Chapter 1

Introduction

1.1 Fuzzy Mathematics

Fuzzy mathematics is the study of fuzzy structures, or structures that involve fuzziness, that is, such mathematical structures that at some points replace the two classical truth values 0 and 1 with a larger structure of degrees. Often, the real unit interval [0, 1] is employed as the system of degrees, but other options are common as well- a finite set, an arbitrary lattice, algebra of some kind or other. The degrees are intended to provide more flexibility to a fuzzy mathematical structure than the two truth degrees provide to the corresponding classical (“crisp”) mathematical structure. Fuzzy mathematics is just a kind of mathematics developed in this frame work. Hence in a certain sense, fuzzy mathematics is the kind of mathematical theory which contains wider content than the classical theory.

1.2 Fuzzy Set Theory

In traditional set theory, membership of an object belonging to a set can only be one of two values: 0 or 1. An object either belongs to a set completely or it does not belong at all. No partial membership is allowed. Crisp sets handle black-and-white concepts well, such as "chairs," ; "ships," and "trees" where little ambiguity exists. They are not sufficient, however, to realistically describe vague concepts.
In our daily lives, there are countless vague concepts that we humans can easily describe, understand, and communicate with each other but traditional mathematics, including the set theory, fails to handle in a rational way. The concept "young" is an example. For any specific person, his or her age is precise. However, relating a particular age to "young" involves fuzziness and is sometimes confusing and difficult. What age is young and what age is not? The nature of such questions is deterministic and has nothing to do with stochastic concepts such as probability or possibility.

A hypothetical crisp set "young" is given in Figure 1.1. This set is unreasonable because of the abrupt change of the membership value from 1 to 0 at 25. Although a different cutoff age at which membership value changes from 1 to 0 may be used, a fundamental problem exists. Why is it that a 24.9-year-old person is completely "young," while a 25.1-year-old person is not "young" at all? No crisp set can realistically capture, quantitatively or even qualitatively, the essence of the vague concept "young" to reasonably match what "young" means to human beings. This simple example is not
meant to discredit the traditional set theory. Rather, the intention is to demonstrate that crisp sets and fuzzy sets are two different and complementary tools, with each having its own strengths, limitations, and most effective application domains.

Fuzzy set theory was proposed by Professor Zadeh[94] at the University of California at Berkeley in 1965 to quantitatively and effectively handle problems of this nature. The theory has laid the foundation for computing with words. Fuzzy sets theory generalizes 0 and 1 membership values of a crisp set to a membership function of a fuzzy set. Using the theory, one relates an age to "young" with a membership value ranging from 0 to 1; 0 means no association at all, and 1 indicates complete association. For instance, one might think that age 10 is "young" with membership value 1, age 30 with membership value 0.75, age 50 with membership value 0.1, and so on. That is, every age/person is "young" to a certain degree. By plotting membership values versus ages, we generate a fuzzy set "young." All possible ages, say 0 to 110, form a universe of discourse. From this example, a definition of fuzzy sets naturally follows. Fuzzy set: A fuzzy set consists of a universe of discourse and a membership function that maps every element in the universe of discourse to a membership value between 0 and 1.

Fuzzy set theory was applied to control systems theory and engineering almost immediately after its birth. Advances in modern computer technology continuously backs up the fuzzy framework for coping with engineering systems of a broad spectrum, including many control systems that are too complex or too imprecise to tackle by conventional control theories and techniques.
Research on the theory of fuzzy sets has been witnessing an exponential growth; both within mathematics and in its applications. This range from traditional mathematical subjects like logic, topology, algebra, analysis etc. to pattern recognition, information theory, artificial intelligence, operations research, neural networks, planning etc. Consequently, fuzzy set theory has emerged as a potential area of interdisciplinary research.

The following definitions and results of fuzzy subsets which are in [94].

**Definition 1.2.1.** A fuzzy subset $\mu$ of $X$ is defined as a function $\mu : X \rightarrow [0, 1]$ for each $x \in X$, the value of $\mu(x)$ describes a degree of membership of $x$ in $\mu$. A fuzzy subset is completely characterized by its membership function $\mu$.

**Definition 1.2.2.** Let $\mu$ be a fuzzy subset of a set $X$. For $t \in [0, 1]$, the set $X_{\mu}^t = \{ x \in X \mid \mu(x) \geq t \}$ is called a $t$-level subset of the fuzzy subset of $\mu$.

