Introduction

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CHAPTER I
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INTRODUCTION

General topology plays a significant role in space time geometry as well as in different branches of mathematics. The notions of sets and functions in topological spaces are highly developed and used extensively in many practical and engineering problems. Rosen and Peters [160] have used topology as a body of mathematics that could unify diverse areas of computer aided geometric design and engineering design research. Levine [87] and Njastad [118] introduced the notions of generalized closed sets and α-open sets. Balachandran et al. [14] defined a new class of mappings called generalized continuous mappings and investigated their properties. In the last decade, the modifications of g-closed sets have been introduced by using δ-open sets, α-open sets, π-open sets, pre-open sets and their properties have been obtained in [4,29,42,104]. Zaitsev [183] introduced the concept of π-closed sets and defined a class of topological spaces called quasi normal spaces. This thesis mainly deals with the study of a new type of set in a topological space called πgα-closed set, its respective continuous maps, irresolute maps, closed maps, homeomorphisms and its extension to bitopological settings.

In this chapter, the recent developments of topology contributed by various authors are mentioned and definitions cited by them are presented. Section 1 starts with the definition of strong and weak forms of open sets and closed sets. Section 2 gives the notion of strong and weak forms of continuous functions. Section 3 presents irresolute functions while Section 4 is devoted to closed and open maps. Section 5 explains the concept of some generalizations of homeomorphisms. In Section 6 the notion of bitopological spaces is explained. Section 7 outlines the contributions of the author. The last section describes the various notations used in the thesis. Throughout the thesis X, Y, Z denote the topological spaces (X,τ), (Y,σ) and (Z,η) respectively on which no separation axioms are assumed unless otherwise mentioned.
1.1 Strong And Weak Forms Of Open And Closed Sets

The strong forms of open sets namely regular open sets, strong regular open sets, \(\theta\)-open sets, \(\delta\)-open sets, \(\pi\)-open sets have been introduced and investigated by Stone [172], Tong[178], Velicko [181], Zaitsev [183] respectively. Mashour et al. [106], Levine [86,87], Njastad[118], Abd El-Monsef et al. [1], Arya and Nour[11], Bhattacharya and Lahiri[19], Palaniappan and Rao[144], Maki et al.[104] and Dontchev [33] have introduced preopen sets, semi-open sets and \(g\)-open sets. \(\alpha\)-open sets, \(\beta\)-open sets, generalized semi-open sets, semi-generalized open sets, regular generalized open sets, generalized preopen sets and generalized semi-pre-open sets respectively. These are some of the weak forms of open sets and the complements of these are called the same type of closed sets respectively. Cameron [23] has formulated the concept of regular semi-open sets which is weaker than regular open sets. Ganster and Reilly [58], Balachandran et al. [15] have introduced locally closed sets and generalized locally closed sets, semi-generalized locally closed sets and semi-locally closed sets which are weaker than open and closed sets. Arockiarani [4] has introduced generalized-\(\delta\)-closed sets and generalized \(\theta\)-\(C\)-closed sets. Various types of sets have been studied in [37, 39, 40, 47, 48, 60, 83, 100, 101, 158]. We recall the definitions of some of them which are pre-requisites for our present study.

**Definition 1.1.1:** A subset \(A\) of a space \((X,\tau)\) is called

i) preopen [106] if \(A \subseteq \text{int}(\text{cl}(A))\) and preclosed if \(\text{cl}(\text{int}(A)) \subseteq A\)

ii) \(\alpha\)-open [118] if \(A \subseteq \text{int}(\text{cl}(\text{int}(A)))\) and \(\alpha\)-closed if \(\text{cl}(\text{int}(\text{cl}(A))) \subseteq A\).

iii) regular open [172] if \(A = \text{int}(\text{cl}(A))\) and regular-closed if \(A = \text{cl}(\text{int}(A))\).

iv) \(\beta\)-open [1] if \(A \subseteq \text{cl}(\text{int}(\text{cl}(A)))\) and \(\beta\)-closed if \(\text{int}(\text{cl}(\text{int}(A))) \subseteq A\).

v) semi-open [86] if \(A \subseteq \text{cl}(\text{int}(A))\) and semi-closed if \(\text{int}(\text{cl}(\text{cl}(A))) \subseteq A\).

