Chapter 2

Oscillation of First Order Neutral Delay Difference Equations with Positive and Negative Coefficients
SECTION I

(The content of this part has been published in the 'Far East Journal of Mathematical Sciences', Vol 41, Number 2, (2010), 217-231.)

2.1 Introduction

An equation which expresses the value $a_n$ of a sequence $\{a_n\}$ as a function of the term $a_{n-1}$ is called a first order difference equation. If there exists a function $f$ such that $a_n = f(n)$, $n = 1, 2, 3, \ldots$, then we will have solved the difference equation.

Given constants $\alpha$ and $\beta$ and a difference equation of the form

$$x_{n+1} = \alpha x_n + \beta, \ n = 0, 1, 2, 3, \ldots,$$

is called a first order linear difference equation.

Consider

$$x_n = \alpha x_{n-1} + \beta, \ n = 0, 1, 2, 3, \ldots$$

$$= \alpha (\alpha x_{n-2} + \beta) + \beta$$

$$= \alpha^2 (\alpha x_{n-3} + \beta) + \beta (\alpha + 1).$$
In general,

\[ x_n = \alpha^n x_0 + \beta(\alpha^{n-1} + \alpha^{n-2} + \ldots + 1). \]

Hence \( x_n = \alpha^n x_0 + \beta \left( \frac{1 - \alpha^n}{1 - \alpha} \right), \quad n = 0, 1, 2, 3, \ldots \) is the solution of the first order linear difference equation \( x_{n+1} = \alpha x_n + \beta \), when \( \alpha \neq 1 \).

When \( \alpha = 1 \), the solution is \( x_n = x_0 + n\beta, \quad n = 0, 1, 2, 3, \ldots \).

A first order difference equation is a recursively defined sequence in the form

\[ y_{n+1} = f(n, y_n), \quad n = 0, 1, 2, 3, \ldots \]

It also comes from the differential equation \( y' = g(n, y(n)) \), recalling the limit definition of the derivative this can be written as \( \lim_{n \to \infty} \frac{y(n + h) - y(n)}{h} \). If we think of \( h \) and \( n \) as integers, then the smallest \( h \) without 0 is 1. Now, the differential equation becomes

\[ y(n + 1) - y(n) = g(n, y(n)). \]

Now letting \( f(n, y(n)) = y(n) + g(n, y(n)) \) and putting into sequence notation gives \( f(n, y(n)) = y_{n+1} \). If the first order difference equation depends only on \( y_n \), then

\[ y_1 = f(y_0), \quad y_2 = f(y_1) = f^2(y_0). \]
Hence in general, \( y_n = f^n(y_0) \).

Non-linear difference equation over a set \( S \) is written in the form

\[
f_0(t)y_{t+k} + f_1(t)y_{t+k-1} + \ldots + f_n(t)y_t = g(t), \quad t = 0, 1, 2, 3, \ldots
\]

where \( f_0, f_1, f_2, \ldots, f_n \) and \( g \) are each functions of \( y_t \) defined over all values of \( t \) in the set \( S \). Solutions to nonlinear difference equations are rarely known and no general technique exists for their discovery.

Suppose a certain population of owls is growing at the rate of 2\% per year. Let \( x_0 \) represent the size of the initial population of owls and \( x_n \) the number of owls \( n \) years later.

Then

\[
x_{n+1} = x_n + 0.02x_n = 1.02x_n, \quad n = 0, 1, 2, 3, \ldots
\]

(2.1.1)

Hence the number of owls in any given year is equal to the number of owls in the previous year plus 2\% of the number of owls in the previous year. Equation (2.1.1) is an example of first order difference equation. It relates the number of owls in a given year with the number of owls in the previous year. Hence we know the value of a specific \( x_n \) once we know the value of \( x_{n-1} \). To get the sequence started we have to know the value of \( x_0 \). For example if initially we have a population of \( x_0 = 100 \) owls and we want to know what will be the population
after 4 years, we may compute \( x_1 = 1.02x_0 = 102, \)

\[
x_2 = 1.02x_1 = 104.04, \\
x_3 = 1.02x_2 = 106.12, \\
x_4 = 1.02x_3 = 108.24.
\]

Thus we would expect about 108 owls in the population after 4 years. We may work backwards to find \( x_4 \) explicitly in terms of \( x_0 \).

\[
x_4 = 1.02x_3 = (1.02)^2x_2 = (1.02)^3x_1 = (1.02)^4x_0.
\]

If we do this in general, then the solution of the difference equation \( x_{n+1} = 1.02x_n \) is \( x_n = (1.02)^nx_0 \). By replacing 1.02 with an arbitrary constant \( \alpha \), we get the general result that the solution of the difference equation

\[
x_{n+1} = \alpha x_n, \ n = 0, 1, 2, 3, ...
\]

is given by

\[
x_n = \alpha^n x_0, \ n = 0, 1, 2, 3, ...
\]

In this section, we deal with the oscillatory criteria for all solutions of first order neutral delay difference equations with positive and negative coefficients.
Here we consider the neutral delay difference equation of the form

\[ \Delta(x_n - c_n x_{n-k}) + p_n f(x_{n-l}) - q_n g(x_{n-m}) = 0, \]

where \( n \in N(n_0) \), \( n_0 \) is a non-negative integer and \( \Delta \) is the forward difference operator defined by \( \Delta y_n = y_{n+1} - y_n \). \( \{p_n\}, \{q_n\} \) are positive real sequences and \( \{c_n\} \) is a real sequence. \( f \) and \( g \) are continuous functions such that \( uf(u) \neq 0 \), \( ug(u) \neq 0 \), \( \forall u \neq 0 \) and \( k, l, m > 0 \).

