CHAPTER II

HYDROMAGNETIC CONVECTION IN A RAPIDLY ROTATING FLUID LAYER
IN THE PRESENCE OF HALL CURRENT
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2.1 Introduction

The problem of the generation of magnetic fields by motions in a conducting fluid is known as the dynamo problem. An increasing number of theoretical investigations of this problem at different levels of mathematical complexity have appeared in recent years. Most of the work has been focused on the kinematic dynamo problem. The kinematic theory of the dynamo is related essentially to the investigation of the properties of the induction equation. For the hydromagnetic dynamo, the induction equation is added to the usual system of hydrodynamic equations. A most interesting example of a hydromagnetic dynamo is provided by the dynamo functioning in the liquid core of the Earth which gives rise to the Earth's magnetic field. It is generally believed that the existence of the geomagnetic field is the manifestation of a finite amplitude instability of the Earth's core. The desire to understand better the magnetohydrodynamics of the interiors of the Earth and planets has recently motivated a number of studies on convective motions in hydromagnetic rotating systems. Since most cosmical bodies are rotators, it is felt
the study of convective motions in a rotating electrically conducting fluid is essential for an understanding of the evolution of these fields.

The thermal instability of rotating magnetic systems has been discussed by a number of authors including Chandrasekhar (1961). Busse (1973) has constructed a dynamo in a Benard layer in which convective motions are supplemented by a unidirectional plane poiseuille flow set up by an externally applied pressure gradient. Childress and Soward (1972) and Soward (1974) examined a model which is perhaps closer to the solar dynamo in which the motions are set up in a highly rotating Benard layer. An alternative model of the geodynamo has been proposed by Busse (1975) in which there is no significant azimuthal flow and the magnetic field is maintained simply by small-scale convective motions.

The convective instability of a layer of Boussinesq fluid heated uniformly from below has long been recognised as a problem of crucial importance in many fields of fluid mechanics, and any discussion of energy transport in the Earth's atmosphere, its oceans, mantle and core or the outer layers of the sun, must take account of this instability. In attempts to understand the phenomenon theoretically the model adapted by Rayleigh (1916) is that of a horizontal fluid layer of thickness $d$, subject to a uniform adverse
temperature gradient $\beta$ applied across the layer. His aim was basically to describe the experimental observations of Benard (1900). A review of all the work done on the Benard layer, including the effects of a uniform magnetic field $\vec{B}_0$ or a uniform angular velocity $\vec{\omega}$ antiparallel to the gravitational force, can be found in Chandrasekhar (1961) and in Weiss (1964).

The simultaneous action of rotation and a uniform magnetic field when $\vec{B}_0$ and $\vec{\omega}$ are both vertical was studied by Chandrasekhar (1961, Chapter 5) when the bounding planes are free. The parameters characterizing the flow are the Prandtl number $\nu$, the magnetic Prandtl number $\nu_m$, the Hartmann number $M$, the Taylor number $T$ and the Rayleigh number $R$. They are defined by

$$
p = \frac{\nu}{\kappa}, \quad \nu_m = \frac{\nu}{\eta}, \quad \eta = \frac{1}{\mu \sigma_c}, \quad R = \frac{g \alpha \beta d^4}{\nu \kappa}, \quad M^2 = \frac{d^2 |\vec{B}_0|^2}{\mu \rho_0 \nu \eta}, \quad T = \frac{4 |\vec{\omega}|^2 d^4}{\nu^2}
$$

where $\nu$ is the kinematic viscosity, $\kappa$ the thermal diffusivity, $\mu$ the magnetic permeability, $\rho_0$ the mean density, $\eta$ the magnetic diffusivity, $\sigma_c$ the electrical conductivity, $g$ gravitational acceleration and $\alpha$ the coefficient of volume expansion.
Eltayeb (1972) examined the linear stability of the hydromagnetic rotating layer when the principle of exchange of stabilities is valid, for different types of boundaries. He considered four different models characterized by the relative directions of angular velocity and magnetic field. For each model, the critical mode was located for all relations $T = T(M)$ in the double limit $T, M \to \infty$. When the principle of exchange of stabilities is valid, he found that $R_c = O(T^{1/2})$ for $T = O(M^4)$ in all the models. In other words when $T$ is large, the presence of a magnetic field facilitates convection (Eltayeb and Roberts 1970).

2.2 Author's Contribution

In the present study we shall investigate the effect of the Hall current on the linear stability of a hydromagnetic rotating layer. That the Hall current could play a significant role in the dynamo problems referred earlier is seen from the following estimate of the Hall parameter $m = \frac{\sigma|B_o|}{en_e}$. In the Earth's interior for example, $|\vec{B}_o| = 10^{-3}$ webers/m$^2$, $\sigma = 8 \times 10^5$ mho/m, $e = 1.602 \times 10^{-19}$ coulombs, $n_e = 10^{22}$/m$^3$ yields $m = 0.8$. While in the case of the sun it is about 5 with $|\vec{B}_o| = 10^{-4}$ webers/m$^2$, $\sigma = 8 \times 10^5$ mho/m, $n_e = 10^{14}$/m$^3$. 
The basic equations of the problem are given in Section 2.3 along with the necessary boundary conditions. We analyze the problem for large $T,M$ using the boundary layer approach. The rotating fluid is made up of a main stream and Ekman-Hartmann layers. The influence of Hall current on Rayleigh number has been calculated for a wide range of values of $T = O(M^{4+\beta})$, $(4+\beta > 0)$ and for varying $m$, in Sections 2.4 to 2.6. In Section 2.4 it is assumed that the magnetic field and the axis of rotation are both vertical, while in Section 2.5 the magnetic field is taken in the horizontal direction but rotation in the vertical direction. The directions of both the magnetic field and rotation are assumed to be horizontal and inclined at an angle $\phi$, in Section 2.6.

2.3 The Basic Equations and Boundary Conditions

Consider a horizontal layer of electrically conducting fluid rotating with uniform angular velocity $\vec{\Omega}$ in the presence of a uniform magnetic field $\vec{B}_0$. $O(x',y',z')$ is the rotating frame with $Oz'$ vertically upwards and $Ox'$ and $Oy'$ in any two perpendicular horizontal directions. The heat is conducted from the lower plane $z' = -d/2$ to the upper plane $z' = d/2$ at a constant temperature gradient $-\beta$ in the vertical direction. We are interested in
finding the marginal state at which the convection can first occur.

The equations of the problem in their general form relative to the rotating frame are

\[ \rho'[\frac{\partial \mathbf{u}'}{\partial t'} + (\mathbf{u}' \cdot \mathbf{V}') \mathbf{u}'] + 2(\mathbf{\Omega} \times \mathbf{u}') \]

\[ = -\nabla' \rho' \mathbf{V}' \times (\nabla' \times \mathbf{u}') + \frac{1}{\mu'} (\nabla' \times \mathbf{B}') \times \mathbf{B}' - \rho' g \hat{z} \quad (2.1) \]

\[ \frac{\partial \rho'}{\partial t'} + \nabla' \cdot (\rho' \mathbf{u}') = 0 \quad (2.2) \]

\[ \frac{\partial \mathbf{E}'}{\partial t'} + (\mathbf{u}' \cdot \nabla') \mathbf{E}' = \kappa (\nabla' \times \mathbf{B}') \times \mathbf{B}' \quad (2.3) \]

\[ \nabla' \times \mathbf{B}' = \mu' j' \quad (2.4) \]

\[ \nabla' \times \mathbf{E}' = -\frac{\partial \mathbf{B}'}{\partial t'} \quad (2.5) \]

\[ \nabla' \cdot \mathbf{B}' = 0 \quad (2.6) \]

\[ j' = \sigma_c [(\mathbf{E}' + \mathbf{u}' \times \mathbf{B}') - \frac{1}{\tau_0} \mathbf{j} \times \mathbf{B}'] \quad (2.7) \]

\[ \rho' = \rho_0 [1 - \alpha (\mathbf{T}_e - T_0)] \quad (2.8) \]

where \( \mathbf{u}' \) is the velocity, \( -g \hat{z} \) is the external force due to gravity where \( \hat{z} \) is a unit vector parallel to the z-axis, \( \rho \)
is the pressure, \( T_e \) is the temperature, \( T_o \) is the temperature when \( p' = p_o \), \( e \) is the electric charge, \( n_e \) is the number density of electrons and the other quantities have their usual meanings.

Using the Boussinesq approximation neglecting density variations equations (2.1) and (2.2) after using (2.8) reduce to

\[
\left( \frac{\partial}{\partial t} - \gamma V'^2 \right) U' + (U' \cdot V')U' + 2 \mathcal{E} x U' = 0
\]

(2.9)

\[
\n' \cdot U' = 0
\]

(2.10)

where

\[
\pi = \frac{p}{p_o} + \frac{1}{2} B'^2 - g z
\]

(2.11)

Equations (2.4), (2.5), (2.6) and (2.7) yield the modified induction equation

\[
\frac{\partial B'}{\partial t'} = \eta V'^2 B' + V' \times (U' \times B') - \frac{1}{\mu n_e} V' \times [(V' \times B') \times B']
\]

(2.12)

The equations of the problem under consideration are (2.3), (2.6), (2.9) to (2.12). Taking an initial stationary solution of the equations we examine small perturbations about that solution. A stationary solution which satisfies the
above equations, in the rotating frame is

\[ \vec{u}' = 0, \vec{B}' = \vec{B}_0, T_e = T_0 - \beta z \] (2.13)

If the perturbations in velocity, temperature, magnetic field and pressure are respectively \( \vec{u}', \theta', \vec{b}', \text{and} \vec{\omega}' \), the basic linearised equations neglecting squares and products of the perturbed quantities may be written as

\[ \left( \frac{\partial}{\partial t'} - \gamma \nabla'^2 \right) \vec{u}' + 2 \nabla \times \vec{u}' = -\nabla' \vec{\omega}' + \frac{1}{\mu \rho_0} (\vec{B}_0 \cdot \nabla') \vec{b}' + \alpha g \theta'^2 \] (2.14)

\[ \frac{\partial \vec{b}'}{\partial t'} = \eta \nabla'^2 \vec{b}' + (\vec{B}_0 \cdot \nabla') \vec{u}' - \frac{1}{\mu \kappa n_e} (\vec{B}_0 \cdot \nabla') (\nabla' \times \vec{b}') \] (2.15)

\[ \frac{\partial \theta'}{\partial t'} = \beta \vec{w}' + \kappa \nabla'^2 \theta' \] (2.16)

\[ \nabla' \cdot \vec{u}' = 0 = \nabla' \cdot \vec{b}' \] (2.17)

where \( \vec{w}' \) is the vertical component of velocity \( \vec{u}' \). The above equations may be cast in dimensionless form by the transformations

\[ (x', y', z') = d(x, y, z), \quad t' = \frac{d^2}{\gamma} t, \quad \theta' = \frac{\gamma \beta d}{\kappa} \theta \] (2.18)

\[ \vec{u}' = \frac{\gamma}{d} \vec{u}, \quad \vec{b}' = \frac{\gamma |\vec{B}_0|}{\eta} \vec{b}, \quad \vec{\omega}' = \frac{d^3}{\gamma^2} \vec{\omega} \]