**Definition 1.2.3.** A fuzzy set of a set $X$ is called a fuzzy point if and only if it takes the value 0 for all $y \in X$ except one, say, $x \in X$. If its value at $x$ is $t$, $(0 < t \leq 1)$ then we denote this fuzzy point by $x_t$.

**Definition 1.2.4.** Two fuzzy subsets $\mu$ and $\tau$ of a set $X$ are said to be disjoint if there exist no $x \in X$ such that $\mu(x) = \tau(x)$.

**Definition 1.2.5.** Let $\mu$ be any fuzzy subset of $X$. Then $\overline{\mu}$ (complement of $\mu$) is a fuzzy subset of $X$ defined by $\overline{\mu}(x) = \overline{\mu(x)} = 1 - \mu(x)$ for all $x \in X$. 
Definition 1.2.6. Let $\mu$, $\tau$ be two fuzzy subsets of $X$, then the following are hold:

(i) $\mu \subseteq \tau$ if $\mu(x) \leq \tau(x)$ for all $x \in X$

(ii) $\mu \subset \tau$ if $\mu(x) \leq \tau(x)$ for all $x \in X$ and there exists at least one $x \in X$ such that $\mu(x) < \tau(x)$

(iii) $\mu = \tau$ if $\mu(x) = \tau(x)$ for all $x \in X$.

Definition 1.2.7. Let $\mu$, $\tau$ be any two fuzzy subsets of $X$ then $(\mu \cap \tau)$ is fuzzy subset of $X$ defined by $(\mu \cap \tau)(x) = \mu(x) \land \tau(x)$ for all $x \in X$, and $(\mu \cup \tau)$ is the fuzzy subset of $X$ defined by $(\mu \cup \tau)(x) = \mu(x) \lor \tau(x)$ for all $x \in X$.

1.3 Fuzzy Relations

The study of fuzzy relations was started by Zadeh[95] in 1971. In that celebrated paper, he introduced the concept of fuzzy relation, defined the notion of equivalence, and gave the concept of fuzzy orderings. The concept of fuzzy order was introduced by generalizing the notion of reflexivity, antisymmetry and transitivity, thereby facilitating the derivation of known results in various areas and stimulating the discovery of new ones.

This concept has been applied to various areas including fuzzified graphic theory. There are many applications in which a fuzzy relation on a fuzzy subset is quite useful. Fuzzy relations play an important role in fuzzy modeling, fuzzy diagnosis, and fuzzy control. They also have applications in fields such as psychology, medicine, economics, and sociology.
The following definitions and results of fuzzy relations which are in [56] and [68].

**Definition 1.3.1.** Let $S$ and $T$ be two sets and $\mu$ and $\tau$ be fuzzy subsets on $S$ and $T$ respectively. Then the fuzzy relation $\rho$ from the fuzzy subset $\mu$ into the fuzzy subset $\tau$ is a fuzzy subset on $S \times T$ such that $\rho(x, y) \leq \mu(x) \land \tau(y)$ for all $x \in S$ and $y \in T$.

**Note.** The restriction $\rho(x, y) \leq \mu(x) \land \tau(y)$ for all $x \in S$ and $y \in T$ allows $\rho^t$ to be a relation from $\mu^t$ into $\tau^t$ for all $t \in [0, 1]$.

**Note.** There are three special cases of fuzzy relations which are extensively found.

(i) If $S=T=X$ and $\mu=\tau$. In this case $\rho$ is said to be a fuzzy relation on $\mu$ and $\rho$ is a fuzzy subset of $X \times X$ such that $\rho(x, y) \leq \mu(x) \land \mu(y)$ for all $x, y \in X$.

(ii) $\mu(x)=1.0$ for all $x \in S$ and $\tau(y) = 1.0$ for all $y \in T$. In this case, $\rho$ is said to be a fuzzy relation from $S$ into $T$.

(iii) $S=T$, $\mu(x) = 1.0$ for all $x \in S$ and $\tau(y) = 1.0$ for all $y \in T$. In this case, $\rho$ is said to be a fuzzy relation on $S$.

**Definition 1.3.2.** A fuzzy binary relations defined on the set $X$ is a fuzzy subset of the direct product $X \times X$. We regard relations as preference relation considering $\rho(x, y)$ as a degree of preference of $x$ over $y$. If $\rho(x, y)$ is a fuzzy relation $X$ then its converse is the relation $\rho^{-1}(x, y) = \rho(y, x)$ the complementary of $\rho$ is the relation $\rho^c$; $\rho^c(x, y) = 1 - \rho(x, y)$, and its dual $\rho^d$ is defined by $\rho^d(x, y) = \rho^{-1}(x, y)$ for each $x, y \in X$. 

