, $(\mu(b) \ L \ \rho^{-1}(b, c)) = \max \max \{0.6 + 0.5 - 1, 0\} = 0.1 \leq 0.6 = \mu(b)$. Thus, $b$ is not external domination under the composition $L$ and $b \in \text{Ned}(\rho, L)$. The edge $cd$, we get $(\mu(c) \ L \ \rho(c, d)) = \max \max \{0.2 + 0.5 - 1, 0\} = 0 \leq 0.2 = \mu(c)$ and $(\mu(c) \ L \ \rho^{-1}(c, d)) = \max \max \{0.8 + 0.5 - 1, 0\} = 0.3 \leq 0.8 = \mu(c)$. Hence, we get $c$ is not external domination under the composition $L$ and $c \in \text{Ned}(\rho, L)$. The edge $ad$, we get $(\mu(d) \ L \ \rho(d, a)) = \max \max \{0.5 + 0.2 - 1, 0\} = 0 \leq 0.5 = \mu(d)$. Similarly, $(\mu(d) \ L \ \rho^{-1}(d, a)) = \max \max \{0.5 + 0.2 - 1, 0\} = 0 \leq 0.5 = \mu(d)$. Thus, we get $d$ is said to be not external domination under $L$ and $d \in \text{Ned}(\rho, L)$. The edge $db$, we get $(\mu(d) \ L \ \rho(d, b)) = \max \max \{0.5 + 0.4 - 1, 0\} = 0 \leq 0.5 = \mu(d)$. Similarly, we have $(\mu(d) \ L \ \rho(d, b)) = \max \max \{0.5 + 0.4 - 1, 0\} = 0 \leq 0.5 = \mu(d)$. Thus, we say that $d$ is not external domination under the composition $L$ and $d \in \text{Ned}(\rho, L)$. We use
similar method to the edges $ba, cb, dc, ad$ and $bd$ we get, $b, c, d, a$ and $b$ are not external domination under the composition $L$ and the vertices $a, b, c$ and $d$ are in $\text{Ned}(\rho, L)$.

**Example 2.4.3.** Let $G = (\mu, \rho)$ be a fuzzy graph where $X = \{a, b, c\}$, $\mu : X \rightarrow [0, 1]$ and $\rho : X \times X \rightarrow [0, 1]$ with $\mu(a) = 0.9, \mu(b) = 1.0, \mu(c) = 0.7, \rho(a, b) = 0.9, \rho(b, c) = 0.7$ and $\rho(c, a) = 0.7$ defined as shown in the Figure 2.3.

![Figure 2.3: Ned (\rho, L) fuzzy graph](image)

The edge $ab$, we get $L(\mu(a), a, b) = \max\{0.1 + 0.5 - 1, 0\} = 0 \leq 0.1 = \mu(a)$

Similarly, $(\mu(a) L (a, b)^{-1}) = \max\{0.9 + 0.9 - 1, 0\} = 0.8 \leq 0.9 = \mu(a)$. Thus, $a$ is not external domination under the composition $L$ and $a \in \text{Ned}(\rho, L)$. The edge $bc$, we get, $L(\mu(b), b, c) = \max\{0 + 0.7 - 1, 0\} = 0 \leq 0 = \mu(b)$. Similarly, $(\mu(b) L (b, c)^{-1}) = \max\{1 + 0.7 - 1, 0\} = 0.7 \leq 1 = \mu(b)$. Thus, $b$ is not external domination under the composition $L$ and $b \in \text{Ned}(\rho, L)$. From the edge $ca$, we have, $(\mu(c) L (c, a) = \max\{0.3 + 0.7 - 1, 0\} = 0 \leq 0.3 = \mu(c)$ and $(\mu(c) L (c, a)^{-1}) = \max\{0.7 + 0.7 - 1, 0\} = 0.4 \leq 0.7 = \mu(c)$. Hence, we have $c$ is not external domination under the composition $L$ and $c \in \text{Ned}(\rho, L)$. Similar
method to apply the edges $ba, ca,$ and $ac$, we have $b, c$ and $a$ are not external domination under the composition $\overline{L}$ and the vertices $a, b, c \in \text{Ned}(\rho, \overline{L})$.

**Proposition 2.4.4.** Let $G = (\mu, \rho)$ be a fuzzy graph without loops and with underlying set $X$ where $\mu : X \rightarrow [0, 1]$ and $\rho : X \times X \rightarrow [0, 1]$, the following are satisfied.

(i) The set $\text{Ned}(\rho, \overline{L})$ is a sub weak lattice of the weak lattice $(\mu(X), \overline{L}, \underline{L})$

(ii) $\text{Ned}(\rho, \overline{L})$ contains any constant $k \cdot 1$ of the set $\varphi(X)$.