Then the linearised equations in the presence of Hall current
can be written in the following non-dimensional form.

\[
\left( \frac{\partial}{\partial t} - \nabla^2 \right) \mathbf{u} + \Gamma \left( \frac{1}{2} \left( \mathbf{\hat{u}} \times \mathbf{u} \right) \right) = -\nabla \mathbf{\omega} + M^2 (\mathbf{\hat{B}} \cdot \nabla) \mathbf{b} + R \mathbf{e} \quad (2.19)
\]

\[
(p_m \frac{\partial}{\partial t} - \nabla^2) \mathbf{b} = (\mathbf{\hat{B}} \cdot \nabla) \mathbf{u} - m (\mathbf{\hat{B}} \cdot \nabla) (\mathbf{V} \times \mathbf{b}) \quad (2.20)
\]

\[
(p \frac{\partial}{\partial t} - \nabla^2) \mathbf{\theta} = \mathbf{w} \quad (2.21)
\]

\[
\nabla \cdot \mathbf{u} = 0 = \nabla \cdot \mathbf{b} \quad (2.22)
\]

The unit vectors \( \mathbf{\hat{u}} \) and \( \mathbf{\hat{B}} \) are parallel to the directions of rotation and magnetic field respectively and the parameters \( p, p_m, M, T \) and \( R \) are as defined in Section 2.1. Assume the perturbations in the form

\[
F(x,y,z,t) = F(z) \exp[i(kx+ly+zt)]
\]

where \( F \) stands for any of the above mentioned variables. We shall treat \( \mathbf{\theta}, \mathbf{u}, \mathbf{b} \) and \( \mathbf{\omega} \) as functions of \( z \) only and suppress the exponential dependence.

Taking curl on both sides of (2.19) we get

\[
(i\sigma - \nabla^2)(\nabla \times \mathbf{u}) - T^{1/2} \left( \mathbf{\hat{u}} \cdot \nabla \right) \mathbf{u} = M^2 (\mathbf{\hat{B}} \cdot \nabla)(\nabla \times \mathbf{b}) + R(\nabla \mathbf{\theta} \times \mathbf{\hat{z}})
\]

\[
\quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \}
\[
(ip_m \sigma - \nabla^2)(i \sigma - \nabla^2)(\nabla \times \mathbf{u}) - \frac{1}{2} (ip_m \sigma - \nabla^2)(\hat{\mathbf{\Omega}} \cdot \mathbf{v})
\]

\[= M^2(\hat{\mathbf{B}} \cdot \mathbf{v})^2(\nabla \times \mathbf{u}) + R(ip_m \sigma - \nabla^2)(\nabla \times \mathbf{\hat{z}})
\]

Equating the vertical components on both sides of (2.24) we get

\[
L \xi = \frac{1}{2} (ip_m \sigma - \nabla^2)(\hat{\mathbf{\Omega}} \cdot \mathbf{v}) \mathbf{w} + m^2(\hat{\mathbf{B}} \cdot \mathbf{v})^2 \mathbf{v} \cdot \mathbf{b}
\]

where

\[
L = (ip_m \sigma - \nabla^2)(i \sigma - \nabla^2) - M^2(\hat{\mathbf{B}} \cdot \mathbf{v})^2,
\]

\(\xi\) and \(\mathbf{b}\) are the vertical components of vorticity \((= \nabla \times \mathbf{u})\) and the magnetic field \(\mathbf{b}\) respectively. Applying the operators \(\hat{\mathbf{z}}\) and \(\hat{\mathbf{z}} \cdot \text{curl}\) to equation (2.20) we get respectively

\[
(ip_m \sigma - \nabla^2)\mathbf{b} = (\hat{\mathbf{B}} \cdot \mathbf{v}) \mathbf{w} - m(\hat{\mathbf{B}} \cdot \mathbf{v}) \xi
\]

\[
(ip_m \sigma - \nabla^2)\xi = (\hat{\mathbf{B}} \cdot \mathbf{v}) \xi + m(\hat{\mathbf{B}} \cdot \mathbf{v}) \mathbf{v}^2 \mathbf{b}
\]

where \(\xi\) is the vertical component of the electric current \(\mathbf{j}\).

Multiply (2.23) by \(m\) and use (2.20) to obtain

\[
M^2(ip_m \sigma - \nabla^2)\mathbf{b} = M^2(\hat{\mathbf{B}} \cdot \mathbf{u}) + \frac{1}{2} (ip_m \sigma - \nabla^2)(\hat{\mathbf{\Omega}} \cdot \mathbf{v}) \mathbf{w} - m(i \sigma - \nabla^2)(\nabla \times \mathbf{u})
\]

\[+ m R(\mathbf{\nabla} \times \mathbf{\hat{z}})
\]

Equating the vertical components on both sides of (2.29) we obtain
Apply the operator $\hat{z}\cdot \text{curl}$ to (2.23) to obtain

$$(i\sigma - \nabla^2)^{1/2} W + T^{1/2} (\hat{\nabla} \cdot \nabla) \xi = M^2 (\hat{B} \cdot \nabla)^2 \xi - Ra^2 \Phi$$

(2.31)

Operating on both sides of (2.31) by $(i\sigma - \nabla^2)$ we get after using (2.21)

$$(i\sigma - \nabla^2)(i\sigma - \nabla^2)^{1/2} W + T^{1/2} (i\sigma - \nabla^2)(\hat{\nabla} \cdot \nabla) \xi$$

$$= M^2 (i\sigma - \nabla^2)^2 (\hat{B} \cdot \nabla)^2 b - Ra^2 W$$

(2.32)

Apply the operator $(i\sigma - \nabla^2)$ to (2.25) and use (2.30) to yield

$$L_1 \xi = T^{1/2} (i\sigma - \nabla^2)^2 (\hat{\nabla} \cdot \nabla) W + m^2 M^2 (\hat{B} \cdot \nabla)^2 W$$

$$+ m^2 T^{1/2} (\hat{B} \cdot \nabla)^2 (\hat{\nabla} \cdot \nabla) W$$

(2.33)

where

$$L_1 = (i\sigma - \nabla^2)L + m^2 (i\sigma - \nabla^2)(\hat{B} \cdot \nabla)^2 W$$

(2.34)

Eliminating $b$ between (2.30) and (2.32) we get

$$(i\sigma - \nabla^2)(i\sigma - \nabla^2)(i\sigma - \nabla^2)^{1/2} W + T^{1/2} (i\sigma - \nabla^2)^2 (\hat{\nabla} \cdot \nabla) \xi$$

$$= M^2 (i\sigma - \nabla^2)^2 (\hat{B} \cdot \nabla)^2 W + mT (i\sigma - \nabla^2)(\hat{B} \cdot \nabla)(\hat{\nabla} \cdot \nabla) W$$

$$- m (i\sigma - \nabla^2)(i\sigma - \nabla^2)(\hat{B} \cdot \nabla)^2 \xi - Ra^2 (i\sigma - \nabla^2) W$$

(2.35)
Apply the operator \( L_1 \) to (2.35) to eliminate \( \zeta \) between (2.33) and (2.35) which yields

\[
(ip_\sigma - \nabla^2)[L^2 \nabla^2 + T(ip_m \sigma - \nabla^2)(\hat{\nabla} \cdot \nabla)]W
\]

\[
= -Ra^2(ip_m \sigma - \nabla^2)LW - 2mM^2 T \frac{1}{2} (ip_\sigma - \nabla^2)(\hat{\nabla} \cdot \nabla)^3(\hat{\nabla} \cdot \nabla)\nabla^2W
\]

\[
-m^2(ip_\sigma - \nabla^2)[(i\sigma - \nabla^2)^2 \nabla^2 + T(\hat{\nabla} \cdot \nabla)^2(\hat{\nabla} \cdot \nabla)^2]W
\]

\[
-m^2Ra^2(i\sigma - \nabla^2)(\hat{\nabla} \cdot \nabla)^2\nabla^2W
\]  

(2.36)

We can eliminate \( b \) from equations (2.27) and (2.28) to give

\[
[(ip_m \sigma - \nabla^2)^2 + m^2(\hat{\nabla} \cdot \nabla)^2]_b = (ip_m \sigma - \nabla^2)(\hat{\nabla} \cdot \nabla) + m(\hat{\nabla} \cdot \nabla)^2\nabla^2W
\]

\( ... \)  

(2.37)

Applying the operator \( L_1 \) to (2.37) and eliminating \( \zeta \) using (2.33) we obtain

\[
[(ip_m \sigma - \nabla^2)^2 + m^2(\hat{\nabla} \cdot \nabla)^2]L_1 \xi = T \frac{1}{2} (ip_m \sigma - \nabla^2)^3(\hat{\nabla} \cdot \nabla)(\hat{\nabla} \cdot \nabla)W
\]

\[
+ m^2(ip_m \sigma - \nabla^2)(\hat{\nabla} \cdot \nabla)^4\nabla^2W + mT \frac{1}{2} (ip_m \sigma - \nabla^2)^2(\hat{\nabla} \cdot \nabla)^3(\hat{\nabla} \cdot \nabla)\nabla^2W
\]

\[
+ m(\hat{\nabla} \cdot \nabla)^2\nabla^2L_1 W
\]  

(2.38)

Operating on both sides of (2.30) by \( L_1 \) and using (2.33) eliminate \( \zeta \) to get

\[
M^2(ip_m \sigma - \nabla^2)L_1 b = M^2(\hat{\nabla} \cdot \nabla)L_1 W + mT \frac{1}{2} (\hat{\nabla} \cdot \nabla)L_1 W
\]

\[
- mT \frac{1}{2} (ip_m \sigma - \nabla^2)^2(i\sigma - \nabla^2)(\hat{\nabla} \cdot \nabla)W
\]

\[
- m^2M^2(i\sigma - \nabla^2)(\hat{\nabla} \cdot \nabla)^3\nabla^2W - m^2T \frac{1}{2} (i\sigma - \nabla^2)(\hat{\nabla} \cdot \nabla)^3\nabla^2W
\]

\( ... \)  

(2.39)
Equation (2.21) can be written as

\[(i\pi - \nabla^2)\theta = W\]           \hspace{1cm} \text{(2.40)}

For the above equations we have the notation

\[a^2 = k^2 + \ell^2, \quad D = \frac{d}{dz}, \quad \nabla^2 = D^2 - a^2\] \hspace{1cm} \text{(2.41)}

Equations (2.33), (2.36), (2.38) to (2.40) are to be solved subject to certain boundary conditions. Equation (2.36) is of twelfth order and for \(p_m = 0\) it reduces to a tenth-order equation. Since the above said equations involve only the vertical components after solving for them the horizontal components can easily be obtained.

The boundary conditions are as follows.

