Chapter 6

Fuzzy Prime and Anti Fuzzy Prime Implicative Filters of Lattice Wajsberg Algebras

6.1 Introduction

This chapter deals with the notion of fuzzy prime implicative filter and highlights the relation between prime implicative filter and fuzzy prime implicative filter in lattice Wajsberg algebras (W-algebras). We show that a necessary and sufficient condition for a fuzzy subset $\mu$ to be a fuzzy prime implicative filter. Further, we prove that fuzzy prime characteristics function to be prime filter.

Next, we propose the notion of an anti fuzzy prime implicative filter of lattice W-algebras. We use some conditions to the set of anti fuzzy implicative filter to be a prime implicative filter. Further, we obtain the complement of characteristics function to be anti fuzzy prime implicative filter.
6.2 Preliminaries

In this section, we use some definitions which are given in the chapter 5 (From definition 5.2.1 to definition5.2.9).

**Definition 6.2.1**[24] Let $A$ be W-algebra. An implicative filter $F$ of $A$ is called a prime implicative filter of $A$ if it satisfies the axiom for all $x, y \in A$, $(x \lor y) \in F$ then $x \in F$ or $y \in F$.

**Definition 6.2.2**[by definition 5.3.1] A fuzzy subset $\mu$ of lattice W-algebra $A$ is called a fuzzy implicative filter of $A$ if it satisfies the following axioms.

(i) $\mu(l) \geq \mu(x)$ for all $x \in A$

(ii) $\mu(y) \geq \min \{ \mu(x \rightarrow y), \mu(x) \}$ for all $x, y \in A$.

**Definition 6.2.3**[by definition 5.4.1] A fuzzy subset $\mu$ of lattice W-algebra $A$ is said to be an anti fuzzy implicative filter of $A$ if it satisfies the following axioms.

(i) $\mu(l) \leq \mu(x)$

(ii) $\mu(y) \leq \max \{ \mu(x \rightarrow y), \mu(x) \}$ for all $x, y \in A$.

6.3 Fuzzy Prime Implicative Filter of Lattice Wajsberg Algebras

**Definition 6.3.1.** A non constant fuzzy implicative filter $\mu$ of a lattice W-algebra $A$ is said to be prime if $\mu(x \lor y) \leq \max \{ \mu(x), \mu(y) \}$ for all $x, y \in A$. 
Example 6.3.2. Let \( A = \{0, \, a, \, b, \, c, \, 1\} \) be a set with Figure 6.1 as partial ordering. Define unary operation “\( \ast \)” and a binary operation “\( \rightarrow \)” on \( A \) as in the Table 6.1 and Table 6.2.

\[
\begin{array}{c|c|c|c|c|c|c}
  x & x' & \rightarrow & 0 & a & b & c & 1 \\
  \hline
  0 & 1 & 0 & 1 & 1 & 1 & 1 & 1 \\
  a & b & a & b & 1 & c & 1 & 1 \\
  b & a & b & a & c & 1 & 1 & 1 \\
  c & c & c & c & c & c & 1 & 1 \\
  1 & 0 & 1 & 0 & a & b & c & 1 \\
\end{array}
\]

Table:6.1 Complement

Table:6.2 Implication

Define \( \vee \) and \( \wedge \) an operation on \( A \) as follow:

\[
(x \vee y) = (x \rightarrow y) \rightarrow y, \\
(x \wedge y) = ((x^* \rightarrow y^*) \rightarrow y^*)^* 
\]

for all \( x, \, y \in A \). Then, we have \( A \) is lattice W-algebra.

Consider the fuzzy subset \( \mu \) on \( A \) is defined by,

\[
\mu(x) = \begin{cases} 
0.8 & \text{if } x \in \{1\} \\
0.6 & \text{if } x \in \{0, \, a, \, b, \, c\} 
\end{cases} \text{ for all } x \in A.
\]

Then, \( \mu \) is fuzzy prime implicative filter of \( A \).

Example 6.3.3. Let \( A = \{0, \, a, \, b, \, c, \, d, \, 1\} \) be a set with Figure 6.2 as partial ordering.

Define unary operation “\( \ast \)” and a binary operation “\( \rightarrow \)” on \( A \) as in the Tables 6.3, 6.4.
Define $\lor$ and $\land$ an operation on $A$ as follows:

$$ (x \lor y) = (x \to y) \to y, $$

$$ (x \land y) = ((x^* \to y^*) \to y^*)^* $$ for all $x, y \in A$. Then, we have $A$ is lattice W-algebra.

