Decomposition Of \( \pi g \alpha \)-Sets

- Introduction
- \( \pi g \alpha \)-Locally Closed Sets
- \( \pi G \alpha \)-LC Continuous And \( \pi G \alpha \)-LC Irresolute Functions
- Decomposition Of \( \pi g \alpha \)-Continuity
CHAPTER V

DECOMPOSITION OF \( \pi g\alpha \)-SETS

5.1 Introduction

The notion of a locally closed set in a topological space was studied by many topologists [58, 82, 171]. Thereafter Balachandran [15], Arockia Rani [6], Nasef [113] and Park [147] studied the weaker forms of locally closed sets. Noiri [129], Ganster and Reilly [59], Nashef [2] established decomposition of \( \alpha \)-continuity, A-continuity, \( \alpha \)-continuity and semi-continuity respectively. In this chapter, we introduce three new classes of sets called \( \pi G\alpha \)-LC\((X,\tau)\), \( \pi G\alpha \)-LC\(\ast\)(\(X,\tau)\), \( \pi G\alpha \)-LC\(\ast\ast\)(\(X,\tau)\) sets along with their respective continuity and irresoluteness. The notions of \( C_\tau \)-sets, \( C_{\tau^+} \)-sets and \( K_{\tau^+} \)-sets, \( K_\tau \)-sets are used to obtain decompositions of \( \pi g \)-continuity, \( \pi g \)-open maps, contra-\( \pi g \)-continuity and decompositions of \( \pi g\alpha \)-continuity, \( \pi g\alpha \)-open maps, contra-\( \pi g\alpha \)-continuity respectively.

5.2 \( \pi g\alpha \)-Locally Closed Sets

In this section we define \( \pi g\alpha \)-locally closed sets which contain the class of \( \alpha \)-LC sets and study some of their properties.

Definition 5.2.1: A subset \( S \) of \((X, \tau)\) is called

a) \( \pi g\alpha \)-locally closed (briefly a \( \pi g\alpha \)-lc set) if \( S = A \cap B \) where \( A \) is \( \pi g\alpha \)-open and \( B \) is \( \pi g\alpha \)-closed in \( X \).

b) a \( \pi g\alpha \)-lc\(\ast\) set if there exist a \( \pi g\alpha \)-open set \( A \) and a closed set \( B \) of \( X \) such that \( S = A \cap B \).

c) a \( \pi g\alpha \)-lc\(\ast\ast\) set if there exist an open set \( A \) and a \( \pi g\alpha \)-closed set \( B \) of \( X \) such that \( S = A \cap B \).

The collection of all \( \pi g\alpha \)-lc sets, \( \pi g\alpha \)-lc\(\ast\) sets and \( \pi g\alpha \)-lc\(\ast\ast\) sets of \((X, \tau)\) will be denoted by \( \pi G\alpha \)-LC\((X,\tau)\), \( \pi G\alpha \)-LC\(\ast\)(\(X,\tau)\) and \( \pi G\alpha \)-LC\(\ast\ast\)(\(X,\tau)\) respectively.
Proposition 5.2.2:  i) If $A \in \text{LC}(X, \tau)$, then $A \in \pi \alpha \text{-LC}(X, \tau)$.

ii) If $A \in \text{LC}(X, \tau)$, then $A \in \pi \alpha \text{-LC}^*(X, \tau)$ and $\pi \alpha \text{-LC}**(X, \tau)$.

iii) If $A \in \pi \alpha \text{-LC}^*(X, \tau)$, then $A \in \pi \alpha \text{-LC}(X, \tau)$.

iv) If $A \in \alpha \text{-LC}(X, \tau)$, then $A \in \pi \alpha \text{-LC}(X, \tau)$.

v) If $A \in \alpha \text{-LC}^*(X, \tau)$, then $A \in \pi \alpha \text{-LC}^*(X, \tau)$.

vi) If $A \in \alpha \text{-LC}**(X, \tau)$, then $A \in \pi \alpha \text{-LC}**(X, \tau)$.

Proof: Obvious.

Remark 5.2.3: Converse of the above need not be true as seen in the following examples.

Example 5.2.4:

i) Let $X = \{a,b,c,d\}$, $\tau = \{\emptyset, X, \{a\}\}$. Then $\text{LC}(X) = \{\emptyset, X, \{a\}, \{b,c,d\}\}$. Then $\pi \alpha \text{-LC}(X) = P(X)$. This shows that a $\pi \alpha$-locally closed set need not be locally closed.

ii) Let $X = \{a,b,c\}$, $\tau = \{\emptyset, X, \{a, b\}\}$. Then $\text{LC}(X, \tau) = \{\emptyset, X, \{a, b\}, \{c\}\}$. Then $\{a\} \in \pi \alpha \text{-LC}^*(X, \tau)$ and $\pi \alpha \text{-LC}**(X, \tau)$ but $\{a\} \notin \text{LC}(X, \tau)$.

Example 5.2.5: a) Let $X = \{a,b,c,d\}$, $\tau = \{\emptyset, X, \{a\}, \{b,c,d\}\}$. Then

i) $\{a, b, d\} \in \pi \alpha \text{-LC}(X, \tau)$ but $\{a, b, d\} \notin \pi \alpha \text{-LC}^*(X, \tau)$.

ii) $\{a, b, c\} \in \pi \alpha \text{-LC}**(X, \tau)$ but $\{a, b, c\} \notin \pi \alpha \text{-LC}^*(X, \tau)$.

b) Let $X = \{a, b, c\}$, $\tau = \{\emptyset, X, \{a\}, \{b, c\}\}$. Then

i) $\{a, b\} \in \pi \alpha \text{-LC}(X, \tau)$ but $\{a, b\} \notin \alpha \text{-LC}(X, \tau)$.

ii) $\{c\} \in \pi \alpha \text{-LC}^*(X, \tau)$ but $\{c\} \notin \alpha \text{-LC}^*(X, \tau)$.

iii) $\{c\} \in \pi \alpha \text{-LC}**(X, \tau)$ but $\{c\} \notin \alpha \text{-LC}**(X, \tau)$.

Remark 5.2.6: The above discussions are summarized in the following diagram.
Proposition 5.2.7: a) Let \((X, \tau)\) be a \(\pi g\alpha\)-space. Then

i) \(\pi G\alpha-LC(X, \tau) = LC(X, \tau)\).

ii) \(\pi G\alpha-LC(X, \tau) \subseteq GLC(X, \tau)\).

iii) \(\pi G\alpha-LC(X, \tau) \subseteq \alpha-LC(X, \tau)\).

b) If \(\pi G\alpha-O(X, \tau) = GO(X, \tau)\), then \(\pi G\alpha-LC(X, \tau) = GLC(X, \tau)\).

c) If \(X\) is a \(\pi g\alpha-T\frac{1}{2}\) space, then \(\pi G\alpha-LC(X, \tau) = \alpha-LC(X, \tau)\).

d) If \(X\) is a \(\pi g\alpha\)-space, then \(\pi G\alpha-LC(X, \tau) = \pi G\alpha-LC*(X, \tau) = \pi G\alpha-LC**(X, \tau)\)