In addition to the above, we assume the following:

1. \( q_{n+1-l-a_1+m} \geq q_n \), for \( a_1 \in N(n_0) \).
2. \( q_{n+1-l-a_2+m} \geq q_n \), for \( a_2 \in N(n_0) \).
3. \( \rho_k = \max(l, m) \).
4. \( H_{n-l+m} = q_{n+1-l-a_2+m} - q_n \).
5. \( (M_1 p_n - q_{n+m}) \geq 0 \) for \( M_1 > 0 \).
6. \( (M_2 q_n - p_{n-m+l}) \geq 0 \) for \( M_2 > 0 \).
7. \( p_n \geq p_{n+1-m-a_3+i} \), for \( a_3 \in N(n_0) \).
8. \( p_n \geq p_{n+1-m-a_4+i} \), for \( a_4 \in N(n_0) \).
9. \( R_{n+l-m} = p_n - p_{n-m+l-a_4+1} \).
10. \( m > \min(l + b_1, l + b_2) \) for \( b_1, b_2 \in N(n_0) \).
2.2. Some Basic Lemmas

(11) \( l > \min(m + d_1, m + d_2) \) for \( d_1, d_2 \in N(n_0) \).

Equation (2.1.2) is said to be oscillatory if all its solutions are oscillatory. In the past few years, there has been an increasing interest in the study of oscillatory and asymptotic behavior of solutions of difference equations. Following this trend, in this chapter, we obtain some sufficient conditions for the oscillation of all solutions of equation (2.1.2). Examples are provided to illustrate the main results.

2.2 Some Basic Lemmas

In this section, we present some lemmas to obtain the sufficient condition for the oscillations of all solutions of equation (2.1.2).

Let us consider the following cases:

(i) \( g(u) = u, \frac{f(u)}{u} \leq M_1 \).

(ii) \( f(u) = u, \frac{g(u)}{u} \geq M_2 \).

**Case I:** Let \( g(u) = u, \frac{f(u)}{u} \leq M_1 \).

Then equation (2.1.2) becomes

\[
\Delta(x_n - c_n x_{n-k}) + p_n f(x_{n-l}) - q_n x_{n-m} = 0. \tag{2.2.1}
\]

Let \( z_n = x_n - c_n x_{n-k} + \sum_{s=n-m}^{n-l-1} q_s x_s \).
Then

\[
\Delta z_n = \Delta (x_n - c_n x_{n-k}) + \Delta (q_n x_{n-m} + \ldots + q_{n+m-l-1} x_{n-l-1})
\]

\[
= \Delta (x_n - c_n x_{n-k}) - q_n x_{n-m} + q_{n+m-l} x_{n-l}
\]

\[
= -p_n f(x_{n-1}) + q_{n+m-l} x_{n-l}
\]

\[
= x_{n-l} \left( -\frac{p_n f(x_{n-l})}{x_{n-l}} + q_{n+m-l} \right)
\]

\[
\geq x_{n-l} (-p_n M_1 + q_{n+m-l}).
\]

Hence,

\[
\Delta z_n + x_{n-l} [M_1 p_n - q_{n+m-l}] \geq 0.
\]

**Lemma 2.2.1 [36]** Suppose there exists a real number \(a_1 \in N(n_0)\) such that

\[
z_{a_1}(n) = c_n - \sum_{s=n-m}^{n-l-a_1} q_{s+m} \geq \frac{c_n}{2}. \tag{2.2.2}
\]

Let \(x_n\) be an eventually negative solution of the difference inequality

\[
\Delta z_n + x_{n-l} [M_1 p_n - q_{n+m-l}] \geq 0 \tag{2.2.3}
\]
and \( z_n \) is defined by

\[
z_n = x_n - c_n x_{n-k} + \sum_{s=n-m}^{n-l-a_1} q_{s+m} x_s. \tag{2.2.4}
\]

Then eventually \( \Delta z_n \geq 0 \) and \( z_n \leq 0 \).

**Proof**

To prove \( \Delta z_n \geq 0 \).

\[
\Delta z_n = \Delta (x_n - c_n x_{n-k}) + \Delta \left( \sum_{s=n-m}^{n-l-a_1} q_{s+m} x_s \right)
\]

\[
\Delta z_n = \Delta (x_n - c_n x_{n-k}) + (q_{n+1-l-a_1+m} x_{n-l-a_1+1} - q_n x_{n-m}) \geq 0.
\]

To prove \( z_n \leq 0 \).

Suppose \( z_n > 0 \), there exists \( \mu > 0 \) such that \( z_n \geq \mu \).

From the equation \( 2.2.4 \),

\[
x_n \geq \mu + c_n x_{n-k} - \sum_{s=n-m}^{n-l-a_1} q_{s+m} x_s.
\]

Let us consider two possible cases:

(i) \( x_n \) is unbounded, then \( \lim_{n \to \infty} x_n = \infty \). Hence there exists a real sequence \( \{ \nu_i \}_{i=1}^{\infty} \) such that \( x(\nu_i) \to \infty \) as \( i \to \infty \).
2.2. Some Basic Lemmas

Let \( x(\nu_i) = \frac{c_n}{2} \min x_n \).

\[
x(\nu_i) \geq \mu + c_n x_{n-k} - \sum_{s=n-m}^{n-l-a_1} q_{s+m} x_s
\]

\[
x_n \geq \mu + \left( c_n - \sum_{s=n-m}^{n-l-a_1} q_{s+m} \right) x(\nu_i) \frac{2}{c_n}
\]

\[
x_n \geq \mu + \frac{c_n}{2} x(\nu_i) \frac{2}{c_n}
\]

\[
x_n \geq \mu + x(\nu_i).
\]

This is a contradiction. Hence \( z_n \leq 0 \).