A finite union of regular open sets is said to be \(\pi\)-open. The \(\alpha\)-closure of \(A \subseteq X\) (briefly \(\alpha\text{cl}(A)\)) is the smallest \(\alpha\)-closed set containing \(A\). The \(\alpha\)-interior of \(A \subseteq X\) (briefly \(\alpha\text{int}(A)\) or \(\tau^\alpha\text{-int}(A)\)) is the largest \(\alpha\)-open set contained in \(A\). Note that \(\alpha\text{int}(A) = A \cap \text{int}(\text{cl}(\text{int}(A)))\) and \(\alpha\text{cl}(A) = A \cup \text{cl}(\text{int}(\text{cl}(A)))\)[3]. The collection \(\alpha(X)\) of
all α-open subsets of X is a topology on X [118] which is finer than the original one and a subset A of X is α-open if and only if A is semi-open and preopen [137].

Definition 1.1.2: A subset A of (X, τ) is called

1) Generalized closed (briefly g-closed) [87] if cl(A) ⊂ U whenever A ⊂ U and U is open.

2) Generalized α-closed (briefly gα-closed) [99] if αcl(A) ⊂ U whenever A ⊂ U and U is α-open.

3) α-Generalized closed (briefly αg-closed) [99] if αcl(A) ⊂ U whenever A ⊂ U and U is open.

4) Generalized preregular closed (briefly gpr-closed) [63] if pcl(A) ⊂ U whenever A ⊂ U and U is regular open.

5) regular generalized closed (briefly rg-closed) [144] if cl(A) ⊂ U whenever A ⊂ U and U is regular open.

6) Generalized preclosed (briefly gp-closed) [104] if pcl(A) ⊂ U whenever A ⊂ U and U is open.

7) πg-closed [42] if cl(A) ⊂ U whenever A ⊂ U and U is π-open.

8) πgp-closed [146] if pcl(A) ⊂ U whenever A ⊂ U and U is π-open.

9) a locally closed set [58] if A = G ∩ F where G is open and F is closed.

10) a generalized locally closed set [15] (briefly gle-set) if A = G ∩ F where G is g-open and F is g-closed.

11) an α-locally closed set [64] (briefly α-lc set) if A = G ∩ F where G is α-open and F is α-closed.

12) an α-lc* set [64] if A = G ∩ F where G is α-open and F is closed in X.

13) an α-lc** set [64] if A = G ∩ F where G is open and F is α-closed in X.

14) an A-set [178] if A = G ∩ F where G is open and F is regular closed in X.

15) a t-set [179] if int(A) = int(cl(A)).

16) a B-set [179] if A = G ∩ F where G is open and F is a t-set in X.

17) an α*-set [118] if int(A) = int(cl(intA)).

18) a C-set [155] if A = G ∩ F where G is g-open and F is a t-set in X.
19) a $C_r$-set \([155]\) if \(A = G \cap F\) where \(G\) is rg-open and \(F\) is a t-set in \(X\).

20) a $C_r^*$-set \([155]\) if \(A = G \cap F\) where \(G\) is rg-open and \(F\) is a $\alpha^*$-set in \(X\).

**Definition 1.1.3:** A space \((X,\tau)\) is called

1) a submaximal space \([34]\) if every dense subset of \(X\) is open in \(X\).

2) a door space \([35]\) if every subset of \(X\) is either open or closed in \(X\).

3) a $T_{1\frac{1}{2}}$-space \([46]\) if every g-closed set is closed.

4) a $T_{\pi g}$-space \([148]\) if every $\pi g$-closed set is g-closed.

5) a semi-pre-$T_{1\frac{1}{2}}$ space \([33]\) if every singleton set is closed or preopen.

6) a partition space \([32]\) or locally indiscrete if every open set is closed.

7) Extremally disconnected \([32]\) if the closure of each open subset of \(X\) is open.

8) Irreducible or hyperconnected \([170]\) if every open subset of \(X\) is dense.