Since the bounding surfaces are maintained at constant temperatures they suffer no change. It is also clear that the normal component of the velocity must vanish on these surfaces. Hence for all types of boundary we must require

\[\theta = 0 \quad \text{and} \quad W = 0 \quad \text{at} \quad z = \pm \frac{1}{2}\] \hspace{1cm} \text{(2.42)}

The condition that no slip occurs on a rigid surface implies that the horizontal components of the velocity, \(u\) and \(v\) vanish. Thus

\[u = 0 \quad \text{and} \quad v = 0\] \hspace{1cm} \text{(2.43)}
in addition to $W = 0$ on a rigid surface. Then it follows from the equation of continuity that

$$D W = 0 \text{ on a rigid surface.}$$

In the case of a free boundary, the vanishing of the tangential stresses with $W = 0$ yield

$$Du = Dv = 0 \text{ on a free surface} \quad (2.44)$$

Again by differentiating the equation of continuity with respect to $z$ we obtain

$$D^2 W = 0 \text{ on a free surface.}$$

Further since $\zeta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$ it follows from (2.43) and (2.44) that

$$\zeta = 0 \text{ on a rigid surface}$$

$$D_\zeta = 0 \text{ on a free surface.}$$

The magnetic boundary conditions depend upon the electrical properties of the medium adjoining the fluid. If the medium adjoining the fluid is electrically non-conducting, then no currents can cross the boundary. Then we must require

$$\chi = 0 \text{ on the bounding surface} \quad (2.45)$$

If the boundary is rigid, there will be no motions at the boundary and for this we must have
43

\[ D\zeta = 0 \] on the bounding surface \hspace{1cm} (2.46)

The boundary conditions stated above are classified as follows:

(a) Free insulating boundaries

\[ W = \zeta = D^2W = D\zeta = \xi = \eta = 0, \quad \text{at} \quad z = + \frac{1}{2} \] \hspace{1cm} (2.47)

(b) Free perfectly conducting boundaries

\[ W = \zeta = D^2W = D\zeta = D\xi = 0, \quad \text{at} \quad z = + \frac{1}{2} \] \hspace{1cm} (2.48)

(c) Rigid insulating boundaries

\[ W = \zeta = DW = \zeta = \xi = 0, \quad \text{at} \quad z = + \frac{1}{2} \] \hspace{1cm} (2.49)

(d) Rigid perfectly conducting boundaries

\[ W = \zeta = DW = \zeta = D\xi = 0 \quad \text{at} \quad z = + \frac{1}{2} \] \hspace{1cm} (2.50)

The continuity of \( b \) must be satisfied in each case. Similar conditions are to be satisfied at the other boundary at \( z = - \frac{1}{2} \).

In all the models, the process of finding the minimum \( R \) is straightforward, for \( R \) depends on \( \alpha \) alone. The critical Rayleigh number \( R_c \) is obtained by solving \( \frac{\partial R}{\partial \alpha} = 0 \). The critical wave number \( \alpha_c \) obtained, depends on the relative magnitudes of \( T \) and \( M \) as both \( T, M \to \infty \). If we make a choice as \( T = O(M^4) \), it is enough to consider the
single limit \( M \to \infty \), for \( T/M^4 \) can be treated as a given constant. The \( T(M) \) relations lead to three possibilities: (i) magnetic field dominant, (ii) fields equi-dominant and (iii) rotation dominant. These three groups will help us to determine a particular boundary, once we find \( R_c \) for each of these ranges and using the continuity condition at the common boundaries.

Having chosen a particular magnitude for \( T(M) \) in the limit \( M \to \infty \), we proceed to find the smallest value of \( R \). Since \( R \) depends on \( a \) alone, we guess the order of magnitude of \( a \) in terms of \( M \) and then minimize \( R = R(M) \). If in the process we get \( a_c = 0 \), then the assumed \( a \) decreases too rapidly with increasing \( M \) and if we get \( a_c = \infty \), \( a \) increases too rapidly with increasing \( M \). For one or more choices of \( a \) as a function of \( M \), the minimum will lie in the range considered and will give a possible \( R_c \). If we exhaust this procedure for all \( a(M) \), we can find the smallest minimum to get the required \( R_c \).

Our problem is to solve (2.36), subject to the appropriate boundary conditions. The solutions of (2.36) are different according to the thickness of different layers when the concerned parameters are functions of \( M \). We obtain the main stream solution assuming \( D = O(1) \) and setting \( M = \infty \) in (2.36). The inner and outer layers are obtained after
taking suitable orders for \( D \) and \( a \). In Model I for free insulating boundaries it is possible to obtain exact solutions satisfying every condition without the aid of boundary layers. However for other types of boundaries we have obtained the boundary layers.

### 2.4 Model I

In this section the applied magnetic field \( \overrightarrow{B}_0 \) and the angular velocity \( \overrightarrow{\Omega} \) are both vertical that is, in the direction parallel to the \( z \)-axis. Hence we have

\[
\hat{B}.V = \hat{\Omega}.V = \frac{d}{dz} = D.
\]

Equations (2.33), (2.36), (2.38) to (2.40) reduce to when \( \sigma = 0 \)

\[
\begin{align*}
{\nabla^4 - (M^2 - m^2 \nabla^2)} \zeta &= -T^{1/2} \nabla^2 DW - m(M^2 + mT^{1/2})D^3 W \\
{\nabla^2 + m^2 (M^2 - m^2 \nabla^2)} \zeta &= T^{1/2} \nabla^2 D^2 W \\
\end{align*}
\]

\[
M^2[\nabla^4 - (M^2 - m^2 \nabla^2)]^2 \nu b = -M^2[\nabla^4 - (M^2 - m^2 \nabla^2)]DW \\
-\frac{1}{2} [\nabla^4 - (M^2 - m^2 \nabla^2)]DW + mT^{1/2} \nabla^2 DW + m^2 D^3 \nu W + m^3 T^{1/2} D^3 \nu W \\
\ldots
\]

(2.53)
\[ \nabla^2 \Theta = -W \]  
(2.54)

\[
\begin{align*}
[\nabla^2 (\nabla^4 - M^2 D^2 - \frac{1}{2} TV D^2)] W &= -Ra^2 (\nabla^4 - M^2 D^2) W \\
-2mM^2 T^{\frac{1}{2}} &V D^4 W - m^2 (\nabla^6 + TD^2) V D^2 W - m^2 Ra^2 V D^2 W 
\end{align*}
\]  
(2.55)

Our aim is to solve for \( \xi \), \( \zeta \) and \( \Theta \) in terms of \( W \) and accordingly equations (2.51) to (2.55) give after some reductions

\[
a^4 \xi = - \frac{1}{(XZ - a^4 Y^2)} (\nabla^2 - a^2 - M^2 + m^2 \nabla^2) D \{[(1 + m^2) D^2 W_1 - m(1 + m^2) D W_2 - (1 + m^2) X W_1 + mX D W_2] \\
- \frac{1}{2} T^{\frac{1}{2}} V D W - m(M^2 + mT) D^2 W \} \]  
(2.56)

\[
(\beta M^2 - a^2) a^4 \xi = \alpha(1 + m^2)^2 D^2 W_3 - \alpha(1 + m^2) [(1 + m^2)^{\frac{1}{2}} + mM^2] D^2 W_4 \\
- \alpha^2 W_4 - \beta(1 + m^2) W_2 + \beta [(1 + m^2)^{\frac{1}{2}} + mM^2] W_4 \]  
(2.57)

\[
Ra^4 M^2 \Theta = [-\nabla^4 - (M^2 - m^2 \nabla^2)^2] [\nabla^4 - M^2 D^2] W - TV D^2 W \\
- \frac{1}{2} T^{\frac{1}{2}} (M^2 - m^2 \nabla^2) D^4 W - mT^{\frac{1}{2}} (M^2 + mT)^{\frac{1}{2}} D^4 W \\
- m^2 (M^2 + mT)^{\frac{1}{2}} V^2 D^2 W - Ra^2 (\nabla^2 - M^2) W - Ra^2 m^2 D^2 W \]  
(2.58)

where \( X = \{(1 + m^2)^{\frac{1}{2}} + mM^2 - m^3 \} (2a^2 + M^2 + m^2 a^2) \)

\[ -(1 + m^2) \{(1 + m^2) a^2 T^{\frac{1}{2}} - m^3 \} \]
\[ Y = (1+m^2) \frac{1}{2} T + m M^2 - m^2 a \]
\[ Z = (1+m^2) a^2 \frac{1}{2} T - m^3 a^4 \]
\[ XZ - a^4 Y^2 = (1+m^2) M^2 a^2 [T + m(T - ma^2)(M^2 + mT)] \]
\[ \alpha = ma^2 M^2 + m M^4 + M^2 (1+m^2) \frac{1}{2} \]
\[ \beta = \left\{ ma^2 M^2 + m M^4 + M^2 (1+m^2) T \right\} (2a^2 + M^4 + ma^2) - m(1+m^2) a^4 M^2 \]
\[ \beta m M^2 - \alpha^2 = (1+m^2) M^4 \left\{ m^2 a^2 (M^2 + mT) \right\} - m M^2 T - (1+m^2) T \]
\[ W_1 = \{ V^6 - (M^2 + mT) \frac{1}{2} V^2 D^2 + Ra^2 \} W \]
\[ W_2 = - \{ T \frac{1}{2} V^2 D + m(M^2 + mT) \frac{1}{2} D^3 \} W \]
\[ W_3 = \{-V^2 (V^4 - M^2 D^2) + mT \frac{1}{2} V^2 D^2 - Ra^2 \} W \]
\[ W_4 = (T + mV^2) D^2 W \]

The rest of the problem is to solve (2.55) for each type of boundary separately.

2.4(a) Free Insulating Boundaries

The boundary conditions to be satisfied are

\[ W = D^2 W = \Theta = D\zeta = \zeta = 0, \text{ at } z = + \frac{1}{2} \quad (2.59) \]

We now proceed to solve equation (2.55) subject to the boundary conditions (2.59). Using (2.54) along with
equations (2.56) to (2.58) the above boundary conditions can be transformed into

\[ W = D^2 W = D^4 W = D^6 W = D^8 W = 0 \text{ at } z = \frac{1}{2} \] (2.60)

Since we are interested only in finding the smallest \( R \), we take the even solution as

\[ W = \cos \pi z \] (2.61)

When we use the solution (2.61) in (2.55), we find that

\[ R_1 = \left\{ \frac{(1+x_1)^2 + M_1^2}{1+x_1} \right\} \times \left[ \frac{(1+x_1)^2 + M_1^2 + m^2(1+x_1)}{1+x_1} \right]^{1/2} \]

where

\[ x_1 = \frac{a^2}{\pi^2}, \quad M_1 = \frac{M}{\pi}, \quad T_1 = \frac{T}{\pi^4} \quad \text{and} \quad R_1 = \frac{R}{\pi^4} \] (2.62)

Differentiating (2.62) with respect to \( x_1 \) and setting \( \frac{\delta R_1}{\delta x_1} = 0 \) we get for large \( M \) and \( T \)

\[ 2x_1^7 + (11 + 4m^2)x_1^6 + 4M_1^2x_1^5 + [(13 + 5m^2) - M_1^2]M_1^2x_1^4 \]

\[ + 2[1 - (m^2 + 2)\tau - 4m\tau^{1/2}]M_1^4x_1^3 + [(1 - m^2) + M_1^2\tau]M_1^4x_1^2 \]

\[ - 2(m^2 + 2)[1 + (m^2 + 1)\tau + 2m\tau^{1/2}]M_1^4x_1 - [(1 + m^2)\tau + 1 + 2m\tau^{1/2}]M_1^6 = 0 \]

\[ \ldots \] (2.64)
It is obvious that the roots of the equation (2.64) depend on the magnitudes of $M_1$ and $T_1$. We specify the relative magnitudes of $T_1$ and $M_1$ and solve (2.64) as $M_1 \to \infty$.