Proof: a) i) Since every \(\pi g\alpha\)-open set is open and every \(\pi g\alpha\)-closed set is closed in \(X\), we have \(\pi G\alpha-LC(X, \tau) \subseteq LC(X, \tau)\) and hence \(\pi G\alpha-LC(X, \tau) = LC(X, \tau)\).

ii) and iii) Since \(LC(X, \tau) \subseteq GLC(X, \tau)\) and \(LC(X, \tau) \subseteq \alpha-LC(X, \tau)\) for any space \(X\) and from i) the proof follows.

b) Let \(A \in \pi G\alpha-LC(X, \tau)\). Then \(A = P \cap Q\) where \(P\) is \(\pi g\alpha\)-open and \(Q\) is \(\pi g\alpha\)-closed in \(X\). By hypothesis, \(P\) is \(g\)-open and \(Q\) is \(g\)-closed. Therefore \(A \in GLC(X, \tau)\) and \(\pi G\alpha-LC(X, \tau) \subseteq GLC(X, \tau)\). Obviously \(GLC(X, \tau) \subseteq \pi G\alpha-LC(X, \tau)\).

Hence \(\pi G\alpha-LC(X, \tau) = GLC(X, \tau)\).

c) Follows from definition 2.3.14 and from the fact that every \(\alpha\)-open set is \(\pi g\alpha\)-open.

d) Obvious.
Remark 5.2.8: Converse of the above Proposition 5.2.7 (b), (c) does not hold as seen in the following example.

Example 5.2.9: Let $X = \{a, b, c, d\}$, $\tau = \{\phi, X, \{b\}, \{a, b\}, \{b, c\}, \{a, b, c\}\}$ then $G_\alpha\text{-LC}(X, \tau) = \alpha\text{-LC}(X, \tau) = \text{GLC}(X, \tau) = P(X)$. But $G_\alpha O(X) = \{\phi, X, \{a\}, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{a, b, c\}\} \neq \pi G_\alpha O(X)$.

$aO(X) = \{\phi, X, \{b\}, \{a, b\}, \{b, c\}, \{a, b, c\}, \{a, b, d\}, \{b, c, d\}\} \neq \pi G_\alpha O(X)$.

Remark 5.2.10: For subsequent results in this chapter we assume that $\pi G_\alpha C(X, \tau)$ is closed under finite intersections.

The hypothesis in Proposition 5.2.7 d) can be weakened as follows.

Proposition 5.2.11: If $\pi G_\alpha O(X, \tau) \subset \text{LC}(X, \tau)$, then $\pi G_\alpha\text{-LC}(X, \tau) = \pi G_\alpha\text{-LC}^*(X, \tau) = \pi G_\alpha\text{-LC}^{**}(X, \tau)$.

Proof: Let $A \in \pi G_\alpha\text{-LC}(X)$. Then $A = P \cap Q$ where $P$ is $\pi g_\alpha$-open and $Q$ is $\pi g_\alpha$-closed. Since $\pi G_\alpha O(X, \tau) \subset \text{LC}(X, \tau)$ implies $\pi G_\alpha C(X, \tau) \subset \text{LC}(X, \tau)$, we have $Q$ is locally closed. Let $Q = M \cap N$ where $M$ is open and $N$ is closed. So $A = (P \cap M) \cap N$ where $P \cap M$ is $\pi g_\alpha$-open and $N$ is closed. Hence $A \in \pi G_\alpha\text{-LC}^*(X)$. For any space $X$, $\pi G_\alpha\text{-LC}^*(X) \subset \pi G_\alpha\text{-LC}(X)$. Thus $\pi G_\alpha\text{-LC}(X) = \pi G_\alpha\text{-LC}^*(X)$. Let $B \in \pi G_\alpha\text{-LC}(X)$. Then $B = P \cap Q$ where $P$ is $\pi g_\alpha$-open and $Q$ is $\pi g_\alpha$-closed. Since $\pi G_\alpha O(X, \tau) \subset \text{LC}(X, \tau)$ implies $P$ is locally closed, we have $P = M \cap N$ where $M$ is open and $N$ is closed. So $A = M \cap (N \cap Q)$ where $M$ is open and $N \cap Q$ is $\pi g_\alpha$-closed. Hence $B \in \pi G_\alpha\text{-LC}^{**}(X)$. For any space $X$, $\pi G_\alpha\text{-LC}^{**}(X) \subset \pi G_\alpha\text{-LC}(X)$. Thus $\pi G_\alpha\text{-LC}(X, \tau) = \pi G_\alpha\text{-LC}^{**}(X, \tau)$.

Now, we obtain a characterization for $\pi G_\alpha\text{-LC}^{*}(X, \tau)$ sets as follows:

Theorem 5.2.12: For a subset $S$ of $(X, \tau)$ the following are equivalent:
1. $S \in \pi G_\alpha\text{-LC}^{*}(X, \tau)$.
2. $S = P \cap \text{cl}(S)$ for some $\pi g_\alpha$-open set $P$.
3. $\text{cl}(S) - S$ is $\pi g_\alpha$-closed.
4. $S \cup (X - \text{cl}(S))$ is $\pi \text{g}_\alpha$-open.

**Proof :** $1 \Rightarrow 2$: Let $S \in \pi G_\alpha - \text{LC}^*(X, \tau)$. Then there exist a $\pi \text{g}_\alpha$-open set $P$ and a closed set $F$ in $(X, \tau)$ such that $S = P \cap F$. Since $S \subset P$ and $S \subset \text{cl}(S)$, we have $S \subset P \cap \text{cl}(S)$.

Also, $S \subset F$ and $F$ is closed implies $P \cap \text{cl}(S) \subset P \cap F = S$. Hence $S = P \cap \text{cl}(S)$.

$2 \Rightarrow 1$: Since $P$ is $\pi \text{g}_\alpha$-open and $\text{cl}(S)$ is closed, $S = P \cap \text{cl}(S) \in \pi G_\alpha - \text{LC}^*(X, \tau)$.

$2 \Rightarrow 3$: Let $S = P \cap \text{cl}(S)$ for some $\pi \text{g}_\alpha$-open set $P$. We have $\text{cl}(S) - S = \text{cl}(S) \cap P^c$, which is $\pi \text{g}_\alpha$-closed.

$3 \Rightarrow 2$: Assume $\text{cl}(S) - S$ is $\pi \text{g}_\alpha$-closed. Let $P = X - (\text{cl}(S) - S)$. Then $P$ is $\pi \text{g}_\alpha$-open and $S = P \cap \text{cl}(S)$.

$3 \Rightarrow 4$: Let $F = \text{cl}(S) - S$. Then $F$ is $\pi \text{g}_\alpha$-closed, by assumption.

$4 \Rightarrow 3$: Let $U = S \cup (X - \text{cl}(S))$. Then $U$ is $\pi \text{g}_\alpha$-open. This implies $X - U = X - (S \cup (X - \text{cl}(S))) = (X - S) \cap \text{cl}(S) = \text{cl}(S) - S$ is $\pi \text{g}_\alpha$-closed.