**Proof.** (i) Let $a, b \in \text{Ned}(\rho, \overline{L})$.

Then, we have $((\mu(a) \overline{L} \rho^{-1}(a, b)) \leq \mu(a))$ and $((\mu(b) \overline{L} \rho^{-1}(a, b)) \leq \mu(b))$ (2.13)

$((\overline{\mu}(a) \overline{L} \rho(a, b)) \leq \overline{\mu(a)})$ and $((\overline{\mu}(b) \overline{L} \rho(b, a)) \leq \overline{\mu(b)})$ (2.14)

To prove: $((\mu(a) \overline{L} \mu(b)) \overline{L} \rho^{-1}(a, b)) \leq (\mu(a) \overline{L} \mu(b))$

Now $(\mu(a) \overline{L} \mu(b)) \overline{L} \rho^{-1}(a, b)) \leq (\mu(a) \overline{L} \rho^{-1}(a, b)) \overline{L} (\mu(b) \overline{L} \rho^{-1}(a, b))$

$$\leq (\mu(a) \overline{L} \mu(b)) \quad \text{(since (2.13))}$$

Similarly, to prove $((\mu(a) \overline{L} \mu(b)) \overline{L} \rho(a, b)) \leq (\mu(a) \overline{L} \mu(b))$ (since (2.14))

Same method to use the operator $\overline{L}$, then, we have the following results

$((\mu(a) \overline{L} \mu(b)) \overline{L} \rho(a, b)) \leq (\mu(a) \overline{L} \mu(b))$, $((\mu(a) \overline{L} \mu(b)) \overline{L} \rho^{-1}(a, b)) \leq (\mu(a) \overline{L} \mu(b))$

Hence, $\text{Ned}(\rho, \overline{L})$ is a sub weak lattice of the weak lattice $(\mu(X), \overline{L}, \underline{L})$.

(ii) Let $k \in [0, 1]$ for all $x \in X$. Then, there exist $y \in X$ such that

$$((k \cdot 1) \overline{L} \rho^{-1}(x, y)) = \max \{ k + \rho^{-1}(x, y) - 1, 0 \}, \text{ since } 0 \leq \rho(x, y) \leq 1 \quad (2.15)$$

$$((\bar{k} \cdot 1) \overline{L} \rho(x, y)) = \max \{ \bar{k} + \rho(x, y) - 1, 0 \} \leq \bar{k} \quad (2.16)$$

From (2.15) and (2.16), we get the condition for an element in $\text{Ned}(\rho, \overline{L})$

Hence, (ii) is proved. ■
Example 2.4.5. Let $G = (\mu, \rho)$ be a fuzzy graph where $X = \{a, b, c, d\}$, $\mu : X \rightarrow [0, 1]$, $\rho : X \times X \rightarrow [0, 1]$ with $\mu(a) = 0.5$, $\mu(b) = 0.6$, $\mu(c) = 0.7$, $\mu(d) = 1.0$, $\rho(a, b) = 0.4$, $\rho(a, c) = 0.3$, $\rho(b, d) = 0.5$, and $\rho(c, d) = 0.6$ defined as shown in the Figure 2.4.

![Figure 2.4: Ned (\rho, L) sub weak lattice fuzzy graph](image)

Consider the edge $ab$ and $a, b \in \text{Ned}(\rho, L)$, we get

$(\mu(a)\overline{L}\rho^{-1}(a, b)) = 0 \leq 0.5 = \mu(a)$ and $(\mu(b)\overline{L}\rho^{-1}(a, b)) = 0 \leq 0.6 = \mu(b)$

Now, $(\mu(a)\overline{L}\mu(b))\overline{L}\rho^{-1}(a, b)) = 0 \leq 0.1 = (\mu(a)\overline{L}\mu(b))$. Therefore, we have $(\mu(a)\overline{L}\mu(b))\overline{L}\rho^{-1}(a, b)) \leq (\mu(a)\overline{L}\mu(b))$. Similarly, we show that $(\mu(a)\overline{L}\mu(b))\overline{L}\rho(a, b)) \leq (\mu(a)\overline{L}\mu(b))$ and $((\mu(a)\overline{L}\mu(b))\overline{L}\rho^{-1}(a, b)) \leq (\mu(a)\overline{L}\mu(b))$. We use similar method and find solution to the remaining edges $ba, ac, ca, cd, dc, bd$ and $db$. Thus, we have Ned $(\rho, L)$ is sub weak lattice of $(\mu(X), \overline{L}, L)$. Hence, (i) of proposition 2.4.4. is satisfied.

Example 2.4.6. If $k \in [0, 1]$ and any relation $\rho$ of a non empty set $X$ such that $0 \leq \rho(x, y) \leq 1$ for all $x, y \in X$, consider $k = 0.6$ and $\rho(x, y) = 0.6$. Then, we have

$((k \cdot 1)\overline{L}\rho^{-1}(x, y)) = \max \{0.6 + 0.6 - 1, 0\} = 0.2 \leq 0.6 = k \quad (2.17)$
From (2.17) and (2.18), we have the conditions (2.15) and (2.16) respectively. Hence, (ii) of proposition 2.4.4. is satisfied.

**Proposition 2.4.7.** Let $G = (\mu, \rho)$ be a fuzzy graph without loops and with underlying set $X$ where $\mu : X \rightarrow [0, 1]$ and $\rho : X \times X \rightarrow [0, 1]$. If $\mu(a) + \mu(b) \leq 1$ for all $a, b \in X$. Then, $a, b \in \text{Ned} \left( \rho, \overline{L} \right)$. Converse is not true.