We let

$$T_1 = KM_1^{4+\beta} \quad \text{and} \quad x_1 = HM_1^{\bar{\alpha}}$$

(2.65)

where $4+\beta > 0$, and $K$ and $H$ are constants of order unity.

Regarding $\beta$ as fixed, the three main possibilities are

(i) $\beta < -\frac{4}{3}$, (ii) $-\frac{4}{3} < \beta < 2$ and (iii) $\beta > 2$.

For each value of $\beta$, all possible values of $\bar{\alpha}$ between $-\infty$ and $+\infty$ are examined till we find all the seven roots of (2.64). We are concerned mainly with the positive roots of (2.64) which gives the minimum value of $R_1$ that is, $R_{1c}$. This $R_{1c}$ must be continuous at $\beta = -\frac{4}{3}$ where (i) and (ii) merge and again at $\beta = 2$ where (ii) and (iii) merge. In this process we get the range of values of $K$, while $H$ is found from (2.64) by taking the leading terms for a given $\bar{\alpha}$.

We consider the three possibilities separately.
Case (i): $\beta \leq -\frac{4}{3}$.

In this case the range of $T_1$ is given by $T_1 \leq \frac{8}{3}$
where $K$ is the positive root of the equation

\[ K^4 - 1.59 K^3 + 3.15 K^2 - 6K + 1.59 = 0 \]

Here the only positive root of (2.64) corresponds to

\[ x_{1c} = \left(\frac{1}{2} M_1^2\right)^{1/3} \]  
(2.66)

The critical Rayleigh number is, to second order

\[ R_{1c} = M_1^2 + \left[ \frac{3+2}{2^{2/3}} + \frac{2mK}{2^{1/3}} \right] M_1 \]  
(2.67)

Case (ii): $-\frac{4}{3} \leq \beta \leq 2$.

That is, when $\frac{8}{3} \leq T_1 \leq \frac{27}{4(1+\sqrt{1+m^2})} M_1^6$.

In this case the positive root yielding the smallest

$R_{1c}$ is given by

\[ x_{1c} = \left[ \frac{(m^{1/2} + 1)^2 + \tau}{\tau} \right]^{1/2} \]  
(2.68)

and

\[ R_{1c} = \left[ (m^{1/2} + 1)^2 + \tau \right] \left[ 1 + \frac{\tau}{\sqrt{(m^{1/2} + 1)^2 + \tau}} \right]^{2} M_1^2 \]  
(2.69)
when $\tau \ll 1$, after considering terms of second order

$$R_{lc} = M_1^2 + \frac{M_1^4}{T_1} + 2(m+1)T_1^{1/2} \quad (2.70)$$

when $\tau \gg 1$, we have in the leading order

$$R_{lc} = (1+\sqrt{1+m^2})^2 \frac{T_1}{M_1^2} \quad (2.71)$$

Case (iii): $\beta \geq 2$.

This case is defined by

$$T_1 \geq \frac{27}{4(1+\sqrt{1+m^2})^6} M_1^6.$$  

For this range the positive root yielding the smallest critical Rayleigh number is

$$x_{lc} = \left(\frac{1}{2} T_1\right)^{1/3} \quad (2.72)$$

and the critical Rayleigh number is, to leading order

$$R_{lc} = 3\left(\frac{1}{2} T_1\right)^{2/3} \quad (2.73)$$

A comparison of the results (2.66) to (2.73) clearly shows that Hall current has no effect on the critical Rayleigh number to the leading order except when $\tau \gg 1$, while its effect is found in the second order terms. We further note that in cases (ii) and (iii), Hall parameter affects the boundaries of the intervals of $T$. The critical mode in
cases (i) and (iii) are identical when the fluid layer is subject to magnetic field in the absence of rotation and rotation in the absence of magnetic field respectively. Since $T = T/M^4$ is independent of viscosity $\nu$, we have from (2.68) and (2.69) both $x_{lc}(a_c)$ and $(R_{lc}/M^2)$ are both independent of viscosity. That is viscosity has no effect into the leading order and the critical temperature gradient, $\beta_c$, is also independent of viscosity. In cases (i) and (iii) the critical wave numbers are viscosity dependent as seen evidently from (2.66) and (2.72).

When $m$ is large, say $m = m_0M_1$.

In this case taking into account dominant terms the modified wave number is given by

\[ 2x_{1}x_{6} + 4m_{2}x_{6} + 2m_{4}x_{5} + (7m_{4} + 5m_{2}M_{1}^{2} - T_{1}^{2})x_{1}^{4} + (3m_{4} + 12m_{2}M_{1}^{2} + 2m_{4} - 4m_{2}M_{1}^{2} - T_{1}^{2})^{1/2} \]

\[ -2m_{2}T_{1}x_{1}^{2} + (m_{4}T_{1} + 2m_{3}M_{1}^{2}T_{1}^{1/2} + m_{4}M_{1}^{2})x_{1}^{2} - (2m_{2}M_{1}^{4} + 2m_{4}T_{1}^{1/2} + 4m_{3}M_{1}^{2}T_{1}^{1/2})x_{1}^{2} \]

\[ -(m_{4}^{4} + m_{4}^{3}T_{1} + m_{4}^{2}T_{1}^{1/2} + 2m_{3}M_{1}^{2}T_{1}^{1/2} + 2m_{2}M_{1}^{2}T_{1}^{1/2} + 2m_{4}^{2}M_{1}^{2}T_{1}^{1/2}) = 0 \]  \hspace{1cm} (2.74) \]

We proceed similar to a discussion earlier and consider the possible ranges of $\beta$, namely $\beta \leq -2$ and $\beta > -2$ separately.

**Case (i):** $\beta > -2$.

The relevant range of $T_{1}$ is given by
The positive root of (2.74) yielding the minimum Rayleigh number \( R_{lc} \) is given by

\[
x_{lc} = \left( \frac{1}{2} T_1 \right)^{1/3}.
\]

(2.75)

The critical Rayleigh number is, to leading order

\[
R_{lc} = 3 \left( \frac{1}{2} T_1 \right)^{2/3}.
\]

(2.76)

**Case (ii):** \( \beta = -2 \).

This case is defined by \( T_1 = \frac{(m_o \sqrt{K}+1)^2}{m_0^2} M_1^2 \). The consistent root of (2.74) is given by

\[
x_{1c}^3 = \left[ \frac{m_o \sqrt{K}+1}{\sqrt{2} m_o} \right]^2 M_1^2
\]

that is,

\[
x_{lc} = \left( \frac{(m_o \sqrt{K}+1)^2}{2m_o^2} M_1^2 \right)^{1/3}
\]

(2.77)

The critical Rayleigh number is, to leading order

\[
R_{lc} = 3 \left\{ \frac{(m_o \sqrt{K}+1)^2}{2m_o^2} M_1^2 \right\}^{2/3}
\]

(2.78)
Case (iii): $\beta < -2$.

The range of $T_1$ is given by $T_1 < m^{-2} M_1^2$. In this case the relevant root of (2.74) is got from

\[ 2m_0^2 x_1^3 - M_1^2 = 0 \]

(or)

\[ x_{1c} = \left( \frac{M_1^2}{2m_0^2} \right)^{1/3} \]  \hspace{1cm} (2.79)

and

\[ R_{1c} = 3 \left( \frac{M_1^2}{2m_0^2} \right)^{2/3} \]  \hspace{1cm} (2.80)

For the above cases, the critical mode in general is given by

\[ x_{1c} = \left[ \frac{1}{\sqrt{2}} \left( \sqrt{T_1} + \frac{M_1}{m_0} \right) \right]^{2/3} \]  \hspace{1cm} (2.81)

and

\[ R_{1c} = 3 \left[ \frac{1}{\sqrt{2}} \left( \sqrt{T_1} + \frac{M_1}{m_0} \right) \right]^{4/3} \]  \hspace{1cm} (2.82)

Unlike in cases when $m = O(1)$, we see that in all cases above, the critical wave numbers are viscosity dependent. The critical mode obtained for $\beta > -2$ when $m$ is large is identical to those obtained for $\beta \geq 2$ when $m$ is order unity, whereas the effect of Hall current is found in cases for which $\beta \leq -2$ as seen from the values in (2.82).

When $m = O(M^s), (s > 0)$.

In this case taking into account dominant terms the modified wave number is given by
In order to determine the positive root of the above equation in the asymptotic limit of large $M$, it is necessary to find the dominant terms of this equation. Comparison of the orders of magnitude of various terms in the coefficients leads to different regions in the $(s, \beta)$ plane as shown in (Figure 1). Approximation of the equation corresponding to each region is obtained and the relevant positive root is calculated after checking the consistency.

For regions I to VIII the suitable combination yields the critical mode given by

$$x_{1c} = \left(\frac{1}{2} T_1\right)^{1/3}$$  \hspace{1cm} (2.84)$$

$$R_{1c} = 3(\frac{1}{2} T_1)^{1/3}$$  \hspace{1cm} (2.85)$$

In region IX the relevant positive root is got from the equation
FIGURE 1 REGIONS OF THE s - β PLANE OCCURRING IN THE DETERMINATION OF CRITICAL RAYLEIGH NUMBER FOR VARYING T AND THE HALL PARAMETER m
\[ 2x_1^3 - M_1^2 = 0 \]

(or) \[ x_{1c} = \left( \frac{1}{2} M_1^2 \right)^{1/3} \] (2.86)  

and \[ R_{1c} = M_1^2 \] (2.87)  

For region X the critical mode is given by

\[ x_{1c} = \frac{1}{2} m^2 \] (2.88)  

and \[ R_{1c} = M_1^2 \] (2.89)  

In regions XI and XII the consistent root is given by

\[ 2m^2 x_1^3 - M_1^4 = 0 \]

(or) \[ x_{1c} = \left( \frac{1}{2m^2} M_1^4 \right)^{1/3} \] (2.90)  

and \[ R_1 = x_1^2 + \frac{M_1^4}{m^2 x_1} \]

(or) \[ R_{1c} = 3 \left( \frac{1}{2m^2} M_1^4 \right)^{2/3} \] (2.91)  

2.4(b) Free, Perfectly Conducting Boundaries

The boundary conditions applicable in this case are

\[ W = D^2 W = \Theta = D^2 \Phi = D^2 \Psi = 0, \text{ at } z = \frac{1}{2} \] (2.92)
We proceed to solve the equation (2.55) after splitting the same as a main stream equation for which \( D = O(1) \) with a suitable order of magnitude for the wave number \( a \). We postulate the order of magnitude for \( D \) and \( a \) to get the equations governing the boundary layers. We obtain the relevant boundary conditions using (2.56) to (2.58) which are to be satisfied by the main stream and must be adjusted by the boundary layers. Finally we solve the equations to find the smallest Rayleigh number \( R \), provided it exists.