(ii) Suppose \( x_n \) is bounded, then \( \lim_{n \to \infty} \sup x_n = L \). Let \( \{\bar{\nu}_i\}_{i=1}^{\infty} \) be a real sequence such that \( \bar{\nu}_i \to \infty \) and \( x(\bar{\nu}_i) \to L \) as \( i \to \infty \).

Let

\[
x(\bar{\nu}_i) = \frac{c_n}{2} \min(x_n) \geq \mu + c_n x_{n-k} - \sum_{s=n-m}^{n-l-a_1} q_{s+m} x_s
\]

\[
\geq \mu + \left( c_n - \sum_{s=n-m}^{n-l-a_1} q_{s+m} \right) x(\bar{\nu}_i) \frac{2}{c_n}
\]

\[
\geq \mu + \frac{c_n}{2} x(\bar{\nu}_i) \frac{2}{c_n}
\]

\[
\geq \mu + x(\bar{\nu}_i)
\]

\[
\sup x(\bar{\nu}_i) \geq \sup(\mu + x(\bar{\nu}_i)).
\]

\( L \geq \mu + L \), which is a contradiction. Hence \( z_n \leq 0 \).
Lemma 2.2.2[36]

Suppose there exists a real number $a_2 \in N(n_0)$ such that

$$z_{a_2}(n) = c_n - \sum_{s=n-m}^{n-1-a_2} q_{s+m} \leq \frac{c_n}{2}. \quad (2.2.5)$$

Let $x_n$ be an eventually negative solution of the equation (2.2.3) and

$$z_n = x_n - c_n x_{n-k} - \sum_{s=n-m}^{n-1-a_2} q_{s+m} x_s. \quad (2.2.6)$$

If the second order difference inequality

$$\Delta^2 x_n - \frac{1}{\rho_k} H_{n-l+m} x_n \geq 0 \quad (2.2.7)$$

doesn’t have eventually negative solution, then $\Delta z_n \geq 0$ and $z_n \geq 0$.

Proof:

Case I: To prove $\Delta z_n \geq 0$.

$$\Delta z_n = \Delta (x_n - c_n x_{n-k}) + \Delta \left( \sum_{s=n-m}^{n-1-a_2} q_{s+m} x_s \right)$$

$$\Delta z_n = \Delta (x_n - c_n x_{n-k}) + (q_{n+1-l-a_2+m} x_{n-l-a_2+1} - q_n x_{n-m}).$$

Hence $\Delta z_n \geq 0$.

To prove $z_n \geq 0$. 
2.2. Some Basic Lemmas

Suppose \( z_n < 0 \), there exists \( T < \rho_k \) such that \( x_n < 0 \).

Let \( M_1 = \max(x_n) \frac{c_n}{2} \). Since

\[
\begin{align*}
z_n &= x_n - c_n x_{n-k} + \sum_{s=n-m}^{n-l-a_2} q_{s+m} x_s, \\
x_n &= z_n + c_n x_{n-k} - \sum_{s=n-m}^{n-l-a_2} q_{s+m} x_s \\
&< c_n x_{n-k} - \sum_{s=n-m}^{n-l-a_2} q_{s+m} x_s, \text{ since } z_n < 0 \\
x_\tau &= c_\tau x_{\tau-k} - \sum_{s=\tau-m}^{\tau-l-a_2} q_{s+m} x_s \\
&\leq \frac{2}{c_\tau} M_1 (c_\tau - \sum_{s=\tau-m}^{\tau-l-a_2} q_{s+m} x_s) \\
&\leq \frac{2}{c_n} M_1 \frac{c_n}{2} \\
&\leq M_1.
\end{align*}
\]

By induction, we can prove \( x_n \leq M_1 \) for \( T + n \rho_k \leq n \leq T + (n + 1) \rho_k \).

Let \( \lim_{n \to \infty} z_n = \alpha \). Then there exist two possible cases:

Case (i) \( \alpha = 0 \).

Let \( T_1 < T \) such that \( z_n \geq \frac{M_1}{2} \), \( x_n \leq \frac{1}{\rho_k} \sum_{T_1}^{n+\rho_k} z_n \).

Case (ii) \( \alpha > 0 \).

Since \( \Delta z_n \geq 0 \) we have, \( z_n \leq \alpha \)

\[
x_n \leq \alpha + c_n x_{n-k} - \sum_{s=n-m}^{n-l-a_2} q_{s+m} x_s
\]
2.2. Some Basic Lemmas

\[ x_n \leq \alpha + \left( c_n - \sum_{s=n-m}^{n-l-a_2} q_{s+m} \right) x_s \]

\[ x_n \leq \alpha + M_1. \]

In general, \( x_n \leq n\alpha + M_1 \), \( \lim_{n \to \infty} x_n = \infty \). Hence there exists \( T_2 < T_1 \) such that

\[ x_n \leq \frac{1}{\rho_k} \sum_{T_2}^{n+\rho_k} z_n. \]

Combining both cases, there exists \( T^* < T_2 \) such that \( x_n \leq \frac{1}{\rho_k} \sum_{T^*}^{n+\rho_k} z_n. \)

\[ x_n = z_n + c_n x_{n-k} - \sum_{s=n-m}^{n-l-a_2} q_{s+m} x_s \]

\[ \leq z_n + \left( c_n - \sum_{s=n-m}^{n-l-a_2} q_{s+m} \right) \frac{1}{\rho_k} \sum_{T^*}^{n} z_n \]

\[ \leq \frac{1}{\rho_k} \sum_{n}^{n+\rho_k} z_n + \frac{1}{\rho_k} \sum_{T^*}^{n} z_n \]

\[ \leq \frac{1}{\rho_k} \sum_{n}^{n+\rho_k} z_n. \]

