CHAPTER 2

MATHEMATICAL MODELLING OF THREE PHASE INDUCTION MOTOR

2.1 INTRODUCTION

The conventional methods of controlling the speed of a three phase induction motor are replaced by static power controllers, mainly converters, which utilize thyristors. The behavior of the induction motor fed from static converters is different from the behavior of motor operating on sinusoidal supply. A study of the transient as well as the steady state behavior of converter fed motor is useful as it provides necessary information regarding voltage and current of the converter which solely determines the efficient and optimum selection of thyristors and communication circuit elements. A dynamic model of the machine which is valid for any instantaneous variation of voltage and current and adequately describing the performance of the machine under steady state and transient operating conditions can be obtained by representing the machine voltages and currents in terms of actual phase variables or in terms of two axis variables or using space phasor theory. The three phase motor is a balanced load because of the absence of zero sequence components and therefore it is possible to represent variables of one phase in terms of the other two thereby reducing the total number of Equations describing the system. Thus, in such a model, a three phase machine is replaced by an equivalent two phase model with both phases in quadrature. This helps in eliminating the mutual coupling that exists among
the stator phases and rotor phases. The time varying self and mutual inductances can be made constant by referring all the variables to a common reference frame. This common reference frame can be i) fixed to stator, ii) fixed to rotor or iii) fixed to the synchronously rotating magnetic field.

The realization of a variable frequency induction motor drive with high performance comparable to that of a separately excited DC motor requires a suitable mathematical model. The dynamic model of induction motor will be useful in estimating the flux of the motor, to determine the torque angle, the phase difference between the flux axis and stator current vector, to fix up the flux axis and to enable the stator current to decompose into two components, one along the flux axis and the other perpendicular to the flux axis.

Basically, induction motor can be looked on as a transformer with a moving secondary, where the coupling coefficients between the stator and rotor phases change continuously with the change of rotor position, $\theta_r$. The machine model can be described by differential Equations with time varying mutual inductances, but such a model tends to be very complex. In order to minimize the complexity, the machine in three phase quantities can be represented as an equivalent machine in two phase quantities. Even though, it is made simple, the problem of time varying quantities still persists. Hence, the required quantities are referred to a common frame of reference and by doing so; the above problem is taken care of. Three different techniques have been proposed for referring the machine quantities to a common reference frame. The techniques proposed by Kron deals with referring both the stator and rotor quantities to a common synchronously rotating reference frame. This makes the motor parameters equivalent to that of a DC motor, which is of paramount importance to the FOC.
The mathematical models of the machine used in this thesis are either in the synchronously rotating reference frame or in the stator reference frame depending upon the situation. The Voltage Source Inverter (VSI) fed model is used for estimating the flux linkages from voltages and currents and for simulating the motor for maintaining direct coupling between the inverter and the motor. Current Source Inverter (CSI) fed models are used for simulations in FOC where mutual coupling does not exist. Both these models which are used in this thesis are dealt in the subsequent sections.

2.2 MODEL FOR VSI FED INDUCTION MOTOR

The voltage current relationship for a voltage fed induction motor in actual variables in the stator reference frame is useful for the analysis of the inverter fed motor. Three phase to two phase transformation of such a model can be achieved if the equivalence, which is based on the principle of power invariance, between three and two phase is established.

The following assumptions are made in the three phase to two phase transformation:

1. The induction motor is symmetrical and linear. This assumption leads to balanced windings and also neglects saturation.

2. Three phase source feeding the motor is balanced.

3. Uniform air gap.

4. Balanced rotor and stator windings, with sinusoidally distributed mmf and hence flux distribution in the air gap is sinusoidal.
The motor voltage equations both for stator and rotor are given in Equations (2.1) and (2.2).