**1.2 Strong And Weak Forms Of Continuous Maps**

Strong and weak forms of continuous maps have been introduced and investigated by several topologists \([5, 7, 14, 43, 58, 61, 84, 88, 90, 91, 92, 95, 106, 114, 117, 123, 132, 143, 145, 153, 173]\). Various interesting problems arise when one considers continuity, a stronger form of continuity or a weaker form of continuity. One of them which has been of great interest to general topologists is its decompositions. The strong forms of continuous maps have been discussed by Noiri\([131]\), Levine\([84]\), Arya and Gupta\([8]\), Munshi and Bassan\([112]\), Reilly and Vamanamurthy \([157]\). They have introduced strong continuous maps, strongly-$\theta$-continuous maps, completely continuous maps, super continuous maps and clopen continuous maps. Noiri\([126]\) has studied $\delta$-continuity which is independent of the concept of continuity. Biswas\([20]\), Hussain\([67]\), Ganster and Reilly\([58]\), Noiri\([124]\), Mashour et al. \([108]\), Monsef et al. \([1]\), Tong\([177]\), Devi et al. \([30]\), Baker\([13]\) have introduced and studied simple continuity, almost continuity, LC-continuity, weak continuity, $\alpha$-continuity, $\beta$-continuity, semi-weak continuity and
weak almost continuity, $\alpha$-generalized and generalized $\alpha$-continuity and contra-almost $\beta$-continuity. Semi-continuous maps have been studied by various authors [21, 26, 44, 62, 65, 71, 86, 107, 119, 121, 127, 128, 134]. Balachandran et al. [14], Sundaram et al. [174] and Arockiarani et al. [5] have defined and studied $g$-continuous maps and generalized locally continuous maps, $sg$-continuous maps and $rg$-continuous maps respectively. Recently Beceren and Noiri [18] have introduced the notion of strongly pre-continuous functions and Zorultana et al. [184] obtained their characterizations. Levine [85] proved that a map is continuous if and only if it is weakly continuous and weak*continuous. Various topologists [12, 53, 66, 151, 154] investigated decomposition of continuity. Here we define some of these maps which are used in our study.

**Definition 1.2.1:** Let $f : (X, \tau) \rightarrow (Y, \sigma)$ be a map. Then $f$ is said to be
a) strongly continuous [84] if $f^{-1}(V)$ is clopen in $X$ for every subset $V$ of $Y$.
b) $\alpha$-continuous [108] if $f^{-1}(V)$ is $\alpha$-closed in $X$ for every closed set $V$ of $Y$.
c) $\beta$-continuous [1] if $f^{-1}(V)$ is $\beta$-closed in $X$ for every closed set $V$ of $Y$.
d) $\pi$-continuous [42] if $f^{-1}(V)$ is $\pi$-closed in $X$ for every closed set $V$ of $Y$.
e) almost continuous [164] if $f^{-1}(V)$ is open in $X$ for every regular open set $V$ of $Y$.
f) almost $\pi$-continuous [42] if $f^{-1}(V)$ is $\pi$-closed in $X$ for every regular closed set $V$ of $Y$.
g) completely continuous [8] if $f^{-1}(V)$ is regular open in $X$ for every open set $V$ of $Y$.
h) slightly continuous [70] if $f^{-1}(V)$ is open in $X$ for each clopen set $V$ of $Y$.
i) regular set-connected [38] if $f^{-1}(V)$ is clopen in $X$ for every regular open subset $V$ of $Y$.
j) contra-continuous [36] if $f^{-1}(V)$ is closed in $X$ for every open set $V$ of $Y$.
k) contra-$\alpha$-continuous [68] if $f^{-1}(V)$ is $\alpha$-closed in $X$ for every open set $V$ of $Y$.
l) contra-$\pi g$-continuous [50] if $f^{-1}(V)$ is $\pi g$-closed in $X$ for every open set $V$ of $Y$.
m) contra-$\pi gp$-continuous [148] if $f^{-1}(V)$ is $\pi gp$-closed in $X$ for every open set $V$ of $Y$.
n) LC-continuous [58] if $f^{-1}(V) \in \text{LC}(X, \tau)$ for each open set $V$ of $Y$.
o) sub-LC-continuous [58] if there is a base $B$ for $(Y, \sigma)$ such that
\[ f^{-1}(V) \in \text{LC}(X, \tau) \] for each $V \in B$.
p) strongly $\alpha$-continuous [17] if $f^{-1}(V)$ is $\alpha$-open in $X$ for every semi-open set $V$ of $Y$.
q) regular continuous [144] if $f^{-1}(V)$ is regular closed in $X$ for every closed set $V$ of $Y$.
1.3 Irresolute Maps