**Remark 5.2.13**: It is not true that $S \in \pi G_\alpha - \text{LC}^*(X, \tau)$ if and only if $S \subset \text{int}(S \cup (X - \text{cl}(S)))$. Let $S = \{b, c\}$ be a subset of the topological space $(X, \tau)$ given in Example 5.2.5(a). Then $S \subset \text{int}(S \cup (X - \text{cl}(S)))$ but $S \in \pi G_\alpha - \text{LC}^*(X, \tau)$.

**Definition 5.2.14**: A topological space $(X, \tau)$ is called $\pi \text{g}_\alpha$-submaximal if every dense subset in $(X, \tau)$ is $\pi \text{g}_\alpha$-open.

**Proposition 5.2.15**: a) Let $(X, \tau)$ be a topological space. If $X$ is submaximal, then it is $\pi \text{g}_\alpha$-submaximal.

b) A topological space $(X, \tau)$ is $\pi \text{g}_\alpha$-submaximal if and only if $\pi G_\alpha - \text{LC}^*(X, \tau) = P(X)$.

**Proof :** a) Obvious.

b) **Necessity**: Let $S \in P(X)$ and $U = S \cup (X - \text{cl}(S))$. Then $\text{cl}(U) = X$. $U$ is dense in $X$ and $X$ is $\pi \text{g}_\alpha$-submaximal implies $U$ is $\pi \text{g}_\alpha$-open. By Theorem 5.2.12, $S \in \pi G_\alpha - \text{LC}^*(X, \tau)$.

**Sufficiency**: Let $S$ be a dense subset of $(X, \tau)$. Then $S \cup (X - \text{cl}(S)) = S \cup \phi = S$. Now
$S \in P(X)$ implies $S \in \pi G \alpha - LC^*(X, \tau)$. By Theorem 5.2.12, $S \cup (X - cl(S)) = S$ is $\pi g \alpha$-open. Hence $(X, \tau)$ is $\pi g \alpha$-submaximal.

**Remark 5.2.16:** Converse of Proposition 5.2.15 a) is not true as seen in the following example.

**Example 5.2.17:** Let $X = \{a, b, c\}, \tau = \{\emptyset, X, \{a\}, \{b, c\}\}$. Let $A = \{a, b\}$. Then $A$ is dense in $X$ such that $A$ is $\pi g \alpha$-open but not open.

**Proposition 5.2.18:** For a subset $S$ of $(X, \tau)$ if $S \in \pi G \alpha - LC^{**}(X, \tau)$, then there exists an open set $P$ such that $S = P \cap cl(S)$ where $cl(S)$ is the $\pi g \alpha$-closure of $S$.

**Proof:** Let $S \in \pi G \alpha - LC^{**}(X, \tau)$. Then there exist an open set $P$ and a $\pi g \alpha$-closed set $F$ of $(X, \tau)$ such that $S = P \cap F$. Since $S \subset P$ and $S \subset cl(S)$, we have $S \subset P \cap cl(S)$. Since $cl(S) \subset F$, we have $P \cap cl(S) \subset P \cap F \subset S$. Thus $S = P \cap cl(S)$.

**Theorem 5.2.19:** Let $A$ and $B$ be any two subsets of $(X, \tau)$.

a) If $A \in \pi G \alpha - LC(X, \tau)$ and $B$ is $\pi g \alpha$-open or $\pi g \alpha$-closed, then $A \cap B \in \pi G \alpha - LC(X, \tau)$.

b) If $A \in \pi G \alpha - LC^{**}(X, \tau)$ and $B$ is closed or open, then $A \cap B \in \pi G \alpha - LC^{**}(X, \tau)$.

**Proof:** a) $A \in \pi G \alpha - LC(X, \tau)$ implies $A \cap B = (G \cap F) \cap B$ for some $\pi g \alpha$-open set $G$ and $\pi g \alpha$-closed set $F$. If $B$ is $\pi g \alpha$-open then $A \cap B = (G \cap B) \cap F \in \pi G \alpha - LC(X, \tau)$. If $B$ is $\pi g \alpha$-closed, then $A \cap B = G \cap (B \cap F) \in \pi G \alpha - LC(X, \tau)$.

b) If $A \in \pi G \alpha - LC^{**}(X, \tau)$, then there exist an open set $G$ and a $\pi g \alpha$-closed set $F$ of $(X, \tau)$ such that $A \cap B = (G \cap F) \cap B$. If $B$ is open, then $A \cap B = (G \cap B) \cap F \in \pi G \alpha - LC^{**}(X, \tau)$. If $B$ is closed, then $A \cap B = G \cap (F \cap B) \in \pi G \alpha - LC^{**}(X, \tau)$.

**Theorem 5.2.20:** If $A \in \pi G \alpha - LC^*(X, \tau)$ and $B \in \pi G \alpha - LC^*(X, \tau)$, then $A \cap B \in \pi G \alpha - LC^*(X, \tau)$.

**Proof:** If $A, B \in \pi G \alpha - LC^*(X, \tau)$ then by Theorem 5.2.12, there exist $\pi g \alpha$-open sets $P$ and $Q$ such that $A = P \cap cl(A)$ and $B = Q \cap cl(B)$. $P \cap Q$ is also $\pi g \alpha$-open. Then $A \cap B = (P \cap Q) \cap cl(A) \cap cl(B) \in \pi G \alpha - LC^*(X, \tau)$.
Proposition 5.2.21: Let $A$ and $Z$ be any two subsets of $(X, \tau)$ and let $A \subset Z$. If $Z$ is regular open and $\pi \alpha$-closed in $(X, \tau)$ and if $A \in \pi \alpha$-LC* $(Z, \tau / Z)$, then $A \in \pi \alpha$-LC* $(X, \tau)$.

Proof: If $A \in \pi \alpha$-LC* $(Z, \tau / Z)$ then by Theorem 5.2.12, there is a $\pi \alpha$-open set $G$ in $(Z, \tau / Z)$ such that $A = G \cap \text{cl}_{Z}(A)$ where $\text{cl}_{Z}(A) = Z \cap \text{cl}(A)$. By Proposition 2.2.19, $G$ is $\pi \alpha$-open in $X$. We have $A = (G \cap Z) \cap \text{cl}(A) \in \pi \alpha$-LC* $(X, \tau)$.

Remark 5.2.22: The following examples show that one of the assumptions in the above theorem. That is, $Z$ is regular open in $(X, \tau)$ cannot be removed.

Example 5.2.23: Let $X = \{a, b, c, d\}$, $\tau = \{\emptyset, X, \{b\}, \{c, d\}, \{b, c, d\}\}$. Let $Z = A = \{a, b, d\}$. $\tau / Z = \{\emptyset, \{b\}, \{b, d\}, Z\}$ where $Z$ is not regular open in $X$. Then $A \in \pi \alpha$-LC* $(Z, \tau / Z)$ but $A \notin \pi \alpha$-LC* $(X, \tau)$.

Theorem 5.2.24: Let $A$ and $Z$ be any two subsets of $(X, \tau)$ and let $A \subset Z$ such that $Z$ is $\pi \alpha$-closed and regular open in $X$. Then

1) if $A \in \pi \alpha$-LC $(Z, \tau / Z)$, then $A \in \pi \alpha$-LC $(X, \tau)$.