\[
\begin{pmatrix}
V_a \\
V_b \\
V_c
\end{pmatrix} = \begin{pmatrix}
R_s + p(L_g + L_{ls}) & p(-L_g/2) & p(-L_g/2) \\
p(-L_g/2) & R_s + p(L_g + L_{ls}) & p(-L_g/2) \\
p(-L_g/2) & p(-L_g/2) & R_s + p(L_g + L_{ls})
\end{pmatrix} \begin{pmatrix}
i_a \\
i_b \\
i_c
\end{pmatrix} + pL_g \begin{pmatrix}
\cos \delta & \cos(\delta + 4\pi/3) & \cos(\delta + 2\pi/3) \\
\cos(\delta + 2\pi/3) & \cos \delta & \cos(\delta + 4\pi/3) \\
\cos(\delta + 4\pi/3) & \cos(\delta + 2\pi/3) & \cos \delta
\end{pmatrix} \begin{pmatrix}
i_A \\
i_B \\
i_C
\end{pmatrix} \tag{2.1}
\]

\[
\begin{pmatrix}
V_A \\
V_B \\
V_C
\end{pmatrix} = pL_g \begin{pmatrix}
\cos \delta & \cos(\delta + 4\pi/3) & \cos(\delta + 2\pi/3) \\
\cos(\delta + 2\pi/3) & \cos \delta & \cos(\delta + 4\pi/3) \\
\cos(\delta + 4\pi/3) & \cos(\delta + 2\pi/3) & \cos \delta
\end{pmatrix} \begin{pmatrix}
i_a \\
i_b \\
i_c
\end{pmatrix} + \begin{pmatrix}
R_r + p(L_g + L_{ls}) & p(-L_g/2) & p(-L_g/2) \\
p(-L_g/2) & R_r + p(L_g + L_{ls}) & p(-L_g/2) \\
p(-L_g/2) & p(-L_g/2) & R_r + p(L_g + L_{ls})
\end{pmatrix} \begin{pmatrix}
i_A \\
i_B \\
i_C
\end{pmatrix} \tag{2.2}
\]

where \(V_a, V_b, V_c\) are the stator voltages per phase, \(V_A, V_B, V_C\) are the rotor voltages per phase, \(R_s, L_{ls}\) are the stator resistance and leakage reactance per phase respectively, \(R_r, L_{ls}\) are the rotor resistance and leakage reactance per phase respectively, \(L_g\) is the air gap inductance, \(i_a, i_b, i_c\) are the stator currents per phase, \(i_A, i_B, i_C\) are the rotor currents per phase, \(p\) is the differential operator, \(\delta\) is the flux angle. Since, three phase induction motors are in general symmetrical, the sum of instantaneous values of phase voltages or currents is zero. Using this fact, the phase ‘c’ current can be expressed in terms of the other two phase currents as given Equation (2.3)

\[
i_c = -(i_a + i_b) \tag{2.3}
\]

Substituting Equation (2.3) in Equation (2.1) and simplifying, we get Equation (2.4)
\[
\begin{pmatrix}
V_s \\
V_b \\
V_A \\
V_B
\end{pmatrix} =
\begin{pmatrix}
0 & \frac{-2}{\sqrt{3}} L_m \sin(\delta - \frac{\pi}{3}) & \frac{2}{\sqrt{3}} L_m \sin\delta & r_s + pL_s \\
\frac{2}{\sqrt{3}} L_m \sin\delta & \frac{2}{\sqrt{3}} L_m \sin\delta & r_s + pL_s & 0 \\
\frac{-2}{\sqrt{3}} L_m \sin\delta & \frac{-2}{\sqrt{3}} L_m \sin\delta & \frac{2}{\sqrt{3}} L_m \sin\delta & r_s + pL_s
\end{pmatrix}
\begin{pmatrix}
i_a \\
i_b \\
i_A \\
i_B
\end{pmatrix}
\]

where \(L_m = (2/3)L_g\), \(L_s = L_m + L_{ls}\) and \(L_r = L_m + L_{lr}\) are the mutual inductance, stator inductance and the rotor inductance respectively. Since, the inductances and currents are time varying in nature, Equation (2.4) cannot be directly applied to determine the behavior of the motor. This difficulty is overcome by referring both stator and rotor equations to stator reference frame. The rotating rotor coils can be replaced by a set of pseudo stationary coils fixed along the axes of the stator coils. These coils carry time varying currents \(i_{2a}\), \(i_{2b}\) and \(i_{2c}\). Currents \(i_{2a}\) and \(i_{2b}\) are related to \(i_A\) and \(i_B\) as given in Equation (2.5).