Consider a fuzzy subset $\mu$ on $A$ is defined by,

$$ \mu(x) = \begin{cases} 
0.5 & \text{if } x \in \{1, a, b\} \\
0.3 & \text{if } x \in \{0, c, d\} 
\end{cases} \text{ for all } x \in A. $$

Then, we have $\mu$ is not fuzzy prime implicative filter of lattice W-algebra $A$.

Since $\mu(c \lor d) = 0.5$ but $\max \{ \mu(c), \mu(d) \} = 0.3$

Thus, we get $\mu(c \lor d) \not\geq \max \{ \mu(c), \mu(d) \}$.

**Proposition 6.3.4.** If $\mu$ is fuzzy implicative filter of lattice W-algebra $A$. Then the set $A_{\mu} = \{ x \in A / \mu(x) = \mu(1) \}$ is implicative filter of $A$.

**Proof.** Let $\mu$ be a fuzzy implicative filter of lattice W-algebra $A$

Consider the set $A_{\mu} = \{ x \in A / \mu(x) = \mu(1) \}$

Then, for any $x \in A \Rightarrow \mu(x) \in A_{\mu} \Rightarrow \mu(1) \in A_{\mu}$
If, \((\mu(x) \rightarrow \mu(l)) \in A_\mu\)

\[\Rightarrow (\mu(x) \rightarrow \mu(l)) = 1\]

\[\Rightarrow (\mu(x) \leq \mu(y)), \text{ since } x \leq y \text{ if and only if } x \rightarrow y = 1\]

\[\Rightarrow (\mu(l) \leq \mu(y))\]  \hspace{1cm} (6.1)

Since \(\mu\) is fuzzy implicative filter of lattice \(W\)-algebra, \((\mu(l) \geq \mu(y))\) \hspace{1cm} (6.2)

From (6.1) and (6.2), we have \(\mu(y) = \mu(l)\). Thus, \(\mu(y) \in A_\mu\).

Hence, \(A_\mu\) is implicative filter of lattice \(W\)-algebra \(A\).  

**Proposition 6.3.5.** Let \(\mu\) be a fuzzy prime implicative filter of lattice \(W\)-algebra \(A\).

Then the set \(A_\mu = \{x \in A / \mu(x) = \mu(l)\}\) is a prime implicative filter of \(A\).

**Proof.** We have, \(A_\mu\) is a implicative filter of \(A\) (Proposition 6.3.4)

For any \(x, y \in A\), we have \((x \vee y) \in A_\mu\)

Then, we get \(\mu(1) = \mu(x \vee y) \leq \max\{\mu(x), \mu(y)\} = \mu(x)\) or \(\mu(y)\)

Thus, \(\mu(x) = \mu(1)\) or \(\mu(y) = \mu(1)\). Hence, we get \(\mu(x) \in A_\mu\) or \(\mu(y) \in A_\mu\).

Therefore, \(A_\mu\) is prime implicative filter of \(A\).  

**Proposition 6.3.6.** A fuzzy subset \(\mu\) is a fuzzy prime implicative filter of lattice \(W\)-algebra \(A\), if and only if the set \(S = \{x \in A / \mu(x) = 1\}\) is either empty or a fuzzy prime implicative filter of \(A\).

**Proof.** Let \(\mu\) be fuzzy prime implicative filter of lattice \(W\)-algebra \(A\)

Then, we get \(\mu(x \vee y) \leq \max\{\mu(x), \mu(y)\}\) for all \(x, y \in A\)

Consider the set \(S = \{x \in A / \mu(x) = 1\}\)
If $S \neq \phi$. We have to prove that $S$ is fuzzy prime implicative filter of $A$

Since $S \neq \phi$, for any $x, y \in A$ and $\mu(x), \mu(y) \in S$ such that $\mu(x)=1$ and $\mu(y)=1$

Now $\mu(x \lor y) \leq \max \{ \mu(x), \mu(y) \}=1$

Thus, we have $\mu(x \lor y) \in S$ for all $x, y \in A$

Hence, $S$ is fuzzy prime implicative filter of $A$.

Conversely, if $S \neq \phi$ and $S$ is fuzzy prime implicative filter of $A$.