Definition 1.3.3. Let $\rho : S \times T \rightarrow [0, 1]$ be a fuzzy relation from a fuzzy subset $\mu$ of $S$ into a fuzzy subset $\tau$ of $T$ and $\lambda : T \times U \rightarrow [0, 1]$ be a fuzzy relation from a fuzzy subset $\tau$ of $T$ into a fuzzy subset $\sigma$ of $U$. Define $\rho \circ \lambda : S \times U \rightarrow [0, 1]$, such that $(\rho \circ \lambda)(x, z) = \vee \{ \rho(x, y) \land \lambda(y, z) / y \in T \}$ for all $x \in S, z \in U$. Then, $(\rho \circ \lambda)$ is called the composition of $\rho$ and $\lambda$.

1.4 Fuzzy Graphs

It is quite well known that graphs are simply models of relations. A graph is a convenient way of representing information involving relationship between objects. The objects are represented by vertices and relations by edges. When there is vagueness in the description of the objects or in its relationships or in both, it is natural that we need to design a 'Fuzzy Graph Model.'

Application of fuzzy relations are widespread and important; especially in the field of clustering analysis, neural networks, computer networks, pattern recognition, decision making and expert systems. In each of these, the basic mathematical structure is that of a fuzzy graph.

We know that a graph is a symmetric binary relation on a nonempty set $V$. Similarly, a fuzzy graph is a symmetric binary fuzzy relation on a fuzzy subset. The first definition of a fuzzy graph was by Kaufmann[37], based on Zadeh's fuzzy relations [95]. But it was Azriel Rosenfeld[68] who considered fuzzy relations on fuzzy sets and developed the theory of fuzzy graphs. During the same time Yeh et al.[86] have also introduced various connectedness concepts in fuzzy graphs.
The following definitions and results of fuzzy graphs which are in [56] and [68].

**Definition 1.4.1.** A fuzzy graph $G$ on a nonempty set $X$ is an ordered pair $(\mu, \rho)$ of functions where $\mu: X \rightarrow [0, 1]$ and $\rho: X \times X \rightarrow [0, 1]$ such that $\rho(x, y) \leq \mu(x) \land \mu(y)$ for all $x, y \in X$. Here $\mu$ and $\rho$ are the fuzzy vertex set and the fuzzy edge set of $G$, respectively.

**Definition 1.4.2.** The fuzzy graph $H = (\nu, \tau)$ is called a partial subgraph of $G = (\mu, \rho)$ if $\nu \subseteq \mu$ and $\tau \subseteq \rho$. In particular, we call $H = (\nu, \tau)$ a fuzzy subgraph of $G = (\mu, \rho)$ if $P \subseteq X$, $\nu(x) = \mu(x)$ for all $x \in P$ and $\tau(x, y) = \rho(x, y)$ for all $x, y \in P$. For any $t$, $0 \leq t \leq 1$, we have $\mu^t = \{ x \in X \mid \mu(x) \geq t \}$ and $\rho^t = \{ (x, y) \in X \times X \mid \rho(x, y) \geq t \}$.

Since $\rho(x, y) \leq \mu(x) \land \mu(y)$ for all $x, y \in X$, we get $\rho^t \subseteq \mu^t \times \mu^t$ so that $(\mu^t, \rho^t)$ is a graph with vertex set $\mu^t$ and edge set $\rho^t$ for all $t \in [0, 1]$.

**Note.** Let $G = (\mu, \rho)$ be a fuzzy graph. If $0 \leq s_1 \leq s_2 \leq 1$, then $(\mu^{s_2}, \rho^{s_2})$ is a subgraph of $(\mu^{s_1}, \rho^{s_1})$.

**Note.** Let $H = (\nu, \tau)$ be a partial fuzzy subgraph of $G = (\mu, \rho)$. For any threshold $t$, $0 \leq t \leq 1$, $(\nu^t, \tau^t)$ is subgraph of $(\mu^t, \rho^t)$.

**Definition 1.4.3.** For any subset $\nu$ of $X$ such that $\nu \subseteq \mu$, the partial fuzzy subgraph of $(\mu, \rho)$ induced by $\nu$ is the maximal partial fuzzy subgraph of $(\mu, \rho)$ that vertex set $\nu$.