Crossley and Hildebrand [27] introduced and investigated irresolute maps which are stronger than semi-continuous maps but are independent of continuous maps. Since then several types of irresolute maps have been introduced by many authors [14, 24, 37, 51, 80, 96], D.Maio and Noiri[98], Cammarota and Noiri[23], Ganster and Reilly[58], Maheshwari and Thakur [94], Dontchev et al. [37], Balachandran et al. [14], Devi et al. [30] and Arockiarani [4] have introduced and studied quasi-irresolute, strongly irresolute maps, weak irresolute maps and $\theta$-irresolute maps, $L$-$\epsilon$-irresolute maps, $\alpha$-irresolute maps, $\delta g$-irresolute maps, $g c$-irresolute maps, $\alpha g$-irresolute maps and $g a$-irresolute maps and rg-irresolute maps respectively. Here we define few types of irresolute maps.

**Definition 1.3.1:** A function $f: (X,\tau) \rightarrow (Y,\sigma)$ is called

a) $\beta$- irresolute[96] if $f^{-1}(F)$ is $\beta$-closed in $X$ for every $\beta$-closed set $F$ of $Y$.

b) $\alpha$-irresolute[94] if $f^{-1}(F)$ is $\alpha$-closed in $X$ for every $\alpha$-closed set $F$ of $Y$.

c) pre-irresolute[159] if $f^{-1}(F)$ is preclosed in $X$ for every preclosed set $F$ of $Y$.

d) an R-map[25] if $f^{-1}(F)$ is regular closed in $X$ for every regular closed set $F$ of $Y$.

e) gc-irresolute[14] if $f^{-1}(F)$ is g-closed in $X$ for every g-closed set $F$ of $Y$.

f) strongly $\alpha$-irresolute[55] if $f^{-1}(F)$ is open in $X$ for every $\alpha$-open set $F$ of $Y$.

g) almost $\alpha$-irresolute[16] if $f^{-1}(F)$ is $\beta$-open in $X$ for every $\alpha$-open set $F$ of $Y$.

1.4 Closed Maps And Open Maps

Preclosed mappings were introduced and studied by Sen and Bhattacharyya[162]. Malghan [105] introduced and investigated some properties of generalized closed maps. Biswas[20], Das[28] and Noiri[120,122,136], Mashour et al.[106,108], Mrsevic et al.[110], Crossley et al.[27], Devi et al.[29], Noiri et al.[138] and Arockiarani [4] have defined and studied semi-open maps and semi-closed maps, $\alpha$-open maps, preopen maps
and weak preopen maps, $\delta$-open maps and $\delta$-closed maps, pre-semi-open maps, $\alpha g$-closed maps, $gp$-closed maps and $\delta g$-closed maps and $rg$-closed maps respectively. Definitions of some open and closed maps are listed below.

**Definition 1.4.1:** Let $f: (X, \tau) \rightarrow (Y, \sigma)$ be a map. A map $f$ is said to be

a) $rc$-preserving if $f(U)$ is regular closed in $Y$ for every regular closed set $U$ of $X$.

b) $\alpha$-open if $f(U)$ is $\alpha$-open in $Y$ for every open set $U$ of $X$.

c) preopen if $f(U)$ is preopen in $Y$ for every open set $U$ of $X$.

d) pre-$\alpha$-open if $f(U)$ is $\alpha$-open in $Y$ for every $\alpha$-open set $U$ of $X$.

e) p-open if $f(U)$ is preopen in $Y$ for every preopen set $U$ of $X$.