By induction, \( x_n \leq \frac{1}{\rho_k} \sum_{n}^{n+\rho_k} z_n. \)

Let \( y_n = \sum_{T^*}^{n+\rho_k} z_n \). Since \( z_n < 0, y_n < 0, \Delta y_n = z_n, \Delta^2 y_n = \Delta z_n. \)

\[ \Delta z_n \geq (q_{n+1-l-a_2+m} - q_n) x_{n-m} \]

\[ \Delta z_n \geq (H_{n-l+m}) \frac{1}{\rho_k} \sum_{T^*}^{n+\rho_k} z_n \]
2.2. Some Basic Lemmas

\[
\Delta z_n - (H_{n-l+m}) \frac{1}{\rho_k} y_n \geq 0
\]

\[
\Delta^2 y_n - (H_{n-l+m}) \frac{1}{\rho_k} y_n \geq 0.
\]

This contradicts the given condition of lemma 2.2.2. Hence \(z_n \geq 0\).

**Case II**

Let \(f(u) = u; \frac{g(u)}{u} \geq M_2\). Then equation (2.1.2) becomes

\[
\Delta(x_n - c_n x_{n-k}) + p_n x_{n-l} - q_n g(x_{n-m}) = 0. \tag{2.2.8}
\]

Let \(z_n = x_n - c_n x_{n-k} - \sum_{s=n-l}^{n-m-1} p_{s+l} x_s\). Then

\[
\Delta z_n = \Delta(x_n - c_n x_{n-k}) - \Delta(p_n x_{n-l} + \ldots + p_{n-m+l-1} x_{n-m-1})
\]

\[
= \Delta(x_n - c_n x_{n-k}) - p_{n-m+l} x_{n-m} + p_n x_{n-l}
\]

\[
= q_n g(x_{n-m}) - p_{n-m+l} x_{n-m}
\]

\[
= x_{n-m} \left( \frac{q_n g(x_{n-m})}{x_{n-m}} - p_{n-m+l} \right).
\]

Hence \(\Delta z_n - x_{n-m} [M_2 q_n - p_{n-m+l}] \geq 0\).

**Lemma 2.2.3[36]**

Suppose there exists a real number \(a_3 \in N(n_0)\) such that

\[
z_{a_3}(n) = c_n + \sum_{s=n-l}^{n-m-a_3} p_{s+l} \geq \frac{c_n}{2}. \tag{2.2.9}
\]
2.2. Some Basic Lemmas

Let $x_n$ be an eventually negative solution of the difference inequality

$$\Delta z_n - x_{n-m}[M_2q_n - p_{n-m+l}] \geq 0, \quad (2.2.10)$$

and $z_n$ is defined by

$$z_n = x_n - c_n x_{n-k} - \sum_{s=n-l}^{n-m-a_3} p_{s+l} x_s. \quad (2.2.11)$$

Then eventually $\Delta z_n \geq 0$ and $z_n \leq 0$.

Proof

To prove $\Delta z_n \geq 0$.

$$\Delta z_n = \Delta(x_n - c_n x_{n-k}) - \Delta \left( \sum_{s=n-l}^{n-m-a_3} p_{s+l} x_s \right)$$

$$= \Delta(x_n - c_n x_{n-k}) + (p_n x_{n-l} - p_{n-m-a_3+l+1} x_{n-m-a_3} + 1)$$

$$\geq 0.$$

To prove $z_n \leq 0$.

Suppose $z_n > 0$, there exists $\mu > 0$ such that $z_n \geq \mu$. From the equation (2.2.10),

$$x_n \geq \mu + c_n x_{n-k} + \sum_{s=n-l}^{n-m-a_3} p_{s+l} x_s.$$

Let us consider two possible cases:
2.2. Some Basic Lemmas

(i) Suppose \( x_n \) is unbounded, then \( \lim_{n \to \infty} x_n = \infty. \)

Hence there exists a real sequence \( \{\nu_i\}_{i=1}^\infty \) such that \( x(\nu_i) \to \infty \) as \( i \to \infty. \)

Let

\[
x(\nu_i) = \frac{c_n}{2} \min(x_n)
\geq \mu + c_n x_{n-k} + \sum_{s=n-l}^{n-m-a_3} p_{s+l} x_s
\geq \mu + \left( c_n + \sum_{s=n-l}^{n-m-a_3} p_{s+l} \right) x_s
\geq \mu + \frac{c_n}{2} x(\nu_i) \frac{2}{c_n}
\geq \mu + x(\nu_i)
\]

which is a contradiction. Hence \( z_n \leq 0. \)

(ii) Suppose \( x_n \) is bounded, then \( \lim_{n \to \infty} \sup x_n = L. \) Let \( \{\nu_i\}_{i=1}^\infty \) be a real sequence such that \( \nu_i \to \infty \) and \( x(\nu_i) \to L \) as \( i \to \infty. \)

Let

\[
x(\nu_i) = \frac{c_n}{2} \min(x_n) \geq \mu + c_n x_{n-k} + \sum_{s=n-l}^{n-m-a_3} p_{s+l} x_s
\geq \mu + \left( c_n + \sum_{s=n-l}^{n-m-a_3} p_{s+l} \right) x_s
\geq \mu + \frac{c_n}{2} x(\nu_i) \frac{2}{c_n}
\geq \mu + x(\nu_i)
\]

\[
\sup x(\nu_i) \geq \sup(\mu + x(\nu_i)).
\]
2.2. Some Basic Lemmas

Hence \( L \geq \mu + L \), which is a contradiction. Hence \( z_n \leq 0 \).