\[
\begin{pmatrix}
i_{2a} \\
i_{2b}
\end{pmatrix} = \frac{2}{\sqrt{3}} \begin{pmatrix}
-sin(\theta-\frac{\pi}{3}) & -sin\theta \\
sin\theta & sin(\theta+\frac{\pi}{3})
\end{pmatrix}
\begin{pmatrix}
i_A \\
i_B
\end{pmatrix}
\]

where the matrix \(\begin{pmatrix}
-sin(\theta-\frac{\pi}{3}) & -sin\theta \\
sin\theta & sin(\theta+\frac{\pi}{3})
\end{pmatrix}\) is known as transformation matrix ‘T’. Using this transformation matrix and its inverse, the stator Equations of the motor given in Equation (2.4), after simplification, can be expressed as given Equation (2.6).
\[
\begin{pmatrix}
V_a \\
V_b
\end{pmatrix} = \begin{pmatrix} R_s + pL_s & 0 \\
0 & R_s + pL_s \end{pmatrix} \begin{pmatrix} i_a \\
i_b \end{pmatrix} + \begin{pmatrix} L_m p & 0 \\
0 & L_m p \end{pmatrix} \begin{pmatrix} i_{2a} \\
i_{2b} \end{pmatrix} \quad (2.6)
\]

The rotor voltage equations in Equation (2.4) can be transformed to stator reference frame using the relation given Equation (2.7)

\[
\begin{pmatrix}
V_{2a} \\
V_{2b}
\end{pmatrix} = T \begin{pmatrix} V_A \\
V_B \end{pmatrix} \quad (2.7)
\]

Substituting for \((V_A \ V_B)^T\) from Equation (2.4), we get the Equation (2.8)

\[
\begin{pmatrix}
V_{2a} \\
V_{2b}
\end{pmatrix} = T L_m p T^{-1} \begin{pmatrix} i_a \\
i_b \end{pmatrix} + T \begin{pmatrix} R_r + pL_r & 0 \\
0 & R_r + pL_r \end{pmatrix} T^{-1} \begin{pmatrix} i_{2a} \\
i_{2b} \end{pmatrix} \quad (2.8)
\]

This can be further simplified as given Equation (2.9)

\[
\begin{pmatrix}
V_{2a} \\
V_{2b}
\end{pmatrix} = \begin{pmatrix}
L_m p + \frac{1}{\sqrt{3}} L_m \omega_r & \frac{2}{\sqrt{3}} L_m \omega_r & R_r + L_r p + \frac{1}{\sqrt{3}} L_r \omega_r \\
\frac{2}{\sqrt{3}} L_m \omega_r & L_m p + \frac{1}{\sqrt{3}} L_m \omega_r & \frac{2}{\sqrt{3}} L_m \omega_r \\
R_r + L_r p + \frac{1}{\sqrt{3}} L_r \omega_r & \frac{2}{\sqrt{3}} L_m \omega_r & R_r + pL_r \end{pmatrix} \begin{pmatrix} i_a \\
i_b \\
i_{2a} \\
i_{2b} \end{pmatrix} \quad (2.9)
\]

Combining Equations (2.6) and (2.9), we get the Equation (2.10)

\[
\begin{pmatrix}
V_a \\
V_b \\
V_{2a} \\
V_{2b}
\end{pmatrix} = \begin{pmatrix}
R_s + pL_s & 0 & 0 \\
0 & R_s + pL_s & L_m p \\
L_m p + \frac{1}{\sqrt{3}} L_m \omega_r & \frac{2}{\sqrt{3}} L_m \omega_r & R_r + L_r p + \frac{1}{\sqrt{3}} L_r \omega_r \\
\frac{2}{\sqrt{3}} L_m \omega_r & L_m p + \frac{1}{\sqrt{3}} L_m \omega_r & \frac{2}{\sqrt{3}} L_m \omega_r \\
0 & L_m p & R_r + L_r p + \frac{1}{\sqrt{3}} L_r \omega_r \end{pmatrix} \begin{pmatrix} i_a \\
i_b \\
i_{2a} \\
i_{2b} \end{pmatrix} \quad (2.10)
\]
Separating the resistance drops, inductive drops, transformer voltages and speed voltages, we can rewrite Equation (2.10) as given in Equation (2.11).

\[
(V) = (R)(I) + (L)p(I) + (G_p\{\theta\})(I) \tag{2.11}
\]