2) if $A \in \pi \alpha$-LC** $(Z, \tau / Z)$, then $A \in \pi \alpha$-LC** $(X, \tau)$.

Proof: 1) Let $A \in \pi \alpha$-LC $(Z, \tau / Z)$. Then $A = G \cap F$ where $G$ is $\pi \alpha$-open and $F$ is $\pi \alpha$-closed in $(Z, \tau / Z)$. Then by Proposition 2.2.19, $G$ and $F$ are $\pi \alpha$-open and $\pi \alpha$-closed sets in $(X, \tau)$ respectively. Hence $A = G \cap F \in \pi \alpha$-LC $(X, \tau)$.

2) Let $A \in \pi \alpha$-LC** $(Z, \tau / Z)$. Then $A = G \cap F$ where $G$ is open and $F$ is $\pi \alpha$-closed in $(Z, \tau / Z)$. Then by Proposition 2.2.19, $G$ is open and $F$ is $\pi \alpha$-closed in $(X, \tau)$. Hence $A = G \cap F \in \pi \alpha$-LC** $(X, \tau)$.

Proposition 5.2.25: Let $A, B \in \pi \alpha$-LC* $(X, \tau)$. If $A$ and $B$ are separated in $(X, \tau)$, then $A \cup B \in \pi \alpha$-LC* $(X, \tau)$.

Proof: Since $A, B \in \pi \alpha$-LC* $(X, \tau)$ by Theorem 5.2.12, there exist $\pi \alpha$-open sets $P$ and $Q$ of $(X, \tau)$ such that $A = P \cap \text{cl}(A)$ and $B = Q \cap \text{cl}(B)$. Put $U = P \cap (X - \text{cl}(B))$ and $V = Q \cap (X - \text{cl}(A))$. Then $U$ and $V$ are $\pi \alpha$-open subsets of $(X, \tau)$. Then $A = U \cup \text{cl}(A)$,
B = V \cap \text{cl}(B), U \cap \text{cl}(B) = \phi, V \cap \text{cl}(A) = \phi \text{ hold. Consequently.}

A \cup B = (U \cup V) \cap (\text{cl}(A \cup B)) \text{ showing that } A \cup B \in \piGA-LC^*(X, \tau).

Proposition 5.2.26: Let \{Z_i; i \in A\} be a finite \pi-cover of (X,\tau) and let A be a subset of (X,\tau). If \(A \cap Z_i \in \piGA-LC^*(Z_i, \tau / Z_i)\) for each i \in A, then A \in \piGA-LC^*(X, \tau).

Proof: For each i \in A, there exist an open set U_i \in \tau and \piGA-closed set F_i of (Z_i, \tau / Z_i), such that A \cap Z_i = (U_i \cap F_i) \cap Z_i = U_i \cap (F_i \cap Z_i). Then

A = \cup \{A \cap Z_i; i \in A\} = [\cup \{U_i; i \in A\}] \cap [\cup \{Z_i \cap F_i; i \in A\}] \text{ and hence by Proposition 2.2.10, } A \in \piGA-LC^*(X, \tau).

Theorem 5.2.27: Let X, Y be topological spaces which are T_{\pi}-spaces.

i) If A \in \piGA-LC(X,\tau) and B \in \piGA-LC(Y,\sigma), then A \times B \in \piGA-LC(X \times Y, \tau \times \sigma).

ii) If A \in \piGA-LC^*(X,\tau) and B \in \piGA-LC^*(Y,\sigma), then A \times B \in \piGA-LC^*(X \times Y, \tau \times \sigma).

iii) If A \in \piGA-LC^{**}(X,\tau) and B \in \piGA-LC^{**}(Y,\sigma), then A \times B \in \piGA-LC^{**}(X \times Y, \tau \times \sigma).

Proof: i) Let A \in \piGA-LC(X,\tau) and B \in \piGA-LC(Y,\sigma).

Then there exist \piGA-open sets V, V^1 and \piGA-closed sets W, W^1 of (X,\tau) and (Y,\sigma) respectively such that A = V \cap W and B = V^1 \cap W^1. Then

A \times B = (V \cap W) \times (V^1 \cap W^1) = (V \times V^1) \cap (W \times W^1) \text{ holds and hence } A \times B \in \piGA-LC(X \times Y, \tau \times \sigma).

Proofs of (ii) and (iii) are similar to that of (i).

5.3 \piGA-LC Continuous And \piGA-LC Irresolute Functions

In this section, we define \piGA-LC continuous and \piGA-LC irresolute functions and obtain pasting Lemma for \piGA-LC** continuous functions and \piGA-LC** irresolute functions.

Definition 5.3.1: A function f:(X,\tau)\rightarrow(Y,\sigma) is called

i) \piGA-LC continuous if \(f^{-1}(V) \in \piGA-LC(X,\tau)\) for every \(V \in \sigma\).

ii) \piGA-LC*continuous if \(f^{-1}(V) \in \piGA-LC^*(X,\tau)\) for every \(V \in \sigma\).
iii) \( \pi \alpha-\text{LC}^{**} \) continuous if \( f^{-1}(V) \in \pi \alpha-\text{LC}^{**}(X,\tau) \) for every \( V \in \sigma \).

iv) \( \pi \alpha-\text{LC} \) irresolute if \( f^{-1}(V) \in \pi \alpha-\text{LC}(X,\tau) \) for every \( V \in \pi \alpha-\text{LC}(Y,\sigma) \).

v) \( \pi \alpha-\text{LC}^* \) irresolute if \( f^{-1}(V) \in \pi \alpha-\text{LC}^*(X,\tau) \) for every \( V \in \pi \alpha-\text{LC}^*(Y,\sigma) \).

vi) \( \pi \alpha-\text{LC}^{**} \) irresolute if \( f^{-1}(V) \in \pi \alpha-\text{LC}^{**}(X,\tau) \) for every \( V \in \pi \alpha-\text{LC}^{**}(Y,\sigma) \).

**Proposition 5.3.2:** If \( f:(X,\tau) \rightarrow (Y,\sigma) \) is \( \pi \alpha-\text{LC} \) irresolute, then it is \( \pi \alpha-\text{LC} \) continuous.

**Proof:** Let \( V \) be open in \( Y \). Then \( V \in \pi \alpha-\text{LC}(Y,\sigma) \). By assumption, \( f^{-1}(V) \in \pi \alpha-\text{LC}(X,\tau) \). Hence \( f \) is \( \pi \alpha-\text{LC} \) continuous.

**Proposition 5.3.3:** Let \( f:(X,\tau) \rightarrow (Y,\sigma) \) be a function.

1) If \( f \) is LC-continuous, then \( f \) is \( \pi \alpha-\text{LC}^* \) continuous and \( \pi \alpha-\text{LC}^{**} \) continuous.

2) If \( f \) is \( \pi \alpha-\text{LC}^* \) continuous, then \( f \) is \( \pi \alpha-\text{LC} \) continuous.