**Proof.** If $\mu(a) + \mu(b) \leq 1$, for all $a, b \in X$, then by the definition of $\text{Ned} \left( \rho, \overline{L} \right)$ and $(\mu(a) \overline{L} \rho(a, b))$ and $(\mu(a) \overline{L} \rho^{-1}(a, b))$ are always 0. Thus, we have the following results

(i) $(\mu(a) \overline{L} \rho(a, b)) \leq \mu(a))$ and $(\mu(a) \overline{L} \rho^{-1}(a, b)) \leq \mu(a))$

(ii) $(\mu(b) \overline{L} \rho(a, b)) \leq \mu(b))$ and $(\mu(b) \overline{L} \rho^{-1}(a, b)) \leq \mu(b))$.

Hence, we get $a, b \in \text{Ned} \left( \rho, \overline{L} \right)$. ■

**Example 2.4.8.** Let $G = (\mu, \rho)$ be a fuzzy graph where $X = \{a, b, c, d\}$, $\mu : X \rightarrow [0, 1]$, $\rho : X \times X \rightarrow [0, 1]$ with $\mu(a) = 0.6$, $\mu(b) = 0.3$, $\mu(c) = 0.5$, $\mu(d) = 0.4$, $\rho(a, b) = 0.1$, $\rho(b, c) = 0.3$, $\rho(c, d) = 0.2$, $\rho(d, a) = 0.4$ and $\rho(d, b) = 0.3$, defined as shown in the Figure 2.5.

![Figure 2.5: \(\mu(a) + \mu(b) \leq 1\), fuzzy graph](image-url)
Consider the edge $ab$, we have $(\overline{\mu(a) L^\mu a b}) = \max \{0.4 + 0.1 - 1, 0\} = 0$, and $\overline{\mu(a)} = 0.4$. Thus, we get $(\overline{\mu(a) L^\mu a b}) \leq \overline{\mu(a)}$. Similarly, we have $(\overline{\mu(a) L^\mu a b^-1 a b}) \leq \overline{\mu(a)}$. Therefore, $a$ is not external domination under the composition $L$ and $a \in \text{Ned}(\bar{\rho}, \bar{L})$. By considering remaining edges $bc, cd, da$ and $db$ as well as $ba, cb, dc, ad$ and $bd$, we get the vertices $a, b, c$ and $d$ are not external domination under $L$ and $a, b, c, d \in \text{Ned}(\bar{\rho}, L)$.

Converse is not true

**Example 2.4.9.** Let $G = (\mu, \rho)$ be a fuzzy graph where $X = \{a, b, c, d\}$, $\mu : X \to [0, 1]$, $\rho : X \times X \to [0, 1]$ with $\mu(a) = 0.6$, $\mu(b) = 0.5$, $\mu(c) = 0.8$, $\mu(d) = 0.7$, $\rho(a, b) = 0.5$, $\rho(b, c) = 0.4$, $\rho(c, d) = 0.6$ and $\rho(d, a) = 0.4$ defined as shown in the Figure 2.6.

Consider the edge $ab$, we have $(\overline{\mu(a) L^\mu a b}) = \max \{0.4 + 0.5 - 1, 0\} = 0$, and $\overline{\mu(a)} = 0.4$. Therefore, we get $(\overline{\mu(a) L^\mu a b}) \leq \overline{\mu(a)}$. (2.19)

But, $(\overline{\mu(a) L^\mu a b^-1 a b}) = \max \{0.6 + 0.5 - 1, 0\} = 0.1$ and $\mu(a) = 0.6$.

Therefore, $(\overline{\mu(a) L^\mu a b^-1 a b}) \leq \mu(a))$. (2.20)
From (2.19) and (2.20), we have $a$ is not external domination under the composition $\mathbb{L}$ and $a \in \text{Ned}(\rho, \mathbb{L})$. Similarly, all the edges of Figure 2.6, we get the vertices $a, b, c$ and $d$ are not external domination under $\mathbb{L}$ and $a, b, c, d \in \text{Ned}(\rho, \mathbb{L})$. But, we have $\mu(a) + \mu(b) > 1$ for all $a, b \in X$.

2.5 Internally Stable Vertex Set Int $(\rho, \mathbb{L})$

**Definition 2.5.1.** Let $G = (\mu, \rho)$ be a fuzzy graph without loops and with underlying set $X$ where $\mu: X \to [0, 1]$, $\rho: X \times X \to [0, 1]$, vertex $a \in X$ is internally stable under the composition $\mathbb{L} \iff ((\mu(a) \mathbb{L} \rho(a, b)) \leq \overline{\mu(a)})$ and $((\mu(a) \mathbb{L} \rho^{-1}(a, b)) \leq \overline{\mu(a)})$ for some $b \in X$. We denote it by Int $(\rho, \mathbb{L})$.

**Example 2.5.2.** Let $G = (\mu, \rho)$ be a fuzzy graph where $X = \{a, b, c, d\}$, $\mu: X \to [0, 1]$, $\rho: X \times X \to [0, 1]$ with $\mu(a) = 0.7$, $\mu(b) = 0.5$, $\mu(c) = 0.6$, $\mu(d) = 0.8$, $\rho(a, b) = 0.4$, $\rho(b, c) = 0.5$, $\rho(c, d) = 0.3$, $\rho(d, a) = 0.4$, $\rho(d, b) = 0.3$ and $\rho(c, a) = 0.6$ defined as shown in the Figure 2.7.