**To Prove:** Fuzzy subset $\mu$ is a fuzzy prime implicative filter of $A$

For any $x, y \in A$ and $\mu(x), \mu(y) \in S$ such that $\mu(x)=1$ and $\mu(y)=1$

Now $\mu(x \lor y)=1$ and $\max \{ \mu(x), \mu(y) \}=1$

Thus, we get $\mu(x \lor y) \leq \max \{ \mu(x), \mu(y) \}$ for all $x, y \in A$.

Hence, $\mu$ is fuzzy prime implicative filter of lattice W-algebra $A$.

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**Proposition 6.3.7.** Let $F$ be implicative filter of a lattice W-algebra $A$ and let $\alpha < \beta \neq 0$ be elements of $[0, 1)$. Then fuzzy set $\mu : A \rightarrow [0, 1]$ defined by

$$
\mu(x) = \begin{cases} 
\beta & \text{if } x \in F \\
\alpha & \text{otherwise}
\end{cases}
$$

for all $x \in A$, is a fuzzy implicative filter of $A$.

**Proof.** Since $1 \in F$, $\mu(1)=\beta \geq \mu(x)$ for all $x \in A$

Suppose $\mu(y) \geq \min \{ \mu(x \rightarrow y), \mu(x) \}$ for all $x, y \in A$ does not hold.

Then, there exists $x, y \in A$ such that $\mu(x)=\alpha$ and $\min \{ \mu(x \rightarrow y), \mu(x) \} = \beta$.

Thus, we have $\mu(x \rightarrow y) = \beta$ and $\mu(x) = \beta$. Hence, $x \rightarrow y \in F$ and $x \in F$ and so $y \in F$, since $F$ is implicative filter. This is a contradiction.

Therefore, $\mu(x)$ is fuzzy implicative filter of $A$. 
Proposition 6.3.8. Let $S$ be a prime implicative filter of lattice $W$-algebra $A$ and let $\alpha \in [0, 1]$. If $\mu$ is a fuzzy subset in $A$ defined by

$$
\mu(x) = \begin{cases} 
1 & \text{if } x \in S \\
\alpha & \text{otherwise}
\end{cases}
$$

for all $x \in A$. (6.3)

Then $\mu$ is a fuzzy prime implicative filter of $A$.

**Proof.** By the result (proposition 6.3.7), we have $\mu$ is fuzzy implicative filter of $A$

For any $x, y \in A$, we have $(x \vee y) \in S$ then $x \in S$ or $y \in S$

Thus, $\mu(x \vee y) = 1 = \max \{ \mu(x), \mu(y) \}$ (6.4)

If $(x \vee y) \notin S$ then $\mu(x \vee y) = \alpha \leq \max \{ \mu(x), \mu(y) \}$ (6.5)

From (6.4) and (6.5), we have $\mu$ is a fuzzy prime implicative filter of $A$. ■

Corollary 6.3.9. If $S$ is a prime implicative filter of lattice $W$-algebra $A$, then the characteristic function $\chi_S$ is a fuzzy prime implicative filter of $A$.

The following proposition is converse of the corollary 6.3.9

Proposition 6.3.10. If $S$ is a filter of lattice $W$-algebra $A$ such that the characteristics function $\chi_S$ is fuzzy prime implicative filter of $A$, then $S$ is a prime implicative filter of $A$.

**Proof.** For any $x, y \in A$ such that $(x \vee y) \in S$ and $x \notin S$

Then, we get $1 = \chi_S(x \vee y)$, by (6.3) of the proposition 6.3.8

$$
\leq \max \{ \chi_S(x), \chi_S(y) \} = \chi_S(y)
$$

Thus, we have $\chi_S(y) = 1$ so that $y \in S$. Hence, $S$ is a prime filter of $A$. ■
6.4 Anti Fuzzy Prime Implicative Filter of Lattice Wajsberg Algebras

Definition 6.4.1. A non constant fuzzy implicative filter of lattice W-algebra $A$ is called anti fuzzy prime if $\mu(x \land y) \geq \min \{ \mu(x), \mu(y) \}$ for all $x, y \in A$.

Example 6.4.2. Let $A = \{0, a, b, c, 1\}$ be a set with Figure 6.3 as partial ordering. Define unary operation “*” and a binary operation “→” on $A$ as in the Table 6.5 and Table 6.6.