**Lemma 2.2.4[36]** Suppose there exists a real number \( a_4 \in N(n_0) \) such that

\[
  z_{a_4}(n) = c_n + \sum_{s=n-l}^{n-m-a_4} p_{s+l} \leq \frac{c_n}{2}. \quad (2.2.12)
\]

Let \( x_n \) be an eventually negative solution of the equation (2.2.10) and

\[
  z_n = x_n - c_n x_{n-k} + \sum_{s=n-l}^{n-m-a_4} p_{s+l} x_s.
\]

If the second order difference inequality

\[
  \Delta^2 x_n - \frac{1}{\rho_k} R_{n-l+m} x_n \geq 0 \quad (2.2.13)
\]

does not have eventually negative solution, then \( \Delta z_n \geq 0 \) and \( z_n \geq 0 \).

**Proof**

To prove \( \Delta z_n \geq 0 \).

\[
  \Delta z_n = \Delta (x_n - c_n x_{n-k}) - \Delta \left( \sum_{s=n-l}^{n-m-a_4} p_{s+l} x_s \right)
\]

\[
  \Delta z_n = \Delta (x_n - c_n x_{n-k}) + (p_n x_{n-l} - p_{n+1+l-a_4-m} x_{n-m-a_4+1})
\]

\[
  \Delta z_n \geq 0. \quad (2.2.14)
\]
To prove $z_n \geq 0$.

Suppose $z_n < 0$, there exists $T < \rho_k$, such that $x_n < 0$.

$$M_2 = \frac{c_n}{2} \max(x_n).$$

Then

$$z_n = x_n - c_n x_{n-k} - \sum_{s=n-l}^{n-m-a_4} p_{s+l} x_s$$

$$x_T \leq \frac{2}{c_n} M_2 \left( c_T + \sum_{s=T-l}^{T-m-a_4} p_{s+l} \right) x_s$$

$$x_T \leq \frac{2}{c_n} M_2 \frac{c_n}{2}$$

$$x_T \leq M_2.$$

By induction, we can prove $x_n \leq M_2$ for $T + n\rho_k \leq n \leq T + (n+1)\rho_k$.

Let $\lim_{n \to \infty} z_n = \alpha$. Then there exists two possible cases:

**Case (i) $\alpha = 0$.**

Let $T_1 < T$ such that $z_n \geq \frac{M_2}{2}$, $x_n \leq \frac{1}{\rho_k} \sum_{T_1}^{n+\rho_k} z_n$.

**Case (ii) $\alpha > 0$.**

Since $\Delta z_n \geq 0$ we have $z_n \leq \alpha$. 

$$x_n \leq \alpha + c_n x_{n-k} + \sum_{s=n-l}^{n-m-a_4} p_{s+l} x_s$$
\[ x_n \leq \alpha + \left( c_n + \sum_{s=n-l}^{n-m-a_4} p_{s+l} \right) x_s \]
\[ x_n \leq \alpha + M_2. \]

In general, \( x_n \leq n\alpha + M_2 \), \( \lim_{n \to \infty} x_n = \infty \). Hence there exists \( T_2 < T_1 \) such that
\[ x_n \leq \frac{1}{\rho_k} \sum_{T_2}^{n+\rho_k} z_n. \]

Combining both cases, there exists \( T^* < T_2 \) such that \( x_n \leq \frac{1}{\rho_k} \sum_{T^*}^{n+\rho_k} z_n. \)

Let \( y_n = \sum_{T^*}^{n+\rho_k} z_n. \)

Since \( z_n < 0 \), \( y_n < 0 \), \( \Delta y_n = z_n \), \( \Delta^2 y_n = \Delta z_n \).

From the equation (2.2.14),
\[ \Delta z_n \geq (-p_{n+1+l-a_4-m} + p_n)x_n \]
\[ \Delta z_n \geq -(R_{n+l-m}) \left( -\frac{1}{\rho_k} \sum_{T^*}^{n+\rho_k} z_n \right) \]
\[ \Delta z_n - (R_{n+l-m}) \frac{1}{\rho_k} y_n \geq 0 \]
\[ \Delta^2 y_n - (R_{n+l-m}) \frac{1}{\rho_k} y_n \geq 0. \]

This is a contradiction to the given condition of lemma 2.2.4.

Hence \( z_n \geq 0. \)
2.3 Some Oscillatory Results

Theorem 2.3.1

Assume that there exist two real numbers \( b_1, b_2 \in N(n_0) \) such that

\[
z_{b_1}(n) = c_n - \sum_{s=n-m}^{n-l-b_1} q_{s+m} \geq \frac{c_n}{2} \tag{2.3.1}
\]

\[
z_{b_2}(n) = c_n - \sum_{s=n-m}^{n-l-b_2} q_{s+m} \leq \frac{c_n}{2} \tag{2.3.2}
\]

Also assume that the equation (2.2.7) does not have eventually negative solution, then every solution of the equation (2.2.1) oscillates.

Proof

By lemma 2.2.1 and the equation (2.3.1), \( z_n \) is eventually negative. By lemma 2.2.2 and the equation (2.3.2), \( z_n \) is eventually positive. The equation (2.2.1) cannot have both eventually positive solution and eventually negative solution. Hence every solution of the equation (2.2.1) oscillates.

Theorem 2.3.2

Assume that there exist two real numbers \( d_1, d_2 \in N(n_0) \) such that

\[
z_{d_1}(n) = c_n + \sum_{s=n-l}^{n-m-d_1} p_{s+l} \geq \frac{c_n}{2}, \tag{2.3.3}
\]

\[
z_{d_2}(n) = c_n + \sum_{s=n-l}^{n-m-d_2} p_{s+l} \leq \frac{c_n}{2}. \tag{2.3.4}
\]

Also assume that the equation (2.2.13) does not have eventually negative so-
lution and then every solution of the equation (2.2.8) oscillates.