Case (i): \( \beta \leq -\frac{4}{3} \).

The relevant range of \( T \) for this case and for the other two cases as well as for the various cases in §2.4(c) and §2.4(d) are given in Tables 3, 5 and 6 respectively.

Letting \( D = O(1) \) and \( a = O(M^{1/3}) \), equation (2.55) leads to a main stream given by

\[
D^2(M^2D^2+R)W_o = 0 \tag{2.93}
\]

and to the next lower order

\[
D^2(M^2D^2+R)W_o = \frac{1}{a M^2} (M^4D^6+2M^2a^6D^2+a^4TD^2-2mM^2T^{1/2}a^2D^4+Ra^6)W_o
\]

\[
... \tag{2.94}
\]

For \( D = a = O(M^{1/3}) \), the outer layer satisfies
(D^2 - a^2)W_a = 0 \hspace{1cm} (2.95)

Considering terms of next lower order we have

\[ D^2(D^2 - a^2)W_a = \frac{Ra}{M^2}W_a \quad (2.96) \]

When \( D = 0(M) \) and \( a = O(M^{1/3}) \), the inner layer obeys

\[ [(1+m^2)D^4 - 2M^2D^2 + M^4]W_I = 0 \quad (2.97) \]

By taking the next lower order terms, this can be written as

\[ D^2[(1+m^2)D^4 - 2M^2D^2 + M^4]W_I = (5a^2D^4 - 6a^2M^2D^2 + a^2M^4 - 2mM^2D^{1/2} + 4m^2a^2D^4)W_I \quad (2.98) \]

where the subscripts \( 'o' \), \( 'a' \) and \( 'I' \) denotes the solutions in the main stream, in the outer layer and in the inner layer respectively. Hence the 'total' solution \( W \) can be written as

\[ W = W_o + W_a + W_I. \quad (2.99) \]

Assuming \( W_o \) to be even in \( z \), we have

\[ W_o = C_1 \cos k_1z + C_2, \quad W_a = B_1e^{\alpha_1z}, \quad W_I = A_1e^{\alpha z} + A_2e^{*\alpha z} \quad (2.100) \]

where \( C_1, C_2, B_1, A_1 \) and \( A_2 \) are constants. Further
and $a^*$ is the complex conjugate of $a$. The quantities $\epsilon_1$, and $\epsilon$ are given by

$$\epsilon_1 = a(\frac{1}{2} - z) \quad \text{and} \quad \epsilon = M(\frac{1}{2} - z). \quad (2.102)$$

By using equations (2.93) to (2.98) and (2.56) to (2.58), we obtain the following expressions

$$W = W_0 + W_a + W_I,$$

$$D^2 W = D^2 W_0 + a^2 W_a + M^2 \Delta^2 W_I,$$

$$D^2 \phi = \left[ \begin{array}{c} \frac{M^2}{a^4(m^2 - T^{1/2})} \left( a^4 D^2 + a^2 \tau M^2 + \beta D^2 + \frac{Ra^4}{M^2} + \frac{mT^{1/2}R}{M^2} D^2 \right) \\
+ \frac{T^{1/2}}{a^2} D^2 + \frac{mR}{a^4} D^2 \right] W_0 + \frac{mT^{1/2}R}{a^2(T^{1/2} - ma^2)} W_a \\
+ \frac{2(m^2 + 2)M^2}{(1 + m^2)^3 a^4(T^{1/2} - ma^2)} \left( (3m^2) \Delta^2 - 2 \right) W_I + \ldots, \quad (2.103)$$

$$D^2 \psi = \frac{1}{a^4 M^2(T^{1/2} - ma^2)} \left( 3ma^2 T^{1/2} R + \tau R^2 + a^6 M^2 + a^4 T \right) \Delta W_0,$$

$$+ \frac{2mT^{1/2}R}{aM^2(T^{1/2} - ma^2)} \Delta_1 W_a - \frac{M^3}{a^4(T^{1/2} - ma^2)} \left( (12a^4 + 2T - 2ma^2) T^{1/2} \right) \Delta^2,$$

$$- \left\{ \frac{2(3m^2 + 4)}{(1 + m^2)} a^4 + T \right\} \Delta W_I + \ldots, \quad (2.104)$$
\[
\theta = \frac{1}{a^2} W_o - \frac{M^2}{R} W_a + \frac{M^2}{a^4 R} \left[ \frac{m(1+m^2)a^2T}{1+m^2} \right]^{1/2} \frac{-\left(1+m^2\right)T-(3m^2+7)a^4}{1+m^2} W_i + \ldots, \tag{2.105}
\]

where \( \Delta_1 = \frac{d}{d\epsilon} \), \( \Delta = \frac{d}{d\epsilon} \). Also \( D = -a \Delta_1 \), in the outer layer and \( D = -MA \) in the inner layer. Further \( \tau = \tau_o M^8 \).

We apply the boundary conditions (2.92) to the solution (2.99) to find the constants \( C_1, C_2, B_1, A_1 \) and \( A_2 \). (2.100) suggests that two boundary conditions are to be applied to the main stream solution, \( W_o \), one boundary condition to the outer layer solution, \( W_a \), and the remaining two boundary conditions to the inner layer solution \( W_i \).

Using the boundary conditions,

\[
W_o = D^2 W_o = 0 \quad \text{at} \quad z = \frac{1}{2}, \tag{2.106}
\]

to the main stream \( W_o \), we get \( W_o = C_1 \cos (2n+1)\pi z \) \((n = 0, 1, 2, \ldots)\). Since \( n = 0 \) yields the smallest Rayleigh number and choosing \( C_1 = 1 \), we have

\[
W_o = \cos \pi z. \tag{2.107}
\]

To determine \( B_1, A_1 \) and \( A_2 \) we use the conditions \( D\xi = D\xi = \Theta = 0 \) on the boundary.
Here we obtain \( B_1 = O\left(\frac{1}{M_1^{7/3}}\right) \) for all \( \beta \leq -\frac{4}{3} \) and

\[
A_1 (\text{or } A_2) = \begin{cases} 
O \left(\frac{1}{M_1^{5/3}}\right) & \text{when } -\frac{8}{3} < \beta \leq -\frac{4}{3}, \\
O\left(\frac{1}{M_1^{7/3}}\right) & \text{for all } \beta \leq -\frac{8}{3}.
\end{cases}
\]

The critical Rayleigh number \( R_c = \pi^2 M^2 \) to the leading order.

Since this expression for \( R \) is independent of the wave number \( a \), considering second order terms we obtain

\[
a_c = \left(\frac{1}{2} \pi^4 M^2\right)^{1/6}, \quad R_c = \pi^2 M^2 + a^4 + \frac{\pi^2 M^2}{a^2}.
\]

Assuming the solution \( W_0 \) to be odd in \( z \), we have

\[
W_0 = C_1 \sin k_1 z + C_2 z.
\]

When we apply the boundary conditions (2.106) to this solution we get

\[
W_0 = C_1 \sin 2n\pi z, \quad (n = 1, 2, \ldots).
\]

Since the least value of \( n \) gives the smallest Rayleigh number we have for \( n = 1 \) and \( C_1 = 1 \)

\[
W_0 = \sin 2\pi z \quad \text{and} \quad R_c = 4\pi^2 M^2.
\]
The minimization of \( R \) with respect to \( a \) yields

\[
a_c = (8\pi^4 M^2)^{1/6}, \quad R_c = 4\pi^2 M^2.
\]  

(2.114)

Using the boundary conditions

\[
W_o = D W_o = 0 \quad \text{at} \quad z = \frac{1}{2},
\]

which is equivalent to

\[
W_o = D W_o = 0 \quad \text{at} \quad z = \frac{1}{2},
\]  

(2.115)

to the main stream \( W_o \) as in (2.100) we get

\[
k_1 = 2n\pi, \quad W_o = C_1 \left[ \cos 2n\pi z - (-1)^n \right]
\]  

(2.116)

which yield in the leading order

\[
R_c = 4\pi^2 n^2 M^2.
\]  

(2.117)

Since \( n = 1 \) gives the critical Rayleigh number, we have when \( C_1 = 1 \)

\[
W_o = 1 + \cos 2\pi z \quad \text{and} \quad R_c = 4\pi^2 M^2.
\]  

(2.118)

When we consider the second order terms in the expression (2.118) for \( R \) we get the same \( R \) as in (2.113) and again \( a_c \) and \( R_c \) as in (2.114). When the solution selected is odd in \( z \) and when we apply the boundary conditions (2.115) to this solution, the wave number is given by
$$C_1 \sin \left(\frac{k_1}{2}\right) + \frac{C_2}{2} = 0$$

$$k_1C_1 \cos \left(\frac{k_1}{2}\right) + C_2 = 0 .$$

Eliminating $C_1$ and $C_2$ we get

$$\tan \left(\frac{k_1}{2}\right) = \left(\frac{k_1}{2}\right). \quad (2.119)$$

We denote by $\alpha_i$ ($i = 1,2,\ldots$) the positive roots of (2.119) taken in the form

$$\tan \theta = \theta \quad (2.120)$$

The positive root of (2.120) yielding the smallest Rayleigh number is $\alpha_c$, where

$$\alpha_c = 1.43 \pi \quad \text{and} \quad R_c = 4\alpha_c^2 M^2 = 8.18 \pi^2 M^2. \quad (2.121)$$

By applying the remaining boundary conditions $D^2 W = \Theta = D^2 \xi = 0$ we find that

$$B_1 = \begin{cases} O(M^{(2/3)+\beta}) , \text{ when } -8/3 < \beta \leq -4/3 \\ O(M^{-2}) , \text{ when } \beta \leq -8/3 \end{cases} \quad (2.122)$$

$$A_1 \text{ or } A_2 = O(M^{-2}) \quad , \text{ for all } \beta \leq -4/3 .$$

From the expressions for $R_c$, we find that the smallest critical Rayleigh number $R_c$ under both the set of boundary conditions (2.106) and (2.115) belong to the even mode. This
contradicts the result obtained by Eltayeb (1972), for
according to him the odd mode yields the smallest \( R_c \).
However, our results agree with the widely accepted hypo-
thesis that even modes give the smallest critical Rayleigh
number. This contradiction is due to the fact that \( D\xi_o \)
is found proportional to \( DW_o \) in our case instead of \( D^3W_o \)
to the leading order) obtained by Eltayeb.

Case (iia): \(-\frac{4}{3} < \beta < 0\).

In this case the main stream solution, the boundary
layer structure and the critical Rayleigh number, \( R_c \), are
identical to those obtained in case (i) except for the
critical wave number \( a_c \). For after assuming the even
solution to second order we obtain

\[
R = \pi^2 M^2 + a^2 \left( \frac{T}{M^2} \right) + \left( \frac{\pi^4 M^2}{a^2} + 2 \pi^2 \right) \frac{T}{a^2}^{1/2}
\]  

(2.123)

Minimizing \( R \) in the usual way, we get

\[
a_c = \pi \left( \frac{M^4}{T} \right)^{1/4}, \quad R_c = \pi^2 M^2.
\]

(2.124)

Case (iib): \( 0 < \beta < 2 \).