<table>
<thead>
<tr>
<th>$x$</th>
<th>$x^*$</th>
<th>→</th>
<th>0</th>
<th>a</th>
<th>b</th>
<th>c</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>a</td>
<td>c</td>
<td>a</td>
<td>c</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>b</td>
<td>b</td>
<td>b</td>
<td>b</td>
<td>c</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>c</td>
<td>a</td>
<td>c</td>
<td>a</td>
<td>b</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>a</td>
<td>b</td>
<td>c</td>
<td>1</td>
</tr>
</tbody>
</table>

Table: 6.5 Table: 6.6 Implication Figure: 6.3 Lattice diagram

Complement

Define $\lor$ and $\land$ an operation on $A$ as follow:

$(x \lor y) = (x \rightarrow y) \rightarrow y$,

$(x \land y) = ((x^* \rightarrow y^*) \rightarrow y^*)^*$ for all $x, y \in A$. Then, we have $A$ is lattice W-algebra

Consider the fuzzy subset $\mu$ on $A$ is defined by,

$$\mu(x)=\begin{cases} 0.6 & \text{if } x = 1 \\ 0.3 & \text{otherwise} \end{cases}$$

for all $x \in A$.

Then, $\mu$ is anti fuzzy prime implicative filter of $A$. 
Proposition 6.4.3. Let \( \mu \) be an anti fuzzy prime implicative filter of lattice \( W \)- algebra \( A \).

Then the set \( A_\mu = \{ x \in A \mid \mu(x) = \mu(l) \} \) is a prime implicative filter of \( A \).

Proof. We have, \( A_\mu \) is an implicative filter of \( A \) (Proposition 6.3.4)

For any \( x, y \in A \) be such that \( (x \land y) \in A_\mu \)

Then, we have \( \mu(l) = \mu(x \land y) \geq \min \{ \mu(x), \mu(y) \} = \mu(x) \) and \( \mu(y) \)

Thus, \( \mu(x) = \mu(l) \) and \( \mu(y) = \mu(l) \). Hence, we get \( \mu(x) \in A_\mu \) and \( \mu(y) \in \mu \).

Therefore, \( A_\mu \) is prime implicative filter of \( A \). \( \blacksquare \)

Proposition 6.4.4. A fuzzy subset \( \mu \) is an anti fuzzy prime implicative filter of lattice \( W \)- algebra \( A \), if and only if the set \( T = \{ x \in A \mid \mu(x) = 1 \} \) is either empty or a fuzzy prime implicative filter of \( A \).

Proof. Let \( \mu \) be an anti fuzzy prime implicative filter of lattice \( W \)- algebra \( A \)

Then, we get, \( \mu(x \land y) \geq \min \{ \mu(x), \mu(y) \} \) for all \( x, y \in A \)

Consider the set \( T = \{ x \in A \mid \mu(x) = 1 \} \)

If \( T \neq \emptyset \). We have to show that \( T \) is fuzzy prime implicative filter of \( A \)

Since \( T \neq \emptyset \), for any \( x, y \in A \) and \( \mu(x), \mu(y) \in T \) such that \( \mu(x) = 1 \) and \( \mu(y) = 1 \)

Now \( \mu(x \land y) \geq \min \{ \mu(x), \mu(y) \} = 1 \)

Thus, we have \( \mu(x \land y) \in T \) for all \( x, y \in A \)

Hence, \( T \) is fuzzy prime implicative filter of \( A \).

Conversely, if \( T \neq \emptyset \) and \( T \) is fuzzy prime implicative filter of \( A \).

To Prove: Fuzzy subset \( \mu \) is an anti fuzzy prime implicative filter of \( A \)

For any \( x, y \in A \) and \( \mu(x), \mu(y) \in T \) such that \( \mu(x) = 1 \) and \( \mu(y) = 1 \)

Now \( \mu(x \land y) = 1 \) and \( \min \{ \mu(x), \mu(y) \} = 1 \)
Thus, we get \( \mu(x \land y) \geq \min\{\mu(x), \mu(y)\} \) for all \( x, y \in A \)

Hence, \( \mu \) be an anti fuzzy prime implicative filter of lattice W-algebra \( A \).  