3) If \( f \) is \( \pi \alpha-\text{LC}^* \) irresolute, then \( f \) is \( \pi \alpha-\text{LC}^* \) continuous

**Remark 5.3.4:** Converse of the above need not be true as can be seen in the following examples.

**Examples 5.3.5:**

1) Let \( X = \{a,b,c\}, \tau = \{\emptyset, X, \{a\}, \{b\}\} \), \( \sigma = \{\emptyset, X, \{a\}, \{b\}, \{a,b\}\} \). Let \( f:(X,\tau) \rightarrow (X,\sigma) \) be the identity mapping. Then \( f \) is \( \pi \alpha-\text{LC}^* \) continuous and \( \pi \alpha-\text{LC}^{**} \) continuous but not LC-continuous.

2) Let \( X = \{a,b,c,d\}, \tau = \{\emptyset, X, \{b\}, \{c,d\}, \{b,c,d\}\} \), \( \sigma = \{\emptyset, X, \{c\}, \{a,b,d\}\} \) and \( f:(X,\tau) \rightarrow (X,\sigma) \) be the identity mapping. Then \( f \) is \( \pi \alpha-\text{LC} \) continuous but not \( \pi \alpha-\text{LC}^* \) continuous since \( \{a,b,d\} \in (X,\sigma) \) but \( \{a,b,d\} \notin \pi \alpha-\text{LC}^*(X,\tau) \).

3) Let \( X = \{a,b,c,d\}, \tau = \{\emptyset, X, \{b\}, \{c,d\}, \{b,c,d\}\} \), \( \sigma = \{\emptyset, X, \{b\}\} \) and \( f:(X,\tau) \rightarrow (X,\sigma) \) be the identity mapping. Then \( f \) is \( \pi \alpha-\text{LC}^* \) continuous but not \( \pi \alpha-\text{LC}^* \) irresolute since \( \{a, b, d\} \in \pi \alpha-\text{LC}^*(X,\sigma) \) but \( \{a, b, d\} \notin \pi \alpha-\text{LC}^*(X,\tau) \).

**Proposition 5.3.6:** Any map defined on a door space is \( \pi \alpha-\text{LC} \) irresolute.

**Proof:** Let \( (X,\tau) \) be a door space and \( (Y,\sigma) \) be any space. Define a map \( f:(X,\tau) \rightarrow (Y,\sigma) \). Let \( A \in \pi \alpha-\text{LC}(Y,\sigma) \). Then \( f^{-1}(A) \) is either open or closed in \( (X,\tau) \). In both cases \( f^{-1}(A) \in \pi \alpha-\text{LC}(X,\tau) \). Hence \( f \) is \( \pi \alpha-\text{LC} \) irresolute.

84
Theorem 5.3.7: A topological space \((X, \tau)\) is \(\pi\text{g}\alpha\)-submaximal if and only if every function having \((X, \tau)\) as it domain is \(\pi\text{G}\alpha\)-LC*continuous.

Proof: Suppose that \(f:(X, \tau)\to(Y, \sigma)\) is a function. By Theorem 5.2.15 b), we have \(f^{-1}(V) \in P(X) = \pi\text{G}\alpha\)-LC*(\(X, \tau\)) for each open set \(V\) of \((Y, \sigma)\). Therefore \(f\) is \(\pi\text{G}\alpha\)-LC* continuous. Conversely, let every map having \((X, \tau)\) as domain be \(\pi\text{G}\alpha\)-LC* continuous. Let \(Y = \{0, 1\}\) be the Sierpinski space with topology \(\sigma = \{Y, \phi, \{0\}\}\). Let \(V\) be a subset of \((X, \tau)\) and \(f:(X, \tau)\to(Y, \sigma)\) be a function defined by \(f(x) = 0\) for every \(x \in V\) and \(f(x) = 1\) for every \(x \in \bar{V}\). By assumption, \(f\) is \(\pi\text{G}\alpha\)-LC* continuous and hence \(f^{-1}\{0\} = V \in \pi\text{G}\alpha\)-LC*(\(X, \tau\)). Therefore we have \(P(X) = \pi\text{G}\alpha\)-LC*(\(X, \tau\)) and by Theorem 5.2.15 b), \((X, \tau)\) is \(\pi\text{g}\alpha\)-submaximal.

Proposition 5.3.8: If \(f:(X, \tau)\to(Y, \sigma)\) is \(\pi\text{G}\alpha\)-LC** continuous and a subset \(B\) is regular open \(, \pi\text{g}\alpha\)-closed in \((X, \tau)\), then the restriction of \(f\) to \(B\) say \(f/B:(B, \tau/B)\to(Y, \sigma)\) is \(\pi\text{G}\alpha\)-LC** continuous.

Proof: Let \(V\) be an open set of \((Y, \sigma)\). Then \(f^{-1}(V) = G \cap F\) for some open set \(G\) and \(\pi\text{g}\alpha\)-closed set \(F\) of \((X, \tau)\). Now \(G \cap B \in \tau/B\) and \((F \cap B)\) is a \(\pi\text{g}\alpha\)-closed subset of \((B, \tau/B)\). But \((f/B)^{-1}(V) = (G \cap B) \cap (F \cap B)\). Hence \((f/B)^{-1}(V) \in \pi\text{G}\alpha\)-LC**(\(B, \tau/B\)). This implies that \(f/B\) is \(\pi\text{G}\alpha\)-LC** continuous.

We recall the definition of the combination of two functions: Let \(X = A \cup B\) and \(f : A \rightarrow Y\) and \(h : B \rightarrow Y\) be two functions. We say that \(f\) and \(h\) are compatible if \(f \upharpoonright (A \cap B) = h \upharpoonright (A \cap B)\). If \(f: A \rightarrow Y\) and \(h: B \rightarrow Y\) are compatible, then the function \(f \vee h : X \rightarrow Y\) defined as \((f \vee h)(x) = f(x)\) for every \(x \in A\), \((f \vee h)(x) = h(x)\) for every \(x \in B\) is called the combination of \(f\) and \(h\).

Pasting Lemma for \(\pi\text{G}\alpha\)-LC** continuous (resp. \(\pi\text{G}\alpha\)-LC**-irresolute) functions.

Theorem 5.3.9: Let \(X = A \cup B\), where \(A\) and \(B\) are \(\pi\text{g}\alpha\)-closed and regular open subsets of \((X, \tau)\) and \(f : (A, \tau/A) \rightarrow (Y, \sigma)\) and \(h : (B, \tau/B) \rightarrow (Y, \sigma)\) be compatible functions.

a) If \(f\) and \(h\) are \(\pi\text{G}\alpha\)-LC** continuous, then \((f \vee h) : X \rightarrow Y\) is \(\pi\text{G}\alpha\)-LC** continuous.

b) If \(f\) and \(h\) are \(\pi\text{G}\alpha\)-LC** irresolute, then \((f \vee h) : X \rightarrow Y\) is \(\pi\text{G}\alpha\)-LC** irresolute.

85
Proof :a) Let $V \in \sigma$. Then $(f \vee h)^{-1}(V) \cap A = f^{-1}(V)$ and $(f \vee h)^{-1}(V) \cap B = h^{-1}(V)$. By assumption, $(f \vee h)^{-1}(V) \cap A \in \pi \text{Ga-LC}**(A, \tau /A)$ and $(f \vee h)^{-1}(V) \cap B \in \pi \text{Ga-LC}**(B, \tau /B)$. Therefore by Proposition 5.2.26, $(f \vee h)^{-1}(V) \in \pi \text{Ga-LC}** (X, \tau)$ and hence $f \vee h$ is $\pi \text{Ga-LC}**$continuous.

b) Proof is similar to that of a).