Proof

By lemma 2.2.3 and the equation (2.3.3), $z_n$ is eventually negative. By lemma 2.2.4 and the equation (2.3.4), $z_n$ is eventually positive. The equation (2.2.8) cannot have both eventually positive solution and eventually negative solution. Hence every solution of the equation (2.2.8) oscillates.

2.4 Examples

Example 2.4.1

Consider the first order neutral delay difference equation

$$\Delta(x_n - nx_{n-2}) + \frac{9}{4}(n-1)x_{n-1}^3 - \left(\frac{n-1}{4}\right)x_{n-3} = 0, \quad n > 1. \tag{2.4.1}$$

Here

$$k = 2, \ l = 1, \ m = 3, \ c_n = n, \ p_n = \frac{9(n-1)}{4}, \ q_n = \frac{n-1}{4}.$$ 

Let $b_1 = 1, \ b_2 = 0$. Then it is easy to see that all the conditions of Theorem 2.3.1 are satisfied. Hence all the solutions of the equation (2.4.1) are oscillatory.
Example 2.4.2

Consider the first order neutral delay difference equation

\[
\Delta \left( x_n - \left( \frac{10 - 4n}{n^2 - 5n + 6} \right) x_{n-1} \right) + \left( \frac{1}{n-3} \right) x_{n-3} \\
- \left( \frac{2n^3 - 16n^2 + 26n + 2}{(n-1)^3(n^2 - 5n + 6)} \right) x_{n-1}^3 = 0, \quad n > 3. \quad (2.4.2)
\]

Here

\[ k = 1, \quad l = 3, \quad m = 1, \quad p_n = \frac{1}{n-3}, \quad q_n = \frac{2n^3 - 16n^2 + 26n + 2}{(n-1)^3(n^2 - 5n + 6)}, \quad c_n = \frac{10 - 4n}{n^2 - 5n + 6}. \]

Let \( d_1 = 0, \ d_2 = 1. \)

It is easy to see that all the conditions of the Theorem 2.3.2 are satisfied. Hence all the solutions of equation (2.4.2) are oscillatory.
SECTION II

(The content of this section has been published in the journal, 'Bulletin of Pure and Applied Sciences', Vol 30 E (Maths & Stat.) No.1, (2011), 95-100)

2.5 Introduction

In this section oscillation behavior of first order neutral delay difference equations with positive and negative coefficients of the terms involving delay is well developed. First order neutral delay difference equations are gaining interest because they are the discrete analogue of differential equations. They also have physical applications as evident by F. Weil[84]. Keeping this fact in view, an attempt is made here to study the first order neutral delay difference equations with positive and negative coefficients. However we find a very few works about the oscillatory behavior of solutions of first order neutral delay difference equations. Here our object is achieved by using the assumption.

Here we consider the first order neutral delay difference equation (2.1.2), \( \{p_n\} \), \( \{q_n\} \) are positive sequences and \( \{c_n\} \) is a real sequence. \( f \) and \( g \) are continuous functions such that \( uf(u) \neq 0, ug(u) \neq 0, \forall u \neq 0 \) and \( k, l, m > 0 \). In addition to the above, we assume the following:

\[
A_1 : H_{n-a+l-m} = q_{n-2a-m} + p_n + p_{n-a+m} - q_n - q_{n-a-m} - p_{n+l+2a+1-2m} \quad \text{and} \quad H_{n-a+l-m} \geq 0.
\]

\[
A_2 : c_{n-a-m} H_{n-a+l-m} \leq h_1, \quad q_{n-m} H_{n-a+l-m} \leq h_2, \quad p_{n-2a+1-2m} H_{n-a+l-m} \geq h_2,
\]
2.6 Some Basic Lemmas

Lemma 2.2.1(I)

Suppose there exists a real number \( a \in N(n_0) \) such that

\[
z_a(n) = c_n - \sum_{s=n-2a-m}^{n-m-a} q_s x_{s-m} - \sum_{t=n-a-m}^{n-2m-2a+l} p_t x_{t-l} \geq 1.
\]

Let \( x_n \) be an eventually negative solution of the difference inequality,

\[
\Delta(x_n - c_n x_{n-k}) + p_n x_{n-l} - q_n x_{n-m} \geq 0, \tag{2.2'}
\]

where \( z_n = x_n - c_n x_{n-k} + \sum_{s=n-2a-m}^{n-m-a} q_s x_{s-m} + \sum_{t=n-a-m}^{n-2m-2a+l} p_t x_{t-l}. \tag{2.2''} \)

Then eventually \( \Delta z_n \geq 0 \) and \( z_n \leq 0 \).

Proof

To prove \( \Delta z_n \geq 0 \):

\[
\Delta z_n = \Delta(x_n - c_n x_{n-k}) + \Delta \left( \sum_{s=n-2a-m}^{n-m-a} q_s x_{s-m} \right) + \Delta \left( \sum_{t=n-a-m}^{n-2m-2a+l} p_t x_{t-l} \right)
\]
2.6. Some Basic Lemmas

\[ \Delta z_n = \Delta(x_n - c_n x_{n-k}) + q_{n-a-m+1} x_{n-a-2m+1} - q_{n-2a-m} x_{n-2a-2m} + p_{n-2a-2m+l+1} x_{n-2a-2m+1} - p_{n-a-m} x_{n-a-m-l} \]

\[ \Delta z_n \geq -p_n x_{n-l} + q_n x_{n-m} + q_{n-a-m+1} x_{n-a-2m+1} - q_{n-2a-m} x_{n-2a-2m} + p_{n-2a-2m+l+1} x_{n-2a-2m+1} - p_{n-a-m} x_{n-a-m-l} \]

\[ \Delta z_n \geq -H_{n-2a-l-m} x_{n-a-m} \quad (2.2.3') \]

\[ \Delta z_n \geq 0. \quad (2.2.4') \]

To prove \( z_n \leq 0 \):

Suppose \( z_n > 0 \), there exists \( \mu > 0 \) such that \( z_n > \mu \).