Power input to the stator is calculated using Equation (2.12) as given below.

\[
(I)^T(V) = (I)^T(R)(I) + (I)^T(L)p(I) + (I)^T(G_p\{\delta\})(I) \tag{2.12}
\]

The torque developed, \( T_d \), by the motor is calculated using given in Equation (2.13).

\[
T_d = \sqrt{3}P_Lm(i_{2a}i_b-i_{2b}i_a) \tag{2.13}
\]

The flux linkages of the motor are calculated using Equation (2.14) to (2.17) as given below.

\[
\psi_a = L_s i_a + L_{m} i_{2a} \tag{2.14}
\]

\[
\psi_b = L_s i_b + L_{m} i_{2b} \tag{2.15}
\]

\[
\psi_{2a} = L_{r} i_{2a} + L_{m} i_a \tag{2.16}
\]

\[
\psi_{2b} = L_{r} i_{2b} + L_{m} i_b \tag{2.17}
\]

The Equation (2.10) has to be expressed as a state space model as given in Equation (2.18) for implementing it on a digital computer. The state space model is as given in Equation (2.19)

\[
p(I) = (L)^{-1}(V) + (L)^{-1}(R+G\{\omega\})(i) \tag{2.18}
\]
\[
\begin{pmatrix}
i_a \\
i_b \\
i_{2a}
\end{pmatrix} = \frac{1}{\sigma} \begin{pmatrix}
\frac{1}{L_s} & 0 & -\frac{L_m}{L_s L_r} & 0 \\
0 & \frac{1}{L_s} & 0 & -\frac{L_m}{L_s L_r} \\
-\frac{L_m}{L_s L_r} & 0 & \frac{1}{L_s} & 0 \\
0 & -\frac{L_m}{L_s L_r} & 0 & \frac{1}{L_s}
\end{pmatrix} \begin{pmatrix}
V_a \\
V_b \\
V_{2a} \\
V_{2b}
\end{pmatrix} +
\]

where \( \sigma = 1 - \frac{L_m^2}{L_s L_r} \)

The output equation is given by Equation (2.20)

\[
\begin{pmatrix}
\mathbf{i}_A \\
\mathbf{i}_B \\
\mathbf{i}_{2A} \\
\mathbf{i}_{2B}
\end{pmatrix} = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix} \begin{pmatrix}
\mathbf{i}_A \\
\mathbf{i}_B \\
\mathbf{i}_{2A} \\
\mathbf{i}_{2B}
\end{pmatrix}
\] (2.20)

Equation (2.19) has to be expressed in discrete form as given in Equation (2.21) and for using it along with EKF and RTRN.
\[
\begin{pmatrix}
i_a(k+1) \\
i_b(k+1) \\
i_{2a}(k+1) \\
i_{2b}(k+1)
\end{pmatrix} = \frac{1}{\sigma} \begin{pmatrix}
\frac{1}{l_s} t_s & 0 & -\frac{L_{ms}}{L_s L_r} & 0 \\
0 & \frac{1}{l_s} t_s & 0 & -\frac{L_m}{L_s L_r} t_s \\
0 & -\frac{L_m}{L_s L_r} t_s & 0 & \frac{1}{l_s} t_s \\
0 & 0 & 0 & 0
\end{pmatrix} \begin{pmatrix}
V_a(k) \\
V_b(k) \\
V_{2a}(k) \\
V_{2b}(k)
\end{pmatrix} + 
\]

\[
\frac{1}{\sigma} \begin{pmatrix}
\sigma + \left( \frac{R_s}{L_s} + \frac{\omega_r L_m}{\sqrt{3} L_s L_r} \right) t_s \\
\frac{-2 \omega_r L_m}{\sqrt{3} L_s L_r} t_s \\
\frac{2 \omega_r L_m}{\sqrt{3} L_r} t_s \\
2 \omega_r L_m \left( \frac{R_s}{L_s} + \frac{\omega_r L_m}{\sqrt{3} L_r} \right) t_s
\end{pmatrix} \begin{pmatrix}
i_a(k) \\
i_b(k) \\
i_{2a}(k) \\
i_{2b}(k)
\end{pmatrix}
\]