Next we have the theorem concerning the composition of functions.

Theorem 5.3.10 : Let $f: (X, \tau) \to (Y, \sigma)$ and $g: (Y, \sigma) \to (Z, \eta)$ be two functions. Then
a) $g \circ f$ is $\pi \text{Ga-LC}$ irresolute if $f$ and $g$ are $\pi \text{Ga-LC}$ irresolute.
b) $g \circ f$ is $\pi \text{Ga-LC}*$ irresolute if $f$ and $g$ are $\pi \text{Ga-LC}*$ irresolute.
c) $g \circ f$ is $\pi \text{Ga-LC}**$ irresolute if $f$ and $g$ are $\pi \text{Ga-LC}**$ irresolute.
d) $g \circ f$ is $\pi \text{Ga-LC}$ continuous if $f$ is $\pi \text{Ga-LC}$ irresolute and $g$ is $\pi \text{Ga-LC}$ continuous.
e) $g \circ f$ is $\pi \text{Ga-LC}*$ continuous if $f$ is $\pi \text{Ga-LC}*$ continuous and $g$ is $\pi \text{Ga-LC}$ continuous.
f) $g \circ f$ is $\pi \text{Ga-LC}$ continuous if $f$ is $\pi \text{Ga-LC}$ continuous and $g$ is continuous.
g) $g \circ f$ is $\pi \text{Ga-LC}*$ continuous if $f$ is $\pi \text{Ga-LC}*$ irresolute and $g$ is continuous.
h) $g \circ f$ is $\pi \text{Ga-LC}**$ continuous if $f$ is $\pi \text{Ga-LC}**$ irresolute and $g$ is $\pi \text{Ga-LC}**$ continuous.

Definition 5.3.11 : A function $f: (X, \tau) \to (Y, \sigma)$ is called sub $\pi \text{Ga-LC}*$ continuous if there exists a basis $B$ for $(Y, \sigma)$ such that $f^{-1}(U) \in \pi \text{Ga-LC}*(X, \tau)$ for each $U \in B$.

Proposition 5.3.12 : Let $f: (X, \tau) \to (Y, \sigma)$ be a function.
a) If $f$ is sub-$\pi \text{Ga-LC}*$continuous if and only if there is a subbasis $C$ of $(Y, \sigma)$ such that $f^{-1}(U) \in \pi \text{Ga-LC}*(X, \tau)$ for each $U \in C$.
b) If $f$ is sub-LC-continuous, then $f$ is sub-$\pi \text{Ga-LC}*$continuous.

Proof :a) By assumption, there exists a basis $B$ for $(Y, \sigma)$ such that $f^{-1}(U) \in \pi \text{Ga-LC}*(X, \tau)$ for each $U \in B$. Since $B$ is also a subbasis for $(Y, \sigma)$, the proof is obvious.

Conversely, for a subbasis $C$, let $C_\delta = \{ A \subset Y : A$ is an intersection of finitely many sets belonging to $C \}$. Then $C_\delta$ is a basis for $(Y, \sigma)$. For $U \in C_\delta$, $U = \cap \{ A : A \in \forall \}$ where
A is a finite set. By assumption and Proposition 5.2.20, we have
\[ f^{-1}(U) = \bigcap \{ f^{-1}(F_i) : i \in \Lambda \} \in \pi G\alpha -LC^*(X,\tau). \]

b) follows from the Definition 5.3.11 and the fact that every LC (X,\tau) is \( \pi G\alpha -LC^*(X,\tau) \).

Remark 5.3.13: Converse of Proposition 5.3.12 a) is not true as seen in the following example.

Example 5.3.14: Let \( X = Y = \{a,b,c\} \), \( \tau = \{\emptyset, X,\{a\}\} \) and \( \sigma \) be the topology induced by a base \( B \) of \( Y \). Let \( f: (X,\tau) \to (Y,\sigma) \) be the identity function. If \( B = \{Y,\{c\}\} \), then \( f \) is sub-\( \pi G\alpha -LC^* \) continuous but not sub LC-continuous since \( f^{-1}(\{c\}) = \{c\} \notin LC(X,\tau) \).

5.4 Decomposition Of \( \pi g\alpha \)-Continuity

In this section, we introduce the notions of \( C_\pi \)-sets, \( C_{\pi^*} \)-sets, \( K_\pi \)-sets and \( K_{\pi^*} \)-sets to obtain decompositions of \( \pi g \)-continuity and \( \pi g\alpha \)-continuity.

Definition 5.4.1: A subset \( S \) of \( (X,\tau) \) is called a
1. \( C_\pi \)-set if \( S = G \cap F \) where \( G \) is \( \pi g \)-open and \( F \) is a \( t \)-set
2. \( C_{\pi^*} \)-set if \( S = G \cap F \) where \( G \) is \( \pi g \)-open and \( F \) is a \( \alpha^* \)-set.
3. \( K_\pi \)-set if \( S = G \cap F \) where \( G \) is \( \pi g\alpha \)-open and \( F \) is a \( t \)-set.
4. \( K_{\pi^*} \)-set if \( S = G \cap F \) where \( G \) is \( \pi g\alpha \)-open and \( F \) is a \( \alpha^* \)-set.

Proposition 5.4.2:
1. Every \( B \)-set is a \( C_\pi \)-set.
2. Every \( B^* \)-set is a \( C_{\pi^*} \)-set.
3. Every \( C \)-set is a \( C_\pi \)-set.
4. Every \( C^* \)-set is a \( C_{\pi^*} \)-set.
5. Every \( C_{\pi} \)-set is a \( C_{\pi^*} \)-set.
6. Every \( C_{\pi^*} \)-set is a \( K_\pi \)-set.
7. Every $C^\pi$-set is a $K^\pi$-set.

8. Every $C^\pi$-set is a $C_r^\pi$-set.

9. Every $C^\pi$-set is a $C_r^\ast$-set.

10. Every $C^\pi$-set is a $K^\ast_r$-set.

11. Every $K^\pi$-set is a $K^\pi_n$-set.

Remark 5.4.3: Converse of the above need not be true as seen in the following examples.

Example 5.4.4: Let $X = \{a, b, c\}, \tau = \{\emptyset, X, \{a, b\}\}$. Let $A = \{a, c\}$.

Then $A$ is a $C_n$-set and $C^\pi$-set. But $A$ is neither a $B$-set nor a $C$-set.

Example 5.4.5: Let $X = \{a, b, c, d\}, \tau = \{\emptyset, X, \{a\}, \{b, c\}, \{a, b, c\}\}$. Let $A = \{c, d\}$. Then

$A$ is a $C^\pi$-set, $C_r$-set, $C_r^\ast$-set and $K^\pi$-set. But $A$ is neither a $C_n$-set nor a $K^\pi_n$-set.