f) $g$-closed if $f(U)$ is g-closed in $Y$ for every closed set $U$ of $X$.

g) $ag$-closed if $f(U)$ is $ag$-closed in $Y$ for every closed set $U$ of $X$.

h) $ga$-closed if $f(U)$ is $ga$-closed in $Y$ for every closed set $U$ of $X$.

i) almost-$g$-closed if $f(U)$ is $g$-closed in $Y$ for every regular closed set $U$ of $X$.

j) almost closed if $f(U)$ is closed in $Y$ for every regular closed set $U$ of $X$.

k) almost-$\pi g$-closed if $f(U)$ is $\pi g$-closed in $Y$ for every regular closed set $U$ of $X$.

### 1.5 Generalized Homeomorphisms In Topological Spaces

The notion of homeomorphisms has been studied and generalized by several topologists [4, 27, 29, 102, 116, 130, 167]. Semi-homeomorphisms which are weaker than homeomorphisms have been introduced by Biswas [20] and Crossley and Hildebrand [27]. Neubrunn [118] and Piotrowski [150] have proved that semi-homeomorphism of [20] and semi-homeomorphism of [27] are independent. Semi-generalized homeomorphisms and generalized semi-homeomorphisms, some what homeomorphisms. $\alpha$-homeomorphisms and $\beta$-homeomorphisms, $g$-$\Lambda$-homeomorphism, $g$-homeomorphism and gc-homeomorphism, $rg$-homeomorphisms and $g^{**}$-homeomorphism have been defined and discussed by Devi et al. [29], Gentry and Hoyle [62], Tadros and Abd Allah [175], Umehara and Maki [180], Maki et al. [102], Arockiarani [4], Sai sundarakrishnan and K. Balachandran [161] respectively.
Definition 1.5.1: A map \( f : (X, \tau) \to (Y, \sigma) \) is said to be a semi-homeomorphism if \( f \) is both continuous, semi-open and bijective.

Definition 1.5.2: A map \( f : (X, \tau) \to (Y, \sigma) \) is said to be a semi-homeomorphism if \( f \) is irresolute, pre-semi-open and bijective.

Definition 1.5.3: A map \( f : (X, \tau) \to (Y, \sigma) \) is called a generalized-homeomorphism if \( f \) is g-continuous, g-open and bijective.

Definition 1.5.4: A map \( f : (X, \tau) \to (Y, \sigma) \) is called a gc-homeomorphism if \( f \) is bijective, gc-irresolute and \( f^{-1} \) is gc-irresolute.

Definition 1.5.5: A map \( f : (X, \tau) \to (Y, \sigma) \) is called an rg-homeomorphism if \( f \) is rg-continuous, rg-open and bijective.

Definition 1.5.6: A map \( f : (X, \tau) \to (Y, \sigma) \) is called an rgc-homeomorphism if \( f \) is bijective, rg-irresolute and \( f^{-1} \) is rg-irresolute.

1.6 Bitopological Spaces

Kelley[77] defined a bitopological space as a set equipped with two topologies on a set and initiated a systematic study of bitopological spaces. Following his work, Fukutake[56,57] and Maki et al. [103] have respectively extended the notions of generalized closed sets, semi-open sets and generalized continuous maps in topological spaces to bitopological spaces respectively and studied some of their properties. Maheswari and Prasad[93] and Jelic[76] have extended the concepts of semi-continuity and pre-open sets and pre-continuity in topological spaces to bitopological setting respectively. Arya and Nour[9,10], Mrsevic[109], Mukherjee et al.[111], Patty[149], Popa [152] and Reilly[154] also turned their attention to the study the various concepts of topology in bitopological spaces.

Here we present some of the definitions which are used in our study.
Definition 1.6.1: [56] Let i, j ∈ {1, 2} be fixed integers. A subset A of a bitopological space (X, τ_i, τ_j) is said to be (τ_i, τ_j)-g-closed if τ_i-αcl(A) ⊆ U whenever A ⊆ U and U is τ_i-open.