**Proposition 6.4.5.** Let \( I \) be a prime implicative filter of a lattice W-algebra \( A \), let \( \alpha < \beta \neq 0 \) be elements of \( (0, 1] \). Then fuzzy set \( \mu : A \rightarrow [0, 1] \), defined by

\[
\mu(x) = \begin{cases} \alpha & \text{if } x \in I \\ \beta & \text{otherwise} \end{cases}
\]

for all \( x \in A \), is an anti fuzzy implicative filter of \( A \).

**Proof.** Since \( 1 \in I \), \( \mu(1) = \alpha \geq \mu(x) \) for all \( x \in A \)

Suppose \( \mu(y) \geq \min\{\mu(x \rightarrow y), \mu(x)\} \) for all \( x, y \in A \) does not hold

Then, there exists \( x, y \in A \) such that \( \mu(x) = \beta \) and \( \min\{\mu(x \rightarrow y), \mu(x)\} = \alpha \)

Thus, we have \( \mu(x \rightarrow y) = \alpha \) and \( \mu(x) = \alpha \)

Therefore, \( x \rightarrow y \in I \) and \( x \in I \) so \( y \in I \), since \( I \) is implicative filter.

This is a contradiction. Hence, \( \mu(x) \) is an anti fuzzy implicative filter of \( A \).  

**Proposition 6.4.6.** Let \( T \) be a prime implicative filter of lattice W-algebra \( A \) and let \( \alpha \in (0, 1] \). If \( \mu \) is a fuzzy subset in \( A \) defined by

\[
\mu(x) = \begin{cases} 1 & \text{if } x \in T \\ \alpha & \text{otherwise} \end{cases}
\]

for all \( x \in A \).

Then \( \mu \) is an anti fuzzy prime implicative filter of \( A \).

**Proof.** By the (proposition 6.4.5), we have, \( \mu \) is an anti fuzzy implicative filter of \( A \)

For any \( x, y \in A \), we have \( (x \land y) \in T \) then, \( x \in T \) and \( y \in T \)

Thus, \( \mu(x \land y) = 1 = \min\{\mu(x), \mu(y)\} \) \hspace{1cm} (6.7)

If \( (x \land y) \notin S \), then \( \mu(x \land y) = \alpha \geq \min\{\mu(x), \mu(y)\} \) \hspace{1cm} (6.8)

From (6.7) and (6.8), we have \( \mu \) is an anti fuzzy prime implicative filter of \( A \).
**Proposition 6.4.7.** If $T$ is a filter of lattice W-algebra $A$, then $T$ is a prime implicative filter of $A$ if and only if the complement $\chi_s^*$ of the characteristics function is an anti fuzzy prime implicative filter of $A$.

**Proof.** Obviously, If $T$ is a prime implicative filter of lattice W-algebra, then the characteristics function $\chi_s^*$ is an anti fuzzy prime implicative filter of $A$.

Conversely, the complement $\chi_s^*$ of characteristics function is an anti fuzzy prime implicative filter of $A$, we have to prove that $T$ is a prime implicative filter of $A$.

Suppose $(x \land y) \in T$ and $x \notin T$ for all $x, y \in A$.

Then, $1 = \chi_s^* (x \land y)$ by (6.6) of Proposition 6.4.6.

\[
1 = \chi_s^* (x \land y) \\
\geq 1 - \min \{ \chi_s (x), \chi_s (y) \} \\
= \max \{ 1 - \chi_s (x), 1 - \chi_s (y) \} \\
= \max \{ \chi_s^* (x), \chi_s^* (y) \} \\
= \chi_s^* (y)
\]

Thus, we have $\chi_s^* (y) = 1$. Thus, $y \in T$. Hence, we have $T$ is a prime implicative filter of $A$.  

■
6.5 Conclusion

We have shown that $A_{\mu}$ is implicative filter in which $\mu$ is fuzzy implicative filter of lattice $W$-algebra $A$. By using this result, proved that $A_{\mu}$ is prime implicative filter of $A$, here $\mu$ is prime implicative filter. We discussed the equivalent condition that the set of fuzzy subset either empty or a fuzzy prime implicative filter and showed some properties of fuzzy prime implicative filter. Further, we proved the characteristic function is prime implicative filter.

In anti fuzzy implicative filter, we have investigated that the set of anti fuzzy prime implicative filter is either empty or fuzzy prime implicative filter. Finally, we studied that the complement characteristic function is anti fuzzy prime implicative filter of lattice $W$-algebra $A$. 