Here the outer boundary layer of cases (i) and (iia)
above has broadened to include the mainstream and is there-
fore governed by an equation of higher order. Ekman-Hartmann
layer replaces the Hartmann layer when \( \beta = 0 \). For large
values of \( \beta \), the Ekman-Hartmann layer transforms into an
Ekman layer.

In this case, for $D = a = O(1)$, the main stream satisfies a sixth order equation

$$D^2(T \nabla^4 - RF^2 a^2 + m^2 TD^4 - m^2 Ta^2 D^2) W_0 = 0$$  \hspace{1cm} (2.125)

and the Ekman-Hartmann layer obeying

$$(D^4 + T) W_I = 0.$$  \hspace{1cm} (2.126)

We take the even solution of (2.125) in the form

$$W_0 = C_1 \cos k_1 z + C_2 \cos k_2 z + C_3$$  \hspace{1cm} (2.127)

where $C_1, C_2, C_3$ are constants, and $k_1^2$ and $k_2^2$ are the roots, $\nu$, of the equation

$$(1 + m^2) \nu^2 + (2 + m^2) \nu a^2 + a^2 (a^2 - E) = 0, \ E = (R/M^2/T).$$  \hspace{1cm} (2.128)

Assuming the boundary layer variable $\epsilon$ as

$$\epsilon = T^{1/4} (\frac{1}{2} \ z),$$  \hspace{1cm} (2.129)

we can write the solution of (2.126) in the form

$$W_I = A_1 e^{\alpha \epsilon} + A_2 e^{\alpha^* \epsilon}$$  \hspace{1cm} (2.130)

where $\alpha = (1 + i)/\sqrt{2}$ in the leading order. Further we must have

$$A_1, A_2 = O(M^{-2})$$  \hspace{1cm} (2.131)
Using equations (2.56) to (2.58) we obtain

\[ W = W_0 + \ldots, \]
\[ D^2W = D^2W_0 + \frac{1}{2} \Delta^2W_I + \ldots, \]
\[ D\xi = \frac{2mM^2}{(1+m^2)a^4} \left( \frac{V^2}{T} + \frac{RM^2}{m^2D^2} \right) D^2W_0 + \ldots; \]
\[ D\xi = \frac{1/2}{M^2a^4} \left\{ (1+m^2)D^4 - (2+m^2)a^2D^2 - \frac{Ra^2M^2}{T} \right\} D^2W_0 + \ldots, \quad (2.132) \]
\[ \Theta = - \frac{T}{RM^2a^4} (V^4 + a^2V^2 + m^2D^4 - \frac{RM^2a^2}{T}) W_0 + \ldots. \]

In this case the main stream solution must satisfy the conditions

\[ W_0 = \Theta_0 = D\xi_0 = 0, \text{ at } z = \frac{1}{2}. \quad (2.133) \]

These conditions after using (2.132) are equivalent to

\[ W_0 = DW_0 = \left\{ (1+m^2)D^4 - a^2D^2 \right\} W_0 = 0, \text{ at } z = \frac{1}{2}. \quad (2.134) \]

If we apply conditions (2.134) to \( W_0 \) in (2.127), we find that a non-trivial solution exists if, and only if

\[ \frac{k_1^2 + a^2}{k_1} \tan \frac{k_1}{2} = \frac{k_2^2 + a^2}{k_2} \tan \frac{k_2}{2} \quad (2.135) \]

We calculate \( k_1^2 \) and \( k_2^2 \) as functions of \( a \) from (2.128) for a fixed \( m \). For different values of \( E \) we obtain the corresponding values of \( a \) that satisfy (2.135). We can then plot
the curve of $E$ versus $a$ and determine the value, $a_c$, of $a$ which gives the minimum $E_c$, of $E$, if it exists. The values of $a_c$ and $E_c$ are given in Table 3.

When we take $m = 0$ in (2.134) it reduces to 

$$ (D^4-a^2D^2)W_0 = 0 \text{ at } z = 1/2 $$

and it appears that $D^4W_0$ term has been neglected by Eltayeb in the corresponding case while deriving the conditions for $\Theta_0$ in terms of $W_0$ at $z = 1/2$. This leads to a change in the values of $a_c$ and $E_c$ for $m = 0$.

**Case (iii): $\beta \geq 2$.**

For this range of values of $\beta$, we get a main stream governed by

$$ [D^2 + \frac{(R-a^4)^2}{T}]W_0 = 0 \quad (2.136) $$

The outer layer obeys the equation

$$ (D^2-a^2) \left\{ (1+m^2)D^2-a^2 \right\} W_a = 0 \quad (2.137) $$

and inner layer satisfies

$$ (D^4+T)W_I = 0 \quad (2.138) $$

Applying the boundary condition $W_0 = 0$ at $z = \frac{1}{2}$, we get the solution as

$$ W_0 = \cos \pi z \quad (2.139) $$
As usual minimizing $R$ with respect to $a$ we obtain

$$a_c = \left(\frac{\pi^2 T}{2}\right)^{1/6},$$

$$R_c = 3 \left\{ \left(\frac{\pi^2 T}{2}\right)^{2/3} + \left(\frac{\pi^8 T}{2}\right)^{1/3} \right\}^{1/6}$$

From the above discussion we find that the effect of Hall current is to increase the Rayleigh number when $T$ lies in the range $0(M^4) \leq T \leq O(M^6)$ and for values of $T$ outside this range the critical mode is not affected by the presence of Hall current to the leading order.

When $m$ is large, that is, when $m = m_o M$ and $T = \tau_o M^{4+\beta}$

Case (i): $\beta < -2$.

In this case the main stream equation reduces to

$$[M^4 D^2 + m_o^2 a^2 (R-a^4)] W_o = 0. \quad (2.141)$$

The outer boundary layer satisfies

$$D^2 (D^2-a^2) W_a = 0. \quad (2.142)$$

The Hartmann-layer obeying

$$[m_o^2 M^2 D^4 - 2(1+2m_o a^2) M^2 D^2 + (1+m_o \tau_o M^{1/2} \frac{1+\beta/2}{2} M^4)] W_I = 0. \quad (2.143)$$

The even solution of (2.141) is given by
\[ W_0 = A \cos \left( \frac{VR_1}{M_1} z \right), \text{ where } R_1 = m_0 a^2 (R-a^4). \]

The boundary condition \( W_0 = 0 \) at \( z = \frac{1}{2} \) yields

\[ R_1 = \pi^2 M^2, \]

that is,

\[ R = \frac{\pi^2 M^2}{m_0 a^2} + a^4. \]

Minimizing with respect to \( a \) we obtain

\[ a_c^2 = \left( \frac{\pi^2 M^2}{2m_0^2} \right)^{1/3}, \quad R_c = 3 \left( \frac{\pi^2 M^2}{2m_0^2} \right)^{2/3}. \quad (2.144) \]

**Case (ii):** \( \beta = -2. \)

For this value of \( \beta \), the main stream is given by

\[ [(m_0 \tau_0 + 1)^{1/2} M^2 D^2 + m_0^2 a^2 (R-a^4)] W_0 = 0 \quad (2.145) \]

and the outer and Hartmann layer obey the same form as in (2.142) and (2.143) respectively. Proceeding similar to case (i) above we get the critical mode given by

\[ a_c^2 = \left\{ \frac{(m_0 \tau_0 + 1)^{1/2} \pi^2 M^2}{2m_0^2} \right\}^{1/3}, \quad (2.146) \]
\[ R_c = 3 \left\{ \frac{(m_o \tau_o + 1)^{1/2}}{2 \pi^2 M^2} \right\}^{2/3} \]  \hspace{1cm} (2.147)

**Case (iii):** \(-2 < \beta < 0\).

Here the main stream satisfies

\[ \{T D^2 + a^2 (R - a^4)\} W_o = 0 \]  \hspace{1cm} (2.148)

The outer layer obeying

\[ [m_o^2 T D^4 + (m_o^2 R a^2 - 2 m_o \tau_o) a^{1/2} M^{1+\beta/2} a^2 - 4 m_o^2 a^6 - m_o a T M] D^2 + a^4 \tau_o M^{2+\beta} + \beta m_o^2 (a^4 - R)] W_a = 0 \]  \hspace{1cm} (2.149)

and the Hartmann layer is given by

\[ (D^4 + T) W_I = 0 \]  \hspace{1cm} (2.150)

Assuming the even solution, the boundary condition \( W_o = 0 \) at \( z = 1/2 \) yields the following values of \( a_c \) and \( R_c \).

\[ a_c^2 = \left( \frac{1}{2} \pi^2 T \right)^{1/3}, \hspace{1cm} (2.151) \]

\[ R_c = 3 \left( \frac{1}{2} \pi^2 T \right)^{2/3}. \hspace{1cm} (2.152) \]
\[ R_c = 3\left(\frac{1}{2} \pi^2 T\right)^{2/3} + \frac{2^{4/3} \pi^{4/3}}{m_0^{1/6}} \left(\frac{3\pi}{4}\right) + \frac{T}{M} \]  

Case (iv): \( \beta > 0 \).

In this case the main stream is given by

\[ [TD^2+a^2(R-a^4)]W_o = 0 \]  

(2.154)

The outer boundary layer obeys the equation

\[ [TD^2(D^2-a^2)+a^4(a^4-R)]W_o = 0 \quad (\text{for } \beta \neq 1/2) \]  

(2.155)

\[ [TD^2+a^2(R-a^4)]W_a = 0 \quad (\text{for } \beta = 1/2) \]

The Hartmann layer satisfies

\[ (D^4+T)W_I = 0 \quad \text{when } 0 < \beta < \frac{1}{2} \]

\[ [D^6+T(D^2-a^2)]W_I = 0 \quad \text{when } \beta = \frac{1}{2} \]  

(2.156)

\[ D^2(D^2-a^2)W_I = 0 \quad \text{when } \beta > \frac{1}{2} \]

Following similar method as mentioned earlier we obtain the critical mode as given below.

For \( 0 < \beta < 2 \)

\[ a_c^2 = \left(\frac{1}{2} \pi^2 T\right)^{1/3} \]
\[ R_c = 3\left(\frac{\pi^2 T}{2}\right)^{1/3} + \frac{2^{4/3} \pi^{4/3}}{m_0} T^{1/6} \]

For \( \beta = 2 \)

\[ a_c^2 = \left(\frac{1}{2} \pi^2 T\right)^{1/3} \]

\[ R_c = 3\left(\frac{\pi^2 T}{2}\right)^{2/3} + \left(\frac{7/3 8/3 1/6 2/3 2 1/3}{2^{4/3} \pi m_o \tau_o -2 \pi \tau_o 1/3} \right)M^2 \]

For \( \beta > 2 \)

\[ a_c^2 = \left(\frac{1}{2} \pi^2 T\right)^{1/3} \]

\[ R_c = 3\left(\frac{\pi^2 T}{2}\right)^{2/3} - \pi^2 M^2. \]

From the above values we conclude that the orders of magnitude of the critical wave number and critical Rayleigh number is similar to those obtained for free insulating boundaries when \( m = 0(M) \) and the effect of Hall current is found in the leading order for all \( \beta \leq -2 \) and for \( \beta > -2 \) the effect is found only when we calculate the critical Rayleigh number to second order.