![Figure 2.7: Int $(\rho, \mathbb{L})$ fuzzy graph](image-url)
The edge $ab$, $((\mu(a) L \rho(a, b)) = \max \{ 0.7 + 0.4 - 1, 0 \} = 0.1 \leq 0.3 = \mu(a)$
and $((\mu(a) L \rho^{-1}(a, b)) = \max \{ 0.7 + 0.4 - 1, 0 \} = 0.1 \leq 0.3 = \mu(a)$. Thus, we say that vertex $a$ is internally stable under the composition $L$ and $a \in \text{Int}(\rho, L)$.

The edge $bc$, $((\mu(b) L \rho(b, c)) = \max \{ 0.5 + 0.5 - 1, 0 \} = 0 \leq 0.5 = \mu(b)$
and $((\mu(b) L \rho^{-1}(b, c)) = \max \{ 0.5 + 0.5 - 1, 0 \} = 0 \leq 0.5 = \mu(b)$. Thus, we say that vertex $b$ is internally stable under the composition $L$ and $b \in \text{Int}(\rho, L)$.

The edge $cd$, $((\mu(c) L \rho(c, d)) = \max \{ 0.6 + 0.3 - 1, 0 \} = 0 \leq 0.4 = \mu(c)$
and $((\mu(c) L \rho^{-1}(c, d)) = \max \{ 0.6 + 0.3 - 1, 0 \} = 0 \leq 0.4 = \mu(c)$. Then, we say that vertex $c$ is internally stable under the composition $L$ and $c \in \text{Int}(\rho, L)$.

The edge $da$, $((\mu(d) L \rho(d, a)) = \max \{ 0.8 + 0.4 - 1, 0 \} = 0.2 \leq 0.2 = \mu(d)$
and $((\mu(d) L \rho^{-1}(d, a)) = \max \{ 0.8 + 0.4 - 1, 0 \} = 0.2 \leq 0.2 = \mu(d)$. Then, we say that vertex $d$ is internally stable under the composition $L$ and $d \in \text{Int}(\rho, L)$.

But edge $ad$, $((\mu(a) L \rho(a, d)) = \max \{ 0.7 + 0.4 - 1, 0 \} = 0.1 \leq 0.3 = \mu(a)$
and $((\mu(a) L \rho^{-1}(a, d)) = \max \{ 0.7 + 0.4 - 1, 0 \} = 0.1 \leq 0.3 = \mu(a)$. Then, we say that vertex $a$ is internally stable under the composition $L$ and $a \in \text{Int}(\rho, L)$.

The edge $ca$, $((\mu(c) L \rho(c, a)) = \max \{ 0.6 + 0.6 - 1, 0 \} = 0.2 \leq 0.4 = \mu(c)$
and $((\mu(c) L \rho^{-1}(c, a)) = \max \{ 0.6 + 0.6 - 1, 0 \} = 0.2 \leq 0.4 = \mu(c)$. Then, we say that vertex $c$ is internally stable under the composition $L$ and $c \in \text{Int}(\rho, L)$.

The edge $db$, $((\mu(d) L \rho(d, b)) = \max \{ 0.8 + 0.3 - 1, 0 \} = 0.1 \leq 0.2 = \mu(d)$
and $((\mu(d) L \rho^{-1}(d, b)) = \max \{ 0.8 + 0.3 - 1, 0 \} = 0.1 \leq 0.2 = \mu(d)$. Here, we say that the vertex $d$ is internally stable under the composition $L$ and
Similarly, from the edges $ba, cb, dc, ac$ and $bd$ we show that $b, c, d, a$ and $b$ are internally stable under the composition $L$ and they are in $\text{Int}(\rho, L)$. 

**Proposition 2.5.3.** Let $G = (\mu, \rho)$ be a fuzzy graph without loops and with underlying set $X$ where $\mu : X \to [0, 1]$, $\rho : X \times X \to [0, 1]$ for all $a, b \in X$, the following are satisfied.

(i) $\text{Int}(\rho, L)$ is a $L$-sub weak lattice of the weak lattice $(\mu(X), \overline{L}, L)$

(ii) If $a \in \text{Int}(\rho, L)$ and $\mu(b) \leq \mu(a)$, then $b \in \text{Int}(\rho, L)$.