This is the partial fuzzy graph $(\nu, \tau)$, where $\tau(x, y) = \nu(x) \land \nu(y) \land \rho(x, y)$ for all $x, y \in X$.
1.5 Theoretical Computer Science

In the early days of computer science, a great deal of time and energy was devoted to the development of basic concepts, design of fundamental algorithms, identification and development of major sub-disciplines, and the classification of problems by their difficulty, and the activities that actively engaged many theoretical computer scientists. The role of theoretical computer scientists today is to examine the fundamental problems of the field through modeling, analysis, and experimentation.

Theoretical computer science is the cornerstone of computer science. It “underlies many aspects of construction, explanation, and understanding computers”. In theoretical computer science there are various mathematical topics like logic, set theory, algebraic structures, Boolean algebra, graph theory, basic computability theory and etc.

Graph theory concepts are used in networks, operating systems, and compilers. Set theory concepts are used in software engineering and in databases. As the field of computer science matures, more and more sophisticated analysis techniques are being brought to bear on practical problems. To understand the computational techniques of the future, today’s students will need a strong background in theoretical computer science. While primarily taking a computer science approach, team members also have training and expertise in electrical engineering, physics, pure mathematics, and other areas, and take an interdisciplinary approach that combines the ability to make the most fundamental theoretical advances, with an understanding of how to apply them.
1.6 Residuated Lattices

Residuated lattices are algebraic structures with strong connections to mathematical logic. The algebras under investigation combine the fundamental notions of multiplication, order and residuation, and include many well-studied ordered algebraic structures. Residuated lattices were first considered, albeit in a more restrictive setting than the one we adopt here, by Dilworth et al.[19]. Their investigation stemmed from attempts to generalize properties of the lattice of ideals of a ring. On the other hand, work on residuation, a concept closely related to the notions of categorical adjunction and of Galois connection, was undertaken in algebra, with emphasis on multiplication, and in logic, with emphasis on implication, but without substantial communication between the fields.

Residuated lattices have been further investigated by Krull[46], Dilworth[20], Ward[77], Balbes et al.[7] and Pavelka[61]. Idziak[29] proved that the class of residuated lattices is equational. These lattices have been known under many names: BCK-lattices, full BCK-algebras and integral, residuated, commutative l-monoids. The concept residuated have been studied extensively and include important classes of algebras such as BL-algebras introduced by Hájek[26] as the algebraic counterpart of his Basic Logic, MV-algebras, the algebraic setting for Łukasiewicz propositional logic.

During relatively recent years, studies in relevant logic, linear logic and substructural logic as well as on the algebraic side draw attention to and establish strong connections between the fields. The generality in the definition of residuated lattices is due to Blount et al.[11] who first developed a structure theory for these algebras.
1.7 Wajsberg Algebras

The study of fuzzy subsets and their application to mathematical contexts has important branch of fuzzy mathematics. In 1935, the infinite-valued Łukasiewicz proposition calculus was complete with respect to the axiomatics conjectured by Łukasiewicz itself. Unfortunately, his proof was never published. In 1958, Rose et al.[66] did publish a proof.

In 1966, Imai et al.[30] defined a class of algebras of type (2, 0) called BCK-algebras which generalizes on one hand the notion of algebra of sets with the set subtraction as the only fundamental non-nullary operation, on the other hand the notion of implication algebra.

Meng[51] introduced the notions of implicative ideals and commutative ideals in BCK-algebras and applied them to characterize implicative BCK-algebras and commutative BCK-algebras, respectively. Meng also showed that a non-empty subset of a BCK-algebra is an implicative ideal if and only if it is both a commutative ideal and positive implicative ideal.

The study of fuzzy algebraic structures was started with the introduction of the concept of fuzzy sub-groups in 1971 by Rosenfeld[67]. Since then these ideas have been applied to other algebraic structures such as semi groups, ring, ideals, modules and vector spaces.

Wajsberg algebras (W-algebras) were introduced by Rodriguez[63], [64], as the algebraic counterpart of Łukasiewicz infinite-valued propositional calculi. In constrast to the MV-algebras introduced with the same purpose by Chang[15], [16] the basic
operations of Wajsberg algebras correspond to the implication and negation considered by Łukasiewicz, and the axioms correspond to those used by Wajsberg to axiomatize the infinite-valued propositional calculus after a conjecture of Łukasiewicz. Chang has no implication operator, and his presentation is very complicated and not natural, and the logical contents of subject are not apparent. An approach to the algebraic analysis of Łukasiewicz infinite-valued calculi, similar to that of Rodriguez was followed by Komori[42]. It is also worthwhile to point out that class of Wajsberg algebras coincides with the class of bounded commutative BCK-algebras.