The Equation (2.21), which is in stationary reference frame, is used for controlling the firing of voltage source inverter. The two phase equation referred to any arbitrary reference frame is as given in Equation (2.22).

\[
\begin{pmatrix}
V_{ds} \\
V_{qs} \\
V_{dr} \\
V_{qr}
\end{pmatrix} = \begin{pmatrix}
R_s + p L_s & -\omega_a L_s & L_m p \\
\omega_a L_s & R_s + p L_s & \omega_a L_m \\
L_m p & -(\omega_a - \omega_r) L_m & R_r + L_r p \\
(\omega_a - \omega_r) L_m & L_m p & (\omega_a - \omega_r) L_m
\end{pmatrix} \begin{pmatrix}
i_{ds} \\
i_{qs} \\
i_{dr} \\
i_{qr}
\end{pmatrix}
\]
Equation (2.22) is converted to state space model for using it on digital computer. Here, the stator and rotor current components are considered as state variables where as the stator voltage components are considered as inputs and stator current components are considered as outputs as given Equations (2.23) to (2.25) as given below.

\[
x = (i_{ds} \ i_{qs} \ i_{dr} \ i_{qr})^T \quad (2.23)
\]
\[
u = (V_{ds} \ V_{qs})^T \quad (2.24)
\]
\[
y = (i_{ds} \ i_{ds})^T \quad (2.25)
\]

The state space representation of Equation (2.22) is given in Equations (2.26) and (2.27)

\[
p_x = Ax + Bu \quad (2.26)
\]
\[
y = Cx \quad (2.27)
\]

where $A = \frac{1}{L_sL_r-L_m^2} \begin{pmatrix} -R_s L_r & L_m^2 \omega_r & L_m R_r & L_m L_r \omega_r \\ -L_m^2 \omega_r & -R_s L_r & -L_m L_r \omega_r & L_m R_r \\ -L_m R_s & -L_m L_s \omega_r & -L_s R_r & -L_s L_r \omega_r \\ L_m L_s \omega_r & L_m R_s & L_s L_r \omega_r & -L_s R_r \end{pmatrix}$

\[
B = \frac{1}{L_sL_r-L_m^2} \begin{pmatrix} L_r & 0 \\ 0 & L_r \\ -L_m & 0 \\ 0 & -L_m \end{pmatrix} \quad \text{and} \quad C = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}
\]

The model represented by Equation (2.26) and Equation (2.27) are converted in to discrete form as given in Equations (2.28) and (2.29) respectively.
\[ x(k+1) = F(k)x(k) + G(k) u(k) \quad (2.28) \]

\[ y(k) = Hx(k) \quad (2.29) \]

where \( F(k) = \begin{pmatrix} 1 - a_1 t_s & a_5 t_s \omega_r(k) & a_3 t_s & a_4 t_s \omega_r(k) \\ -a_2 t_s \omega_r(k) & 1 - a_1 t_s & -a_4 t_s \omega_r(k) & a_3 t_s \\ a_6 t_s & -a_7 t_s \omega_r & 1 - a_8 t_s & a_9 t_s \omega_r(k) \\ a_7 t_s \omega_r(k) & a_6 t_s & a_9 t_s \omega_r & 1 - a_8 t_s \end{pmatrix} \]

\[ G(k) = \begin{pmatrix} a_5 t_s & 0 \\ 0 & a_5 t_s \\ -a_{10} t_s & 0 \\ 0 & -a_{10} t_s \end{pmatrix} \]

\[ a_0 = L_N R_L - L_m^2 \quad ; \quad a_1 = \frac{R_s}{a_0} L_r \quad ; \quad a_2 = \frac{L_m^2}{a_0} \]

\[ a_3 = \frac{R_r}{a_0} L_m \quad ; \quad a_4 = \frac{L_m}{a_0} L_r \quad ; \quad a_5 = \frac{L_r}{a_0} \]

\[ a_6 = \frac{R_s}{a_0} L_m \quad ; \quad a_7 = \frac{L_s}{a_0} L_m \quad ; \quad a_8 = \frac{L_s R_r}{a_0} \]

\[ a_9 = \frac{L_s}{a_0} L_r \quad ; \quad a_{10} = \frac{L_m}{a_0} \]