From the equation (2.2.2'),

\[ x_n > \mu + c_n x_{n-k} - \sum_{s=n-2a-m}^{n-m-a} q_s x_{s-m} - \sum_{t=n-a-m}^{n-2a-l} p_t x_{t-l}. \]

Let us consider two possible cases.

(i) \( x_n \) is unbounded, then \( \lim_{n \to \infty} x_n = \infty \).

Hence there exists a real sequence \( \{\nu_i\}, \ i = 1, 2, \ldots \) such that \( x(\nu_i) \to \infty \) as \( i \to \infty \).
Let \( x(\nu_i) = \min(x_n) \).

\[
\min(x_n) \geq \mu + c_n x_{n-k} - \sum_{s=n-2a-m}^{n-m-a} q_s x_{s-m} - \sum_{t=n-a-m}^{n-2m-2a+l} p_t x_{t-l}
\]

\[
x(\nu_i) \geq \mu + \left( c_n - \sum_{s=n-2a-m}^{n-m-a} q_s - \sum_{t=n-a-m}^{n-2m-2a+l} p_t \right) x(\nu_i)
\]

\[
x(\nu_i) \geq \mu + x(\nu_i), \text{ which is a contradiction since } \mu > 0.
\]

Hence \( z_n \leq 0 \).

(ii) Suppose \( x_n \) is bounded, then \( \limsup_{n \to \infty} x_n = L \).

Let \( \{\nu_i\}, i = 1, 2, \ldots \) be a real sequence such that \( x(\nu_i) \to L \) as \( i \to \infty \).

Let \( x(\nu_i) = \min(x_n) \).

\[
x(\nu_i) \geq \mu + c_n x_{n-k} - \sum_{s=n-2a-m}^{n-m-a} q_s x_{s-m} - \sum_{t=n-a-m}^{n-2m-2a+l} p_t x_{t-l}
\]

\[
x(\nu_i) \geq \mu + x(\nu_i)
\]

\[
\sup x(\nu_i) \geq \sup(\mu + x(\nu_i)).
\]

Then \( L \geq \mu + L \), which is a contradiction.

Hence \( z_n \leq 0 \).

**Lemma 2.2.2(I)** Suppose there exists a real number \( b \in N(n_0) \) such that,

\[
z_b(n) = c_n - \sum_{s=n-2b-m}^{n-m-b} q_s x_{s-m} - \sum_{t=n-b-m}^{n-2m-2b+l} p_t x_{t-l} \leq 1.
\]
2.6. Some Basic Lemmas

Let $x_n$ be an eventually negative solution of the equation (2.2.1') and let

$$z_n = x_n - c_n x_{n-k} + \sum_{s=n-2b-m}^{n-m-b} q_s x_{s-m} + \sum_{t=n-b-m}^{n-2m-2b+l} p_t x_{t-l}.$$ 

If the second order difference inequality $\Delta^2 y_n + H_{n-b+l-m} x_{n-b-m} \geq 0$ doesn’t have eventually negative solution, then $\Delta z_n \geq 0$ and $z_n \geq 0$.

**Proof**

Suppose $z_n < 0$, there exists $T < \rho_k$ such that $x_n < 0$.

$$z_n = x_n - c_n x_{n-k} + \sum_{s=n-2b-m}^{n-m-b} q_s x_{s-m} + \sum_{t=n-b-m}^{n-2m-2b+l} p_t x_{t-l}.$$ 

Hence we have,

$$x_n = z_n + c_n x_{n-k} - \sum_{s=n-2b-m}^{n-m-b} q_s x_{s-m} - \sum_{t=n-b-m}^{n-2m-2b+l} p_t x_{t-l}.$$ 

$$x_n < c_n x_{n-k} - \sum_{s=n-2b-m}^{n-m-b} q_s x_{s-m} - \sum_{t=n-b-m}^{n-2m-2b+l} p_t x_{t-l}.$$ 

Let $M_1 = \max(x_n)$.

$$x_T \leq M_1 \left( c_n - \sum_{s=n-2b-m}^{n-m-b} q_s x_{s-m} - \sum_{t=n-b-m}^{n-2m-2b+l} p_t x_{t-l} \right)$$ 

$$x_T \leq M_1.$$ 

By induction we can prove $x_n \leq M_1$ for $T + n\rho_k \leq n \leq T + (n+1)\rho_k$. 

Let $\lim_{n \to \infty} z_n = \alpha$.

i) Let $\alpha = 0$. Then there exists $T_1 < T$ such that $z_n \geq M_1$.

$$x_n \leq \frac{1}{\rho_k} \sum_{T_1}^{n+\rho_k} z_n.$$ 

ii) Let $\alpha > 0$. Since $\Delta z_n \geq 0$, we have $z_n \leq \alpha$.

$$x_n \leq \alpha + c_n x_{n-k} - \sum_{s=n-2b-m}^{n-m-b} q_s x_{s-m} - \sum_{t=n-b-m}^{n-2m-2b+l} p_t x_{t-l}$$

$$x_n \leq \alpha + M_1 \left( c_n - \sum_{s=n-2b-m}^{n-m-b} q_s x_{s-m} - \sum_{t=n-b-m}^{n-2m-2b+l} p_t x_{t-l} \right)$$

$$x_n \leq \alpha + M_1.$$ 

In general, $x_n \leq n\alpha + M_1$.