2.4(c) Rigid Insulating Boundaries

The relevant boundary conditions in this case are

\[ W = \Theta = DW = \eta = \zeta = 0, \text{ at } z = \frac{1}{2}. \]  (2.157)
Case (i): $\beta \leq -\frac{4}{3}$.

The main stream and the boundary layers are identical to those found in (2.93), (2.95) and (2.97).

Following the method of § 2.4(b), we obtain the following expressions

$$W = W_0 + \ldots,$$

$$DW = DW_o - M \Delta W_1 + \ldots,$$

$$\xi = -\frac{M^2}{(T^{1/2} - ma^2)a^4} (M^2D^2 + R)DW_o + \ldots,$$

$$\eta = -\frac{1}{(T^{1/2} - ma^2)a^2} (M^2D^2 + R)W_o + \ldots, \quad (2.158)$$

$$\theta = \frac{1}{a^2} W_o - \frac{M^2}{R} W_a + \ldots.$$

The main stream solution must satisfy the conditions

$$W_o = \xi_o = 0, \quad \text{at} \quad z = \frac{1}{2}, \quad (2.159)$$

which after using (2.158) reduce to

$$W_o = D^2W_o = 0, \quad \text{at} \quad z = \frac{1}{2}. \quad (2.160)$$

This yield the even solution $W_o = \cos \pi z$.

The critical Rayleigh and wave numbers are identical to those of case (i) in § 2.4(a) and of case (i) in § 2.4(b).
Case (iia): \(- \frac{4}{3} \leq \beta \leq 0\).

This case is similar to case (iia) of §2.4(i) and we get the same values for the wave number and critical Rayleigh number.

Case (iib):

For \(\beta\) lying the interval \([0,2]\), the main stream and boundary layer equations are identical to (2.125) and (2.126) of §2.4(b). In the usual way we get the expressions as follows:

\[
W = W_0 + W_I,
\]

\[
\frac{\partial W}{\partial z} = \frac{1}{4} \Delta W_I,
\]

\[
\zeta = \frac{2mR^2}{(1+m^2)^2} \left( \frac{V^2 + \frac{RM^2}{T}}{1+m^2D^2} \right) W_0 + \ldots \tag{2.161}
\]

\[
\zeta = \frac{1}{M^2a^4} \left[ (1+m^2)D^4 - (2+m^2)a^2D^2 - \frac{RM^2a^2}{T} \right] W_0 + \ldots,
\]

\[
\Theta = -\frac{1}{RM^2a^4} \left( V^4 + a^2V^2 + m^2D^4 - \frac{RM^2a^2}{T} \right) W_0 + \ldots.
\]

The main stream solution must satisfy the boundary conditions

\[
W_0 = \Theta_0 = \zeta_0 = 0, \text{ at } z = \frac{1}{2}. \tag{2.162}
\]

Using (2.161), the above conditions reduce to
\[ W_0 = D^2W_0 = D^4W_0 = 0, \quad \text{at} \quad z = \frac{1}{2}. \quad (2.163) \]

The conditions (2.163) when applied to \( W_0 \) as in (2.127) lead to a non-trivial solution for \( C_1, C_2 \) and \( C_3 \) if and only if

\[ C_3 = 0, \quad k_1 = k_2 = (2n+1)\pi. \quad (2.164) \]

Since the smallest \( R \) corresponds to the lowest mode, we take \( n = 0 \) to obtain

\[ R = (\pi^2 + a^2) \left[ (1+m^2)^2 + a^2 \right] \frac{T}{a^2 M^2}. \quad (2.165) \]

When minimized in the usual way, we obtain

\[ a_c = (1+m^2)^{1/4} \pi, \quad (2.166) \]

\[ R_c = (1+m^2)^{2} \pi^2 \frac{T}{M^2}. \quad (2.167) \]

For \( m = 0, .5, 1 \) and 3, the values of \( a_c \) and \( R_c \) are given in Table 5. The magnitudes of \( A_1 \) and \( A_2 \) are found to be such that \( A_1, A_2 = O(T^{-1/4}). \) The remaining conditions \( \xi = 0 = DW \) are adjusted by the Ekman layer.

**Case (iii):** \( \beta \geq 2. \)

Here we get the main stream and boundary layer equations identical to those obtained in case (iii) of § 2.4(b). The main stream solution is of the form \( W_0 = \cos \pi z \) and the values of \( a_c \) and \( R_c \) are similar to those obtained in
We find that the results are identical to the corresponding case for free perfectly conducting boundaries when $m = O(1)$, except when $O(M^4) \leq T \leq O(M^6)$, the Hall effect is to increase the Rayleigh number.

### 2.4(d) Rigid Perfectly Conducting Boundaries

The boundary conditions to be applied here are

$$W = \Theta = DW = \zeta = D\zeta = 0, \text{ at } z = \frac{1}{2}. \quad (2.168)$$

**Case (i):** $\beta \leq -\frac{4}{3}$.

The main stream as well as outer and inner layers are governed by equations as in case (i) of § 2.4(b). We get the expressions for the variables involved in (2.168) to the leading order as follows:

$$W = W_o + \ldots,$$

$$DW = DW_o - M\Delta W_I + \ldots,$$

$$\zeta = -\frac{M^2}{(T^{1/2} - ma^2)a^4} (M^2D^2 + R)DW_o + \ldots, \quad (2.169)$$

$$D\zeta = \frac{1}{M^2a^4(T^{1/2} - ma^2)} [3ma^2 T^{1/2} R + R^2 + M^2a^6 + a^4 T]DW_o + \ldots,$$

$$\Theta = \frac{1}{a^2} W_o - \frac{M^2}{R} W_a + \ldots.$$
The main stream solution is to satisfy the condition

$$W_0 = D \lambda'_{o} = 0, \text{ at } z = \frac{1}{2}$$  \hspace{1cm} (2.170)

which are equivalent to

$$W_0 = DW_0 = 0, \text{ at } z = \frac{1}{2}.$$  \hspace{1cm} (2.171)

Following the same procedure as in case (i) of § 2.4(b) we get the solution as $W_0 = 1 + \cos 2\pi z$ and the critical mode $a_c = \left(8\pi^4 M^2\right)^{1/6}$ with the critical Rayleigh number $R_c = 4\pi^2 M^2$.

**Case (iia):** \(-\frac{4}{3} \leq \beta \leq 0\).

For this range of values of $\beta$ the main stream and boundary layer equations are similar to equations obtained in case (i) except for the critical wave number $a_c$. In this case assuming the even solution for $W_0$, we obtain

$$R = \pi^2 M^2 + a^2 \frac{T}{M^2} + \frac{\pi^4 M^2}{a^2} + 2m\pi^2 T^{1/2}$$

to second order.

Minimizing $R$ in the usual way, we get

$$a_c = \pi \left(\frac{M^4}{T}\right)^{1/4}, \hspace{0.5cm} R_c = \pi^2 M^2.$$  

But to maintain continuity of $R_c$ with the previous range of
\( \beta \) in case (i) we take the odd solution for \( W_0 \) to obtain \( R \) as follows.

\[
R = 4\pi^2 M^2 + \frac{a^2 T}{M^2} + \frac{16\pi^4 M^2}{a^2} + 8\pi^2 M^2 T^{1/2}.
\]

Here the critical mode is given by

\[
a_c = 2\pi\left(\frac{M^4}{T}\right)^{1/4}, \quad R_c = 4\pi^2 M^2.
\]

Case (ii b): 

When \( \beta \) lies in the range \( 0 < \beta < 2 \), the main stream and boundary layer are similar to those of case (ii b) of § 2.4(b). As usual we obtain

\[
W = W_0 + W_1,
\]

\[
DW = DW_0 - T^{1/4} \Delta W_1,
\]

\[
\zeta = \frac{T^{1/2}}{4 a M^2} \left[ a^2 \left\{ (1+m^2) D^2 - a^2 \right\} + \right.
\]

\[
+ \frac{RM^2}{T} \left\{ (1+m^2) D^2 - (2+m^2) a^2 \right\} \right] DW_0 - T^{1/4} \Delta^3 W_1 + \ldots,
\]

\[
D_{\zeta} = \frac{T^{1/2}}{4 a M^2} \left[ (1+m^2) D^4 - (2+m^2) a^2 D^2 - \frac{Ra^2 M^2}{T} \right] DW_0
\]

\[
+ \frac{1}{1+m^2} \left[ T^{1/4} \Delta^3 W_1 - mT^{1/4} \Delta W_1 \right] + \ldots
\]

\[
\Theta = -\frac{T}{RM^2 a^4} \left( \nabla^4 + a^2 \nabla^2 + m^2 D^4 - \frac{RM^2 a^2}{T} \right) W_0
\]

It can be shown that \( A_1 \) and \( A_2 \) must satisfy
In that case we have only two boundary conditions for the main stream, for the other two conditions $D\xi = \zeta = 0$ are dependent in the leading order. We must therefore replace the boundary condition on $D\xi$ by a combination of $\zeta$ and $D\xi$ that is, $\zeta + (1 + m^2)D\xi$. We find

$$\xi + (1 + m^2)D\xi = \frac{1/2}{4}\left(a^2 + \frac{RM^2}{I}\right)[(1 + m^2)D^2 - (2 + m^2)a^2]DW_o - mT_{1/4} \Delta W_I + ..$$

Then we find that the main stream solution (2.127) must satisfy

$$W_o = \theta_o = \zeta_o + (1 + m^2)D\xi_o = 0, \text{ at } z = \frac{1}{2}. \quad (2.175)$$

After using (2.172), the above conditions reduce to

$$W_o = [(1 + m^2)D^4 - a^2D^2]W_o = [(1 + m^2)D^2 - (2 + m^2)a^2]DW_o = 0,$$

$$\text{at } z = \frac{1}{2}. \quad (2.176)$$

The conditions (2.176) when applied to $W_o$ in (2.127) yield a non-trivial solution only when

$$\frac{k_1^2 + a^2}{k_1^3 \tan \frac{k_1}{2}} = \frac{k_2^2 + a^2}{k_2^3 \tan \frac{k_2}{2}} \quad (2.177)$$

where $k_1^2$ and $k_2^2$ are the roots $\nu$ of (2.128) and $E = \frac{RM^2}{I}$. 

$$A_1, A_2 = O(M^{-1}) \quad (2.173)$$
The minimum $E_c$ and the corresponding $a_c$ are obtained for different values of $m$. These values are given in Table 6.

**Case (iii):** $\beta > 2$.

For this range of values of $\beta$, the main stream, boundary layers and the critical mode are identical to those obtained in the corresponding case of § 2.4(b) and § 2.4(c).