**Proof.** (i) Let $a, b \in \text{Int}(\rho, L)$.

Then, we have

$$(\mu(a) \overline{L} \rho(a, b)) \leq \overline{\mu(a)}, \quad ((\mu(b) \overline{L} \rho(a, b)) \leq \overline{\mu(b)}),$$

and

$$(\mu(a) \overline{L} \rho^{-1}(a, b)) \leq \overline{\mu(a)}), \quad ((\mu(b) \overline{L} \rho^{-1}(a, b)) \leq \overline{\mu(b)}).$$

**To Prove:** \((\mu(a) \overline{L} \mu(b) \overline{L} \rho(a, b)) \leq \overline{\mu(a) \overline{L} \mu(b)}\)

Now, we have

\[
(\mu(a) \overline{L} \mu(b) \overline{L} \rho(a, b)) \leq (\mu(a) \overline{L} \rho(a, b)) \overline{L} (\mu(b) \overline{L} \rho(a, b)) \\
\leq \overline{\mu(a) \overline{L} \mu(b)} \quad \text{by (2.21)} \\
\leq (\mu(a) \overline{L} \mu(b))
\]

(2.23)

Similarly, we show that \((\mu(a) \overline{L} \mu(b)) \overline{L} \rho^{-1}(a, b) \leq \overline{(\mu(a) \overline{L} \mu(b))}\) (2.24)

From (2.23) and (2.24) $\text{Int}(\rho, L)$ is sub weak lattice of $(\mu(X), \overline{L}, L)$ with respect to the operator $\overline{L}$.

(ii) If $a \in \text{Int}(\rho, L)$.

Then, we get $((\mu(a) \overline{L} \rho(a, b)) \leq \overline{\mu(a)})$ and $((\mu(a) \overline{L} \rho^{-1}(a, b)) \leq \overline{\mu(a)})$

Given that, $\mu(b) \leq \mu(a)$ for all $a, b \in X$
Then, \( ((\mu(b) \land \rho(a, b)) \leq ((\mu(a) \land \rho(a, b)) \leq \mu(a)) \), since (ii) of proposition 2.3.3,

but, we have \( (\mu(b) \land \rho(a, b)) \leq \mu(b) \) \hspace{1cm} (2.25)

Similarly, we show that \( (\mu(b) \land \rho^{-1}(a, b)) \leq ((\mu(a) \land \rho^{-1}(a, b)) \leq \mu(a)) \) and

\( (\mu(b) \land \rho^{-1}(a, b)) \leq \mu(b) \) \hspace{1cm} (2.26)

Hence, from (2.25), (2.26) and by the definition of \( \text{Int}(\rho, L) \), we get \( b \in \text{Int}(\rho, L) \).

**Example 2.5.4.** Let \( G = (\mu, \rho) \) be a fuzzy graph where \( X = \{a, b, c, d, e\}, \mu: X \rightarrow [0, 1] \)

and \( \rho: X \times X \rightarrow [0, 1] \) with \( \mu(a) = 0.7, \mu(b) = 0.4, \mu(c) = 0.6, \mu(d) = 0.9, \mu(e) = 0.8, \)

\( \rho(a, b) = 0.3, \rho(b, c) = 0.3, \rho(b, d) = 0.4, \rho(d, e) = 0.5 \) and \( \rho(c, e) = 0.5 \), defined as

shown in the Figure 2.8.

![Figure 2.8: \( \text{Int}(\rho, L) \) sub weak lattice fuzzy graph](image)

We consider the edge \( ab, ((\mu(a) \land \mu(b)) \land \rho(a, b)) = 0 \), and \( (\mu(a) \land \mu(b)) = 0.9 \)

Thus, we get \( (\mu(a) \land \mu(b)) \land \rho(a, b)) \leq (\mu(a) \land \mu(b)) \). Similarly, we show that

\( (\mu(a) \land \mu(b)) \land \rho^{-1}(a, b)) \leq (\mu(a) \land \mu(b)) \). From the remaining edges of the graph,
we have the same result. Therefore, \( \text{Int} (\rho, \underline{L}) \) is the sub weak lattice of weak lattice \((\mu(X), \overline{L}, \underline{L})\) under \(\overline{L}\).

**Remark 2.5.5.** \( \text{Int} (\rho, \underline{L}) \) is not a \(L\)-sub weak lattice of the weak lattice \((\mu(X), \overline{L}, \underline{L})\) From the example 2.5.4 we consider the logical operator \(L\) and the edge \(bd\) then, we have \(((\mu(b)L\mu(d))\underline{L} \rho(b,d)) = 0.4\) but \((\mu(a)L\mu(b)) = 0\). Thus, the condition \(((\mu(b)L\mu(d))\underline{L} \rho(b,d)) \leq (\mu(a)L\mu(b))\) is not satisfied. Similarly in all the edges, we get the same result. Hence, \( \text{Int} (\rho, \underline{L}) \) is not a \(L\)-sub weak lattice of the weak lattice \((\mu(X), \overline{L}, \underline{L})\).

**Example 2.5.6.** Let \(G = (\mu, \rho)\) be a fuzzy graph where \(X = \{b, c, d, e\}\), \(\mu : X \rightarrow [0, 1]\), \(\rho : X \times X \rightarrow [0, 1]\) with \(\mu(b) = 0.4\), \(\mu(c) = 0.6\), \(\mu(d) = 0.8\), \(\mu(e) = 0.7\), \(\rho(b, c) = 0.3\), \(\rho(b, d) = 0.4\), \(\rho(d, e) = 0.4\) and \(\rho(c, e) = 0.5\), defined as shown in the Figure 2.9.