Hence we have $\lim_{n \to \infty} x_n = \infty$.

There exists $T_2 < T_1$ such that

$$x_n \leq \frac{1}{\rho_k} \sum_{T_2}^{n+\rho_k} z_n.$$ 

Combining both cases there exists $T^* < T_2$ such that

$$x_n \leq \frac{1}{\rho_k} \sum_{T^*}^{n+\rho_k} z_n.$$
2.6. Some Basic Lemmas

Let \( y_n = \sum_{T^*}^{n+\rho_k} z_n \), since \( z_n < 0 \), we have \( y_n < 0 \).

Hence

\[
\Delta y_n = \Delta \left[ \sum_{T^*}^{n+\rho_k} z_n \right] = z_{n+\rho_k+1} - z_{T^*},
\]

since \( z_{T^*} \) is negligible and \( z_{n+\rho_k+1} = z_n \).

Therefore,

\[
\Delta^2 y_n = \Delta (\Delta y_n) = \Delta (z_{n+\rho_k+1} - z_{T^*})
= (z_{n+\rho_k+2} - z_{T^*} - z_{n+\rho_k+1})
= (z_{n+1+\rho_k+1} - z_{n+\rho_k+1}) \text{ since } z_{T^*} \text{ is negligible and } z_{n+\rho_k+1} = z_n
= z_{n+1} - z_n.
\]

Hence we have, \( \Delta^2 y_n = \Delta z_n \).

From the equation (2.2.3'), \( \Delta z_n \geq -H_{n-b+l-m}x_{n-b-m} \).

Hence \( \Delta^2 y_n \geq -H_{n-b+l-m}x_{n-b-m} \).

That is, \( \Delta^2 y_n + H_{n-b+l-m}x_{n-b-m} \geq 0 \).

This is a contradiction to the given condition of lemma 2.2.2(I). Hence \( z_n \geq 0 \).
2.7 Some Oscillatory Results

Theorem 2.3.1(I)

Assume that \( \left\{ \frac{q_{n-a-m}}{H_{n-a+l-2m}} \right\} \) and \( \left\{ \frac{p_{n-2a+l-2m}}{H_{n-2a+l-2m}} \right\} \) are decreasing sequences and A1, A2 holds. Then every solution of the equation (2.1.2) is oscillatory.

Proof

Suppose to the contrary that equation (2.1.2), has eventually negative solution. Then from the lemma 2.2.2(I),

\[
\Delta z_n \geq 0 \text{ and } z_n \geq 0. \quad (2.3.1')
\]

Now,

\[
\Delta z_n = \Delta (x_n - c_n x_{n-k}) + \sum_{s=n-2a-m}^{n-m-a} q_s x_{s-m} + \sum_{t=n-a-m}^{n-2m-2a+l} p_t x_{t-l}
\]

\[
\Delta \hat{z}_n = -p_n x_{n-l} + q_n x_{n-m} + q_{n-a-m+1} x_{n-a-2m+1} - p_{n-a-m} x_{n-a-m-1} + q_{n-2a-m} x_{n-2a-2m} + p_{n+1-2a+l-2m} x_{n-2a-2m+1}
\]

\[
\Delta \hat{z}_n \geq -H_{n-a+l-m} x_{n-a-m}.
\]

\[
\Delta z_n \geq -H_{n-a+l-m}
\]

\[
\cdot \left[ z_{n-a-m} + c_{n-a-m} x_{n-a-m-k} + \sum_{s=n-2a-m}^{n-m-a} q_s x_{s-m} + \sum_{t=n-a-m}^{n-2m-2a+l} p_t x_{t-l} \right]
\]
\[
\Delta z_n \geq -H_{n-a+l-m} z_{n-a-m} - h_1 H_{n-a-m+l} x_{n-a-m-k} \\
\quad - H_{n-a+l-m} \sum_{s=n-2a-m}^{n-m-a} q_s \frac{H_{s+l-m}}{H_{s+l-m}} x_{s-m} - H_{n-a+l-m} \sum_{t=n-a-m}^{n-2m-2a+l} p_t \frac{H_{t+l-m}}{H_{t+l-m}} x_{t-l}
\]

\[
\Delta z_n \geq -H_{n-a+l-m} z_{n-a-m} + h_1 \Delta z_{n-k} + H_{n-a-m+l} \frac{q_{n-a-m}}{H_{n-a+l-2m}} (z_n - z_{n-a-m}) \\
\quad + H_{n-a+l-m} \frac{p_{n-2a+l-2m}}{H_{n-2a+l-2m}} (z_{n-a-m} - z_{n-l})
\]

\[
\Delta z_n \geq -H_{n-a+l-m} z_{n-a-m} + h_1 \Delta z_{n-k} + h_2 (z_{n-m} - z_{n-a-m}) + h_2 (z_{n-a-m} - z_{n-l})
\]

\[
\Delta z_n \geq -H_{n-a+l-m} z_{n-l} + h_1 \Delta z_{n-k} + h_2 (z_{n-m} - z_{n-a-m}) \\
\quad + h_2 (z_{n-a-m} - z_{n-l})
\]

That is,

\[
(\Delta z_n - h_1 z_{n-k}) + (H_{n-a+l-m} + h_2) z_{n-l} - h_2 z_{n-m} \geq 0.
\]

Hence by lemma 2.2.1(1), \(z_n\) is eventually negative solution. This is a contradiction to the equation \((2.3.1')\). Hence every solution of the equation \((2.1.2)\) oscillates.