Definition 1.6.2: [56] Let i, j ∈ {1, 2} be fixed integers. A bitopological space (X, τ_i, τ_j) is said to be (τ_i, τ_j)-T_1/2 space if every (τ_i, τ_j)-g-closed set is τ_j-closed.

Definition 1.6.3: [103] A map f: (X, τ_i, τ_j) → (Y, σ_1, σ_2) is called D(τ_i, τ_j)-σ_k continuous if the inverse image of every σ_k-closed set is (τ_i, τ_j)-g-closed where D(τ_i, τ_j) denotes the set of all (τ_i, τ_j)-g-closed sets in (X, τ_i, τ_j).

Definition 1.6.4: [103] A map f: (X, τ_i, τ_j) → (Y, σ_1, σ_2) is called generalized bi-continuous (g-bi-continuous) if f is D(τ_i, τ_j)-σ_2-continuous and D(τ_i, τ_j)-σ_1-continuous.

Definition 1.6.5: [103] A map f: (X, τ_i, τ_j) → (Y, σ_1, σ_2) is called bi-continuous if f is τ_1-σ_2 continuous and τ_2-σ_1 continuous.

Definition 1.6.6: [64] Let i, j ∈ {1, 2} be fixed integers. A subset A of a bitopological space (X, τ_i, τ_j) is said to be (τ_i, τ_j)-gpr-closed if τ_j-pcl(A) ⊆ U whenever A ⊆ U and U is τ_j-regular open.

1.7 Contributions Of The Author

In the light of the above work, the author has obtained some interesting generalizations on the following topics.
1. πgα-closed sets and πgα-continuous functions in topological spaces.
2. πgα-closed maps and πgα-homeomorphisms in topological spaces.
4. Decomposition of πgα-sets.
5. Strong forms of πgα-irresolute functions and a weak form of πgα-continuous functions.
6. πgα-closed sets in bitopological Spaces.
1.8 Notations

P(X) \rightarrow \text{Power set of } X

\Lambda \rightarrow \text{index set}

X-A \text{ or } A^c \rightarrow \text{complement of } A

\text{int}(A) \rightarrow \text{interior of } A

\text{cl}(A) \rightarrow \text{closure of } A

\text{cl}_Y(A) \rightarrow \text{closure of } A \text{ with respect to } Y

\alpha\text{cl}_Y(A) \rightarrow \alpha \text{-closure of } A \text{ with respect to } Y

\alpha O(X,\tau) \rightarrow \text{Set of all } \alpha \text{-open subsets of } (X,\tau)

\text{GO}(X,\tau) \rightarrow \text{Set of all } g \text{-open subsets of } (X,\tau)

\text{RO}(X,\tau) \rightarrow \text{Set of all regular open subsets of } (X,\tau)

\alpha C(X,\tau) \rightarrow \text{Set of all } \alpha \text{-closed subsets of } (X,\tau)

\text{GC}(X,\tau) \rightarrow \text{Set of all } g \text{-closed subsets of } (X,\tau)

\text{RC}(X,\tau) \rightarrow \text{Set of all regular closed subsets of } (X,\tau)

\text{LC}(X,\tau) \rightarrow \text{Set of all locally closed subsets of } (X,\tau)

\text{GLC}(X,\tau) \rightarrow \text{Set of all } g \text{-locally closed subsets of } (X,\tau)

\alpha\text{LC}(X,\tau) \rightarrow \text{Set of all } \alpha \text{-locally closed subsets of } (X,\tau)

\alpha\text{LC}^*(X,\tau) \rightarrow \text{Set of all } \alpha\text{le}^* \text{-locally closed subsets of } (X,\tau)

\alpha\text{LC}**(X,\tau) \rightarrow \text{Set of all } \alpha\text{le}^{**} \text{-locally closed subsets of } (X,\tau)

C(X,x) \rightarrow \text{Set of all closed sets of } X \text{ containing } x.

\alpha\text{O}(X,x) \rightarrow \text{Set of all } \alpha \text{-open subsets of } X \text{ containing } x.

D(\tau_i,\tau_j) \rightarrow \text{Set of all } (\tau_i,\tau_j) \text{-g-closed sets of } (X,\tau_i,\tau_j).