![Figure 2.9: \(\mu(b) \leq \mu(a)\), fuzzy graph](image)

We consider the edge \(bd\), \(\mu(b) \leq \mu(d)\) then \((\mu(b)\underline{L} \rho(b, d)) = 0\) and \(\overline{\mu(b)} = 0.6\). Next, we have \((\mu(d)\underline{L} \rho(b, d)) = 0.2\) and \(\overline{\mu(d)} = 0.2\). Therefore, we get
(μ(d) ≤ μ(b, d)) ≤ μ(d). From remaining edges of the Figure 2.9 satisfy the condition that the vertices b, c, d and e are in Int (ρ, L).

**Proposition 2.5.7.** Let G = (μ, ρ) be a fuzzy graph without loops and with underlying set X where μ: X → [0, 1] and ρ: X × X → [0, 1]. If μ(a) + μ(b) ≤ 1, for all a, b ∈ X. Then, a, b ∈ Int (ρ, L). Converse is not true.

**Proof.** If μ(a) + μ(b) ≤ 1, for all a, b ∈ X, then by the definition of Int (ρ, L), we get (μ(a) ρ(a, b)) and (μ(a) ρ−1 (a, b)) are always 0. Thus, we have the following results (i) ((μ(a) L ρ(a, b)) ≤ μ(a)) and ((μ(a) L ρ−1 (a, b)) ≤ μ(a))

(ii) ((μ(b) L ρ(a, b)) ≤ μ(b)) and ((μ(b) L ρ−1 (a, b)) ≤ μ(b)).

Hence, we get a, b ∈ Int (ρ, L). ■

**Example 2.5.8.** Let G = (μ, ρ) be a fuzzy graph where X = {a, b, c}, μ: X → [0, 1] and ρ: X × X → [0, 1], μ(a) = 0.6, μ(b) = 0.4, μ(c) = 0.3, ρ(a, b) = 0.3, ρ(b, c) = 0.3 and ρ(c, a) = 0.5, defined as shown in the Figure 2.10.

![Figure 2.10: μ(a) + μ(b) ≤ 1, fuzzy graph](image)

We consider the edge ab, μ(a) + μ(b) ≤ 1 then, ((μ(a) L ρ(a, b)) = 0 and μ(a) = 0.4. Therefore, ((μ(a) L ρ(a, b)) ≤ μ(a)) and ((μ(a) L ρ−1 (a, b)) ≤ μ(a)). Similarly,
we get \(( (\mu(b)\overline{L}\mu(a, b)) \leq \overline{\mu(b)})\) and \(( (\mu(b)\overline{L}\mu^{-1}(a, b)) \leq \overline{\mu(b)})\). Thus, the vertices \(a\) and \(b\) are internally stable under the composition \(\overline{L}\) and \(a, b \in \text{Int}(\rho, \overline{L})\).

Now we consider the edge \(bc\), \(\mu(b) + \mu(c) = 0.7 \leq 1\) then, \(( (\mu(b)\overline{L}\mu(b, c)) = 0\) and \(\overline{\mu(b)} = 0.6\). Thus, \(( (\mu(b)\overline{L}\mu(b, c)) \leq \overline{\mu(b)})\) and \(( (\mu(b)\overline{L}\mu^{-1}(b, c)) \leq \overline{\mu(b)})\).

From vertex \(c\), we have \(( (\mu(c)\overline{L}\mu(b, c)) \leq \overline{\mu(c)})\) and \(( (\mu(c)\overline{L}\mu^{-1}(b, c)) \leq \overline{\mu(c)})\).

Therefore, the vertices \(b\) and \(c\) are internally stable under \(\overline{L}\) and \(b, c \in \text{Int}(\rho, \overline{L})\).

Finally, consider the edge \(ca\) , \(\mu(c) + \mu(a) = 0.9 \leq 1\) then, \(( (\mu(c)\overline{L}\mu(c, a)) = 0\) and \(\overline{\mu(c)} = 0.7\). Thus, we get \(( (\mu(c)\overline{L}\mu(c, a)) \leq \overline{\mu(c)})\) and \(( (\mu(c)\overline{L}\mu^{-1}(c, a)) \leq \overline{\mu(c)})\).

Finally, with respect to the vertex \(a\), we get \(( (\mu(a)\overline{L}\mu(a, c)) \leq \overline{\mu(c)})\) and \(( (\mu(a)\overline{L}\mu^{-1}(a, c)) \leq \overline{\mu(a)})\). Therefore, the vertices \(c\) and \(a\) are internally stable under the composition \(\overline{L}\) and \(c, a \in \text{Int}(\rho, \overline{L})\). Hence, from graph for all \(a, b \in X\) if \(\mu(a) + \mu(b) \leq 1\) then, we have \(a, b \in \text{Int}(\rho, \overline{L})\).

Converse is not true

**Example 2.5.9.** Let \(G = (\mu, \rho)\) be a fuzzy graph where \(X = \{a, b, c, d\}\), \(\mu : X \rightarrow [0, 1]\), \(\rho : X \times X \rightarrow [0, 1]\) with \(\mu(a) = 0.7, \mu(b) = 0.5, \mu(c) = 0.6, \mu(d) = 0.8, \rho(a, b) = 0.5, \rho(b, c) = 0.5, \rho(c, d) = 0.3\), and \(\rho(d, a) = 0.4\) defined as shown in the Figure 2.11.