The algebras corresponding to the n-valued Łukasiewicz propositional calculi were considered as subvarieties of MV-algebras by Grigolia[25] and subvarieties of Wajsberg algebras by Rodriguez. Font et al.[24] proposed the concept of implicative filters in lattice Wajsberg algebras and investigated some of their properties. Wajsberg algebras are Kleen algebras and, a fortiori, bounded distributive lattices. It is also called Sales algebras.

In 1993, Xu[83] introduced the idea of lattice implication algebras, which combines lattice with implication algebra. He discussed some of their properties. Xu et al.[84] proposed the notion of filter in a lattice implication algebra, and investigative properties. In [85], Xu et al. defined the fuzzy filter in lattice implication algebra, and they discussed some of their properties. Roh et al.[65] introduced the concept of fuzzy implicative filter in lattice implication algebras and they obtained some properties.
1.8 Organization of the Thesis

This thesis is divided into seven chapters. In the first chapter, we discuss the concepts of fuzzy mathematics, fuzzy graphs, theoretical computer science, residuated lattices, fuzzy algebras and Wajsberg algebras. We attempt to study more on the basic ideas of fuzzy sets, fuzzy relations. The general preliminary definitions and results which are used in succeeding chapters are discussed. Due references are given whenever necessary.

In the chapter 2, we investigate some fuzzy logical operators to extend the concept of not external domination and some of their combinations to fuzzy graphs. Here, we consider the logical operators $\bar{L}$, $L$ and the composition $\max - \bar{L}$, (composition $\bar{L}$). Finally, we introduce some notions and study the properties of not externally domination vertices (Ned), internally stable vertex set (Int) of fuzzy graphs.

Chapter 3 is concerned with the development of the concept of externally stable vertices (Ext) in fuzzy graphs. The definition of set of undominated vertices of crisp graphs is motivated to introduce the set of non-dominated fuzzy subsets ($NDFS (\rho, \bar{L})$) and fuzzy set of non-dominated vertices of fuzzy graphs ($FND (\rho)$). Also, we find some properties of ($NDFS (\rho, \bar{L})$) and ($FND (\rho)$).

Chapter 4 begins with discussion of some basic definitions and results of residuated lattices. We introduce the residuated weak lattices by using the logical operators $\bar{L}$, $L$ and the generalization of $\bar{L}$ as $\bar{Inf}$, $\bar{L}$ as $\bar{Sup}$. In the rest of this chapter, we discuss some properties and direct product of residuated weak lattices.
In chapter 5, we consider some basic definitions and results of Wajsberg algebras (W-algebras). In a lattice structure of Wajsberg algebras, it was complete with respect to the axiomatics conjecture. The aim of this chapter is to introduce fuzzy implicative filters, anti fuzzy implicative filters of lattice Wajsberg algebras and obtain some properties. Further we show the extension properties of fuzzy implicative filters and anti fuzzy implicative filters of lattice Wajsberg algebras.

Chapters 6 reveals the introduction of new concepts of fuzzy prime and anti fuzzy prime implicative filters of lattice W- algebras and obtain some properties of fuzzy prime and anti fuzzy prime implicative filters.

Chapter 7 gives the conclusion of our thesis, future research works which are helpful for further development of the results. Finally, we discuss some interesting applications.

The present work is intended to contribute to fuzzy mathematics in theoretical computer science in some fields of graphs, lattices and algebraic systems. Hence, this thesis studies the concepts of ‘Fuzzy graphs’, ‘Residuated weak lattices’ and ‘Implicative filters of lattice Wajsberg algebras’. In the concept of fuzzy graphs, we investigate the several properties of dominated vertices and stability of vertex sets. We introduce the notion of residuated weak lattices and study some of their axioms. Here, we consider the fuzzy subsets as in the illustrations. We present the concepts of fuzzy implicative filters and anti fuzzy implicative filters of lattice Wajsberg algebras. Finally, we introduce the
notions of fuzzy prime and anti fuzzy prime implicative filters of lattice Wajsberg algebras and discuss some of their properties.