### 2.3 MODEL FOR CSI FED INDUCTION MOTOR

For a CSI fed induction motor, the stator currents are known and they are considered as inputs in state space representation. Hence, the rotor Equations completely describe the system. In effect, the order of the system representation becomes two instead of four as in case of VSI fed motor. The rotor flux linkages are the state variables and terminal voltages are the outputs. By suitable manipulations of VSI fed machine Equations, state space Equations of CSI fed motor can be written as given in Equation (2.30).
The terminal voltages are expressed in Equation (2.31)
\[
\begin{bmatrix} V_A \\ V_B \end{bmatrix} = R_s \begin{bmatrix} i_A \\ i_B \end{bmatrix} + \left( 1 - \frac{L_m}{L_s L_r} \right) L_s p \begin{bmatrix} i_A \\ i_B \end{bmatrix} + L_m p \begin{bmatrix} \psi_{2A} \\ \psi_{2B} \end{bmatrix} \tag{2.31}
\]

The discretized form of Equation (2.30) is given in Equation (2.32)
\[
\begin{bmatrix} \psi_{2A}(k+1) \\ \psi_{2B}(k+1) \end{bmatrix} = \begin{bmatrix} b_1 t_s & b_2 t_s & b_3 & b_4 \end{bmatrix} \begin{bmatrix} \psi_{2A}(k) \\ \psi_{2B}(k) \end{bmatrix} + \frac{R_r L_m}{L_r} \begin{bmatrix} i_A(k) \\ i_B(k) \end{bmatrix} \tag{2.32}
\]

where \( b_1 = 1 + \frac{\omega_r(k)}{\sqrt{3}} \cdot \frac{R_r}{L_r} \); \( b_2 = \frac{-2\omega_r}{\sqrt{3}} \); \( b_3 = \frac{2\omega_r}{\sqrt{3}} \); \( b_4 = 1 + \frac{\omega_r(k)}{\sqrt{3}} \cdot \frac{R_r}{L_r} \)

The Equation (2.32) is converted in to d-q axes form as given in Equation (2.33)
\[
\begin{bmatrix} \psi_{dr}(k+1) \\ \psi_{qr}(k+1) \end{bmatrix} = \begin{bmatrix} -R_r & -\omega_r \\ \omega_r & -\frac{R_r}{L_r} \end{bmatrix} \begin{bmatrix} \psi_{dr}(k) \\ \psi_{qr}(k) \end{bmatrix} + \frac{L_m}{L_r} R_r \begin{bmatrix} i_{ds} \\ i_{qs} \end{bmatrix} \tag{2.33}
\]

The discretized form of Equation (2.33) is given in Equation (2.34)
\[
\begin{bmatrix} \psi_{dr}(k+1) \\ \psi_{qr}(k+1) \end{bmatrix} = \begin{bmatrix} 1 - \frac{R_r t_s}{L_r} & -\omega_r t_s \\ \omega_r t_s & 1 - \frac{R_r t_s}{L_r} \end{bmatrix} \begin{bmatrix} \psi_{dr}(k) \\ \psi_{qr}(k) \end{bmatrix} + \frac{R_r L_m t_s}{L_r} \begin{bmatrix} i_{ds}(k) \\ i_{qs}(k) \end{bmatrix} \tag{2.34}
\]

These Equations along with the dynamic Equation of motion given in Equation (2.35) describe the complete performance of the induction motor.
\[
\frac{d\omega_r}{dt} = \frac{B}{J} \omega_r + \frac{1}{J} (T_e - T_d) \frac{P}{2} \tag{2.35}
\]
2.4 CONCLUSIONS

The state space models VSI fed and CSI fed induction motor are necessary for simulating the electromagnetic characteristics of the motor. These models are also useful in designing the controllers and observers in closed loop control. The models may be made available in two of the three phase variables using Park’s transformation. The VSI fed model is suitable for the purpose of simulating the motor so that there is direct coupling between inverter and the motor. The CSI fed model is useful in the simulations like FOC, where mutual coupling between the axes need to be eliminated. The discretized models of the motor are necessary in making use of recursive EKF algorithm.

The parameters of the mathematical model, if optimized for an exclusive application, by adapting energy efficient design, the energy losses can be reduced. The next chapter deals with energy efficient design of a three phase induction motor for optimizing the machine parameters.