![Figure 2.11: \(\mu(a) + \mu(b) > 1\), fuzzy graph](image-url)
The edge $ab$, $((\mu(a) \overline{L} \rho(a, b)) = \max \max \{ 0.7 + 0.5 - 1, 0 \} = 0.2 \leq 0.3 = \mu(a)$ and $((\mu(a) \overline{L} \rho^{-1}(a, b)) = \max \max \{ 0.7 + 0.5 - 1, 0 \} = 0.2 \leq 0.3 = \mu(a)$. Thus, we say that, vertex $a$ is internally stable under the composition $\overline{L}$ and $a \in \text{Int}(\rho, \overline{L})$.

From the edge $ba$, $((\mu(b) \overline{L} \rho(b, a)) = \max \max \{ 0.5 + 0.5 - 1, 0 \} = 0 \leq 0.5 = \mu(b)$ and $((\mu(b) \overline{L} \rho^{-1}(b, c)) = \max \max \{ 0.5 + 0.5 - 1, 0 \} = 0 \leq 0.5 = \mu(b)$. Thus, we say that vertex $b$ is internally stable under the composition $\overline{L}$ and $b \in \text{Int}(\rho, \overline{L})$.

Similarly, we show that from all the edges $a, b, c, d \in \text{Int}(\rho, \overline{L})$. But, $\mu(a) + \mu(b) > 1$ for all $a, b \in X$.

**Note.** From the proposition 2.5.7, the set $\text{Int}(\rho, \overline{L})$ is completely determined by its maximal elements. Let us denote the set of all maximal elements by $\text{Int}_{\max}(\rho, \overline{L})$.

Then, we have the following result.

**Proposition 2.5.10.** Let $G = (\mu, \rho)$ be a fuzzy graph without loops and with underlying set $X$ where $\mu : X \rightarrow [0, 1], \rho : X \times X \rightarrow [0, 1]$ and for any $a \in X$, we have $\text{Int}(\rho, \overline{L}) = \bigcup_{a \in \text{Int}_{\max}(\rho, \overline{L})} [0, \mu(a)]$, where $\text{Int}_{\max}(\rho, \overline{L})$ is the set of solutions.

**Proof.** Let us consider the logical operator $\overline{L}$, the set $\text{Int}(\rho, \overline{L})$ described as mathematical programming problem, Max{$\mu(a) = \{ \mu(x_1), \mu(x_2), \ldots, \mu(x_n) \}$ (2.27) Subject to $\begin{align*}
(\mu(x_j) \overline{L} (\rho(x_i, x_j) \overline{L} \rho(x_j, x_i))) &\leq \overline{\mu(x_j)} \quad \text{(I)} \\
(\mu(x_j) \overline{L} (\rho(x_i, x_j) \overline{L} \rho(x_j, x_i))) &\leq \overline{\mu(x_i)}, \quad \forall i, j \in \{1, 2, \ldots, n\} \quad \text{(II)}
\end{align*}$
(2.28)

From (I) of (2.28), we have $\mu(x_j) \overline{L} (\rho(x_i, x_j) \overline{L} \rho(x_j, x_i))) \leq \overline{\mu(x_j)}$.
\[ \Rightarrow \max \{ \mu(x_i) + \min \{ \rho(x_i, x_j) + \rho(x_j, x_i), 1 \} - 1, 0 \} \leq \mu(x_j) \]

\[ \Rightarrow \max \{ \mu(x_i) + \min \{ (\rho(x_i, x_j) + \rho(x_j, x_i), 1 \} - 1, 0 \} \leq 1 - \mu(x_j) \quad (2.29) \]

From (2.29), if \( \min \{ \rho(x_i, x_j) + \rho(x_j, x_i), 1 \} = 1 \) then we get \( \mu(x_i) + \mu(x_j) \leq 1 \)

If \( \min \{ \rho(x_i, x_j) + \rho(x_j, x_i), 1 \} = \rho(x_i, x_j) + \rho(x_j, x_i) \) then, we have

\[ (2.29) \Rightarrow \max \{ \mu(x_i) + (\rho(x_i, x_j) + \rho(x_j, x_i)) - 1, 0 \} \leq 1 - \mu(x_j) \]

\[ \Rightarrow \mu(x_i) + (\rho(x_i, x_j) + \rho(x_j, x_i)) - 1 \leq 1 - \mu(x_j) \quad \text{(since 0 then result is trivial)} \]

\[ \Rightarrow \mu(x_i) + \mu(x_j) \leq 2 - (\rho(x_i, x_j) + \rho(x_j, x_i)) \]

\[ \Rightarrow \mu(x_i) + \mu(x_j) \leq 1 \quad \text{(since 0 \leq \rho(x_i, x_j) + \rho(x_j, x_i) \leq 1)} \]

Similarly from (II) of (2.28), we have \( \mu(x_i) + \mu(x_j) \leq 1 \)

From (2.28), we have

\[ (\mu(x_j) \overleftarrow{L} (\rho(x_i, x_j) \overleftarrow{L} \rho(x_j, x_i))) \leq \overline{\mu(x_j)} \]