Example 5.4.6: Let $X = \{a, b, c, d\}, \tau = \{\emptyset, X, \{a\}, \{c, d\}, \{a, c, d\}, \{d\}, \{a, d\}\}$. Let

$A = \{a, b, d\}$. Then $A$ is a $K^\pi_n$-set and $K^\pi$-set. But $A$ is neither $C^\pi$-set, nor $C_r^\ast$-set, nor

$C^\pi_n$-set, nor $C$-set.

Remark 5.4.7: $K^\pi_n$-set and $C_r^\pi$-set are independent concepts follows from Examples 5.4.6 and 5.4.5 respectively.

Remark 5.4.8: $K^\pi_n$-set and $C_r^\pi$-set are independent concepts follows from Examples 5.4.6 and 5.4.5 respectively.

Proposition 5.4.9: If $S$ is a $\pi\alpha$-open set, then

i) $S$ is a $K^\pi_n$-set.

ii) $S$ is a $K^\pi$-set.

Remark 5.4.10: Converse of the above need not be true as seen in the following example.
Example 5.4.11: Let $X = \{a, b, c\}, \tau = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\}$. Then $A = \{c\}$ is a $K_\pi$-set and a $K_\pi^*$-set but not $\pi\text{g}\alpha$-open.

The above discussions are summarized in the following diagram:

Proposition 5.4.12: Let $A$ and $B$ be $K_\pi$-sets in $X$. Then $A \cap B$ is a $K_\pi$-set in $X$.

Proof: Since $A$, $B$ are $K_\pi$-sets, $A = G_1 \cap F_1$, $B = G_2 \cap F_2$ where $G_1$, $G_2$ are $\pi\text{g}\alpha$-open and $F_1$, $F_2$ are $t$-sets. Since intersection of two $\pi\text{g}\alpha$-open sets is $\pi\text{g}\alpha$-open and intersection of $t$-sets is a $t$-set, it follows that $A \cap B$ is a $K_\pi$-set in $X$.

Remark 5.4.13: a) The Union of two $K_\pi$-sets need not be a $K_\pi$-set.

b) Complement of a $K_\pi$-set need not be a $K_\pi$-set.

Example 5.4.14: In Example 5.4.5

a) $A = \{a, c\}$ and $B = \{d\}$ are $K_\pi$-sets. $A \cup B = \{a, c, d\}$ is not a $K_\pi$-set.
b) $X - \{a,c\} = \{b,d\}$ is not a $K_\pi$-set.

**Proposition 5.4.15**: Let $A$ and $B$ be $C_\pi$-sets in $X$. Then $A \cap B$ is a $C_\pi$-set in $X$.

**Remark 5.4.16**: The union of two $C_\pi$-sets need not be a $C_\pi$-set and the complement of a $C_\pi$-set need not be a $C_\pi$-set follows from Example 5.4.14.

**Definition 5.4.17**: A function $f: X \rightarrow Y$ is said to be

i) $C_\pi$-continuous if $f^{-1}(V)$ is a $C_\pi$-set for every open set $V$ in $Y$.

ii) $K_\pi$-continuous if $f^{-1}(V)$ is a $K_\pi$-set for every open set $V$ in $Y$.

iii) $C_\pi^*$-continuous if $f^{-1}(V)$ is a $C_\pi^*$-set for every open set $V$ in $Y$.

iv) $K_\pi^*$-continuous if $f^{-1}(V)$ is a $K_\pi^*$-set for every open set $V$ in $Y$.

**Proposition 5.4.18**:

i) Every $C_\pi$-continuous function is $C_\pi^*$-continuous.

ii) Every $C_\pi^*$-continuous function is $K_\pi$-continuous.

iii) Every $K_\pi$-continuous function is $K_\pi^*$-continuous.

iv) Every $C_\pi^*$-continuous function is $K_\pi^*$-continuous.

**Proof**: Follows from Proposition 5.4.2 and Definition 5.4.17.

**Remark 5.4.19**: Converse of the above need not be true as can be seen from the following examples.

**Example 5.4.20**: a) Let $X = \{a,b,c,d\}$, $\tau = \{\phi,X,\{a\},\{b,c\},\{a,b,c\}\}$, $\sigma = \{\phi,X,\{c,d\}\}$ and $f:(X,\tau) \rightarrow (X,\sigma)$ be the identity mapping. Then $f$ is $C_\pi$-continuous but not $C_\pi^*$-continuous.

b) Let $X = \{a,b,c,d\}$, $\tau = \{\phi,X,\{a\},\{c,d\},\{a,c,d\},\{d\},\{a,d\}\}$, $\sigma = \{\phi,\{a,b,d\},X\}$ and $f:(X,\tau) \rightarrow (X,\sigma)$ be the identity mapping. Then $f$ is $K_\pi$-continuous but not $C_\pi^*$-continuous.
c) Let $X = \{a, b, c, d\}$, $\tau = \{\emptyset, \{a\}, \{b, c\}, \{a, b, c\}, X\}$, $\sigma = \{\emptyset, \{c\}, \{c, d\}, X\}$ and $f : (X, \tau) \to (X, \sigma)$ be the identity mapping. Then $f$ is $K_{\pi^*}$-continuous but not $K_{\pi}$-continuous.

d) Let $X = \{a, b, c, d\}$, $\tau = \{\emptyset, X, \{a\}, \{a, c, d\}, \{d\}, \{a, d\}\}$, $\sigma = \{\emptyset, \{a\}, \{a, b, d\}, X\}$ and $f : (X, \tau) \to (X, \sigma)$ be the identity mapping. Then $f$ is $K_{\pi^*}$-continuous but not $C_{\pi^*}$-continuous.

Remark 5.4.21: The above discussions are summarized in the following implications:

\[
\begin{align*}
C_{\pi^*}-continuity & \implies C_{\pi^*}-continuity \\
\downarrow & \downarrow \\
K_{\pi^*}-continuity & \implies K_{\pi^*}-continuity
\end{align*}
\]

Definition 5.4.22: A map $f : X \to Y$ is said to be

i) $K_{\pi}$-open if $f(U)$ is a $K_{\pi}$-set in $Y$ for each open set $U$ in $X$.

ii) $C_{\pi}$-open if $f(U)$ is a $C_{\pi}$-set in $Y$ for each open set $U$ in $X$.

iii) $C_{\pi^*}$-open if $f(U)$ is a $C_{\pi^*}$-set in $Y$ for each open set $U$ in $X$.

iv) $K_{\pi^*}$-open if $f(U)$ is a $K_{\pi^*}$-set in $Y$ for each open set $U$ in $X$.

Definition 5.4.23: A map $f : X \to Y$ is said to be

i) contra- $K_{\pi}$-continuous if $f^{-1}(V)$ is a $K_{\pi}$-set for every closed set $V$ in $Y$.

ii) contra- $C_{\pi}$-continuous if $f^{-1}(V)$ is a $C_{\pi}$-set for every closed set $V$ in $Y$.

iii) contra- $C_{\pi^*}$-continuous if $f^{-1}(V)$ is a $C_{\pi^*}$-set for every closed set $V$ in $Y$.

iv) contra- $K_{\pi^*}$-continuous if $f^{-1}(V)$ is a $K_{\pi^*}$-set for every closed set $V$ in $Y$.