Similarly, we show that

\[ (\mu(x_j) \overleftarrow{L} (\rho(x_i, x_j) \overleftarrow{L} \rho(x_j, x_i))) \leq \overline{\mu(x_j)} \]

Hence, we have the following solution for (2.27)

\[
\begin{cases}
\mu(x_j) \leq (\rho(x_i, x_j) \overleftarrow{L} \rho(x_j, x_i)) \\
\mu(x_j) \leq (\rho(x_i, x_j) \overleftarrow{L} \rho(x_j, x_i)) \\
\mu(x_i) + \mu(x_j) \leq 1
\end{cases}
\quad \text{or}
\begin{cases}
\mu(x_j) \geq (\rho(x_i, x_j) \overleftarrow{L} \rho(x_j, x_i)) \quad \text{then} \\
\mu(x_i) \leq 1 - (\rho(x_i, x_j) \overleftarrow{L} \rho(x_j, x_i)) \\
\mu(x_j) \geq (\rho(x_i, x_j) \overleftarrow{L} \rho(x_j, x_i)) \quad \text{then} \\
\mu(x_i) \leq 1 - (\rho(x_i, x_j) \overleftarrow{L} \rho(x_j, x_i))
\end{cases}
\]

**Example 2.5.11.** Let \( G = (\mu, \rho) \) be a fuzzy graph where \( X = \{a, b, c, d, e\} \),

\( \mu : X \rightarrow [0, 1], \rho : X \times X \rightarrow [0, 1] \) with \( \mu(a) = 0.6, \mu(b) = 0.3, \mu(c) = 0.5, \mu(d) = 0.2, \)

\( \mu(e) = 0.9, \rho(a, b) = 0.2, \rho(b, c) = 0.3, \rho(c, d) = 0.1, \rho(d, a) = 0.2, \rho(b, d) = 0.1 \)

and \( \rho(b, e) = 0.3 \) defined as shown in the Figure 2.12.
From the Figure 2.12, we have \( \text{Max } \mu = \{ \mu(a), \mu(b), \mu(c), \mu(d), \mu(e) \} \)

\[ = \text{ max } \{ 0.6, 0.3, 0.5, 0.2, 0.9 \} = 0.9 \]

Consider edge the \( ab \) using (2.28), we get \(( \mu(a) \bar{L} (\rho(a, b) \underline{L} \rho(b, a)) ) = 0 \leq \mu(b) \)
\( \mu(a) = 0.6 \) and \(( \rho(a, b) \underline{L} \rho(b, a)) = 0.4 \). Then, \( \mu(a) \geq ( \rho(a, b) \underline{L} \rho(b, a)) \). Thus, we get \( \mu(a) \leq 1 - (\rho(a, b) \underline{L} \rho(b, a)) \Rightarrow 0.6 \leq 0.6 \) and \( \mu(a) + \mu(b) = 0.9 \leq 1 \). Then, \( a \in \text{Int}(\rho, [L]) \). Consider the edge \( ba \), \(( \mu(b) \bar{L} (\rho(b, a) \underline{L} \rho(a, b)) ) = 0 \leq \mu(a), \)
\( \mu(b) = 0.3 \), and \(( \rho(b, a) \underline{L} \rho(a, b)) = 0.4 \). Thus, we get \( \mu(b) \leq ( \rho(b, a) \underline{L} \rho(a, b)) \) and \( \mu(a) + \mu(b) = 0.9 \leq 1 \). Hence, \( b \in \text{Int}(\rho, [L]) \). Similarly, we show that from the edges \( bc, cd, da, db \) and \( be \) we get \( a, b, c, d \in \text{Int}(\rho, [L]) \). But from the edge \( eb \), we have \(( \mu(e) \bar{L} (\rho(e, b) \underline{L} \rho(b, e)) ) = 0.5 \leq \mu(b) = 0.7 \), \( \mu(e) \geq (\rho(e, b) \underline{L} \rho(b, e)) \).

But, \( \mu(e) \leq 1 - (\rho(e, b) \underline{L} \rho(b, e)) \Rightarrow 0.9 \leq 0.4 \) and \( \mu(b) + \mu(e) > 1 \). Thus, we have \( e \notin \text{Int}(\rho, [L]) \) and \( \text{Int}(\rho, [L]) = \bigcup_{a \in \text{Int}_{\text{max}}(\rho, [L])} [0, \mu(a)] = [0, 0.6] \).
2.6 Conclusion

In this chapter, the fuzzy extension of some known concepts of crisp graphs has been investigated. Here, we discussed the properties of logical operators $\overline{L}$, $L$. It is shown that the fuzzification of the concepts of not external domination and internally stable vertices in fuzzy graphs by using the logical operators $\overline{L}$, $L$ and the composition $\overline{L}$ which are satisfied the weak lattices and sub weak lattices conditions. We established the conditions that the vertices of fuzzy graphs are in the sets $\text{Ned} (\rho, \overline{L})$ and $\text{Int} (\rho, \overline{L})$. Finally, the set $\text{Int} (\rho, \overline{L})$ is determined by solving a mathematical programming problem with fuzzy graph.