Lemma 5.4.24: A subset $A$ of a space $X$ is

a) $\pi g$-open if and only if $F \subseteq \text{int}(A)$ whenever $F$ is $\pi$-closed and $F \subseteq A$ [42].

b) $\pi g p$-open if and only if $F \subseteq \text{pint}(A)$ whenever $F$ is $\pi$-closed and $F \subseteq A$ [146].

Theorem 5.4.25: A subset $S$ of $X$ is

a) $\pi g$-open if and only if it is both $\pi g p$-open and a $C_{\pi}$-set in $X$. 

91
b) πg-open if and only if it is both πgα-open and a Cπ-set in X.

c) πg-open if and only if it is both πgα-open and a Cπ-set in X.


Sufficiency: Assume that S is both πgp-open and a Cπ-set in X. By assumption, S is a Cπ-set in X implies S = A ∩ B where A is πg-open and B is a t-set. Let F be a π-closed set such that F ⊂ S. Since S is πgp-open, F ⊂ S implies F ⊂ pint(S) ⊂ int(B) Then A is πg-open and F ⊂ S ⊂ A implies F ⊂ int(A). Hence

F ⊂ int(A) ∩ int(B) = int(A ∩ B) = int(S). Hence S is πg-open.

b) Necessity: Obvious

Sufficiency: Let S be both πgα-open and a Cπ-set in X. Since S is a Cπ-set, S = A ∩ B where A is πg-open and B is a t-set. Let F be a π-closed set such that F ⊂ S. Since S is πgα-open, F ⊂ S implies F ⊂ αint(S) ⊂ int(B). Then A is πg-open and F ⊂ S ⊂ A implies F ⊂ int(A). Hence F ⊂ int(A) ∩ int(B) = int(A ∩ B) = int(S).

c) Necessity: Obvious.

Sufficiency: Assume S is both πgα-open and a Cπ*-set in X. Since S is a Cπ*-set, S = A ∩ B where A is πg-open and B is α*-set in X. Let F be a π-closed set such that F ⊂ S. Since S is πgα-open, F ⊂ S implies F ⊂ αint(S) ⊂ int(B). Then A is πg-open and F ⊂ S ⊂ A implies F ⊂ int(A). Hence F ⊂ int(A) ∩ int(B) = int(A ∩ B) = int(S).

**Theorem 5.4.26:** A mapping f : X → Y is

a) πg-continuous if and only if it is both πgp-continuous and Cπ-continuous.

b) πg-continuous if and only if it is both πgα-continuous and Cπ-continuous.

c) πg-continuous if and only if it is both πgα-continuous and Cπ*-continuous.

**Proof:** Follows from Theorem 5.4.25.

**Theorem 5.4.27:** A map f : X → Y is

a) πg-open if and only if it is both πgp-open and Cπ-open.

b) πg-open if and only if it is both πgα-open and Cπ-open.

c) πg-open if and only if it is both πgα-open and Cπ*-open.
Proof: Follows from Theorem 5.4.25.

**Theorem 5.4.28**: A mapping \( f: X \to Y \) is
a) contra-\( \pi g \)-continuous if and only if \( f \) is both contra-\( \pi gp \)-continuous and contra-\( C_{\pi} \)-continuous .
b) contra-\( \pi g \)-continuous if and only if \( f \) is both contra-\( \pi g \alpha \)-continuous and contra-\( C_{\pi} \)-continuous.
c) contra-\( \pi g \)-continuous if and only if \( f \) is both contra-\( \pi g \alpha \)-continuous and contra-\( C_{\pi}^{*} \)-continuous.

Proof: Follows from Theorem 5.4.25.

**Lemma 5.4.29**: [155] Let \( A \) and \( B \) be subsets of a space \( X \). If \( B \) is an \( \alpha * \) set, then 
\[ a\text{int}(A \cap B) = a\text{int}(A) \cap \text{int}(B). \]

**Theorem 5.4.30**: A subset \( S \) of \( X \) is
a) \( \pi g \alpha \)-open if and only if it is both \( \pi g \)-open and a \( K_{\pi} \)-set.
b) \( \pi g \alpha \)-open if and only if it is both \( \pi gp \)-open and a \( K_{\pi}^{*} \)-set.

Proof: a) Necessity: Let \( S \) be \( \pi g \alpha \)-open. For any subset \( A \) of \( X \),
\[ \text{int}(A) \subseteq a\text{int}(A) \subseteq \text{pint}(A). \]
Let \( F \) be a \( \pi \)-closed set such that \( F \subseteq S \). Since \( S \) is \( \pi g \alpha \)-open, \( F \subseteq S \) implies \( F \subseteq a\text{int}(S) \subseteq \text{pint}(S) \) which implies \( S \) is \( \pi gp \)-open. Since \( S = S \cap X \) where \( S \) is \( \pi g \alpha \)-open and \( X \) is a \( t \)-set, \( S \) is a \( K_{\pi} \)-set.

Sufficiency: Let \( S \) be both \( \pi gp \)-open and a \( K_{\pi} \)-set. Since \( S \) is a \( K_{\pi} \)-set, \( S = A \cap B \) where \( A \) is \( \pi g \alpha \)-open and \( B \) is a \( t \)-set. Let \( F \) be a \( \pi \)-closed set such that \( F \subseteq S \). Since \( S \) is \( \pi gp \)-open, \( F \subseteq S \) implies \( F \subseteq \text{pint}(S) = S \cap \text{cl}(S) \subseteq \text{int}(B) \). Then \( A \) is \( \pi g \alpha \)-open and \( F \subseteq S \subseteq A \) implies \( F \subseteq a\text{int}(A) \). Therefore \( F \subseteq a\text{int}(A) \cap \text{int}(B) \subseteq a\text{int}(A \cap B) \subseteq a\text{int}(S) \).

b) Proof: Similar as that of (a).

**Theorem 5.4.31**: A map \( f: X \to Y \) is
a) \( \pi g \alpha \)-continuous if and only if it is both \( \pi gp \)-continuous and \( K_{\pi} \)-continuous.
b) \( \pi g\alpha \)-continuous if and only if it is both \( \pi g p \)-continuous and \( K_{\pi^*} \)-continuous

**Proof:** Follows from Theorem 5.4.30.

**Theorem 5.4.32:** A map \( f: X \rightarrow Y \) is

a) \( \pi g\alpha \)-open if and only if it is both \( \pi g p \)-open and \( K_{\pi} \)-open

b) \( \pi g\alpha \)-open if and only if it is both \( \pi g p \)-open and \( K_{\pi^*} \)-open

**Proof:** Follows from Theorem 5.4.30.

**Theorem 5.4.33:** A map \( f: X \rightarrow Y \) is

a) contra-\( \pi g\alpha \)-continuous if and only if it is both contra-\( \pi g p \)-continuous and contra-\( K_{\pi} \)-continuous.

b) contra-\( \pi g\alpha \)-continuous if and only if it is both contra-\( \pi g p \)-continuous and contra-\( K_{\pi^*} \)-continuous.

**Proof:** Follows from Theorem 5.4.30

\[ \star \star \star \]