From the above results we see that the orders of magnitudes of $a_c$ and $R_c$ are the same as those obtained in the corresponding cases of § 2.4(b) and § 2.4(c) and further for all $\beta \leq 2$ we get higher values for the critical mode.

### 2.5 Model II

In the model considered in this section, the magnetic field $\vec{B}_0$ is assumed in the horizontal direction while the angular velocity $\hat{\omega}$ in the vertical direction. Then we must have

$$\hat{\nabla} \cdot \vec{V} = i k, \quad \hat{\nabla} \cdot \vec{u} = D.$$  

Using the above in equation (2.36), we have for $p_m = 0$ and $\sigma = 0$ (in the case of stationary convection) the
In this case we have restricted our discussion to the case of free perfectly conducting boundaries. The relevant boundary conditions are already stated in Section 2.3.

Case (i): $\beta < 0$.

This case is defined by $T < \frac{4}{3} M^4$. Letting $a = O(1)$, equation (2.179) leads to a main stream given by

$$
( TV^2 + G V^2 + R G a^2 ) \Theta_0 = 0
$$

(2.180)

The solution for $\Theta$ such that $\Theta = 0$ at $z = + \frac{1}{2}$ is

$$
\Theta_0 = \cos \pi z
$$

(2.181)
Using (2.181) in (2.180), we get

\[ R = \frac{\pi^2(a^2 + \ell^2)}{a^2 G} T + \frac{(\pi^2 + a^2)}{a^2} G \]  

(2.182)

Assuming both \( k \) and \( \ell \) are non-zero, we must find the simultaneous roots of \( \frac{\partial R}{\partial k} = \frac{\partial R}{\partial \ell} = 0 \). Such values of \( k \) and \( \ell \) are also given by solving \( \frac{\partial R}{\partial a^2} = \frac{\partial R}{\partial G} = 0 \) simultaneously.

Now \( \frac{\partial R}{\partial G} = 0 \) yields

\[ G^2 = \pi^2 T(\pi^2 + a^2) \]  

(2.183)

and \( \frac{\partial R}{\partial a^2} = 0 \) gives

\[ G^2 = T(a^4 - \pi^4) \]  

(2.184)

Comparing (2.183) and (2.184), we get

\[ a_c = \sqrt{2} \pi \]  

(2.185)

Substituting for \( a \) in (2.184) and using \( G = M^2 k^2 \) we obtain

\[ k_c^2 = \pi^2 \left( \frac{3T}{M^4} \right)^{1/2} \]  

(2.186)

Equation (2.182) gives the following value of the critical Rayleigh number

\[ R_c = 3\pi^2(3T)^{1/2} \]  

(2.187)
Since \( k^2 = a^2 - \ell^2 \leq a^2 \), the results in (2.185) and (2.187) are valid only when \( T < \frac{4M^4}{3} \). This implies that for \( T \geq \frac{4M^4}{3} \), we cannot minimize \( R \) either for non-zero values of \( a \) and \( G \) (or) for non-zero values of \( k \) and \( \ell \), since the only positive root is given by \( a_c \) as in (2.185).

Under the circumstances we have to consider the cases \( k = 0 \) and \( \ell = 0 \) separately. Letting \( k = 0 \), we have \( G = 0 \) and \( a^2 = \ell^2 \).

Then the equation (2.179) for \( \Theta \) takes the form
\[
(\nabla^6 + TD^2)\Theta_o = -Ra^2\Theta_o.
\]
Assuming the solution for \( \Theta_o \) as before, we must have
\[
R = \frac{\kappa^2T}{a^2} + \frac{(\pi^2a^2)^3}{a^2}.
\]
Minimizing \( R \) with respect to \( a^2 \), we get the following values for the critical mode.

\[
a_c = \left(\frac{1}{2} \pi^2 T\right)^{1/3} = \ell_c,
\]
\[
R_c = 3\left(\frac{1}{2} \pi^2 T\right)^{2/3}.
\]

The above values are identical to those obtainable in the absence of magnetic field.
On the other hand if \( \mathcal{L} = 0 \), then \( a^2 = k^2 \). Then the equation (2.179) for \( \theta \) takes the form

\[
(IV^4D^2 + G^2 \nabla^2)\theta_o = (-Ra^2G + m^2k^2IV^2D^2)\theta_o.
\]

Assuming the solution for \( \theta_o \) as given (2.181) we get the following value for \( R \)

\[
R = \frac{\pi^2T}{M^2} \left( \frac{\pi^2+k^2}{k^4} \right) + M^2(\pi^2+k^2) + \frac{m^2\pi^2T}{M^2} \frac{\pi^2+k^2}{k^2}
\]

(2.188)

The minimum \( R \) is obtained by setting

\[
\frac{dR}{dk^2} = 0.
\]

This condition yields

\[
M^4k^6 = \pi^4T \left\{ (m^2+2)k^2+2\pi^2 \right\}.
\]

(2.189)

Using (2.189) in (2.188), the expression for \( R_c \) can be written as

\[
R_c = g(k_c)T^{1/2},
\]

(2.190)

where

\[
g(k) = \frac{\{(1+m^2)k^2+2\pi^2\}(\pi^2+k^2)^2}{k^3 \left\{ (m^2+2)k^2+2\pi^2 \right\}^{1/2}}
\]

(2.191)

and \( k_c \) is a positive root of the equation (2.189). We note that \( k \) is a function of the parameter \( T/M^4 \). As the parameter varies \( g(k) \) attains a minimum value for a given \( m \). This minimum value can be obtained from the equation
g'(k) = 0

This condition using (2.191) leads to the equation

\[(1+m^2)(2+m^2)k^6-(m^4-1)k^4\pi^2-(5m^2+7)k^2\pi^4-6\pi^6 = 0 \quad (2.192)\]

From the above equation we proceed to find the positive root. That is, we solve for \( k_c^2 \) for different values of \( m \). The corresponding minimum \( g(k_c) \) can be found using (2.191). The values of \( k_c^2 \) and \( g(k_c) \) for different \( m \) are given below:

<table>
<thead>
<tr>
<th>( m )</th>
<th>( k_c^2 )</th>
<th>( g(k_c) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1.618 ( \pi^2 )</td>
<td>6.661 ( \pi^2 )</td>
</tr>
<tr>
<td>2</td>
<td>1.306 ( \pi^2 )</td>
<td>9.691 ( \pi^2 )</td>
</tr>
<tr>
<td>5</td>
<td>1.072 ( \pi^2 )</td>
<td>20.771 ( \pi^2 )</td>
</tr>
<tr>
<td>10</td>
<td>0.28 ( \pi^2 )</td>
<td>60.553 ( \pi^2 )</td>
</tr>
</tbody>
</table>

For values of \( T > \frac{4M^4}{3} \) the preferred mode is that of a roll parallel to the magnetic field, and \( k_c \) and \( R_c \) are given by equations (2.189) to (2.191).

**Case (ii):** \( 0 \leq \beta \leq 2 \).

For values of \( T \) lying in the range \( \frac{4M^4}{3} \leq T \leq \frac{27M^6}{4(m+1)^3\pi^2} \), equation (2.189) yield to leading order
\[ kc = \pi \left\{ \frac{(m^2+2)T}{M^4} \right\}^{1/4}. \]

From (2.188) we find the leading order value of the critical Rayleigh number as

\[ R_c = (m^2+1)^2 \frac{T}{M^2}. \quad (2.193) \]

**Case (iii):** \( \beta \geq 2. \)

For this case precisely we must have \( T \geq \frac{27M^6}{4(1+m^2)^2 \pi^2}. \)

We have to consider only the case of \( \ell = 0 \) and this leads from (2.179) to a main stream governed by

\[ (1+m^2) \left[ D_1^2 + \frac{a^2(R-a^4)}{T} \right] \Theta_0 = 0 \quad (2.194) \]

Equation (2.194) is of second order, we need only one boundary condition to be applied to \( \Theta_0 \), which is clearly

\[ \Theta_0 = 0 \text{ at } z = \frac{1}{2}. \]

Hence \( \Theta_0 = \cos \pi z. \)

Using this solution in (2.194), we obtain

\[ R = \frac{\pi^2 T}{a^2} + a^4. \]

We minimize \( R \) in the usual way to get
The above critical values are the same as that obtainable when \( k = 0 \), in the leading order. In other words, in the leading order, the critical Rayleigh numbers of the two modes \( k = 0 \) and \( \ell = 0 \) are equal. This one can usually expect because the magnetic field is a small perturbation in this case, whereas rotation being in the vertical direction, Coriolis forces do not favour any particular orientation for the roll to leading order. In order to get the preferred pattern, it is necessary to consider the expansion to a sufficiently higher order for the magnetic field or Hall current to become effective. For \( \ell = 0 \), the second-order terms yield

\[
\kappa_c^2 = \left( \frac{1}{2} \pi^2 T \right)^{1/3} - \frac{5}{6(1+\mu^2)} M^2 ,
\]

\[
R_c = 3 \left( \frac{1}{2} \pi^2 T \right)^{2/3} - \frac{M^2}{(1+\mu^2)} \left( \frac{1}{2} \pi^2 T \right)^{1/3} ,
\]

while for \( k = 0 \) we can obtain

\[
\ell_c^2 = \left( \frac{1}{2} \pi^2 T \right)^{1/3} - \frac{1}{2} ,
\]

\[
R_c = 3 \left( \frac{1}{2} \pi^2 T \right)^{2/3} + 3 \left( \frac{1}{2} \pi^2 T \right)^{1/3} + o(1) .
\]
Comparison of results (2.196) and (2.197) shows that the minimum value of $R_c$ corresponds to the case of $\ell = 0$. Hence $\ell = 0$ is the preferred mode.

2.6 Model III

In this model, $B_o \rightarrow$ is parallel to the $x$-axis and $\Omega \rightarrow$ is horizontal and makes an angle $\phi$ with $B_o \rightarrow$. Set $f = \sin \phi$ and $g = \cos \phi$. For this case

$\hat{B} \cdot \hat{\Omega} = \cos \phi$, $\hat{B} \cdot V = ik$

$\hat{\Omega} \cdot V = (\cos \phi \hat{i} + \sin \phi \hat{j}) \cdot (ik \hat{i} + i \hat{j})$

$= ik \cos \phi + i \ell \sin \phi$

Taking $k = a \sin \psi$ and $\ell = a \cos \psi$, we can write

$\hat{\Omega} \cdot V = i a \sin (\phi + \psi) = i a b_1$

where $b_1 = \sin (\psi + \phi)$.

When the principle of exchange of stabilities is valid we have $\sigma = 0$. From (2.36) setting $\sigma = 0$ and using the above we get

$$[\nabla^2 (V^4 + M^2 k^2)^2 - TV a^2 b_1^2]W = -Ra^2 (V^4 + M^2 k^2)W - 2mM T^{1/2} k^3 a b_1 V^2 W$$

$$+ m^2 k^2 (V^6 - a^2 b_1^2) V^2 W + m^2 Ra^2 k^2 V^2 W$$

(2.198)
Using $W = -V^2 \theta$ in (2.198), we get

$$
[V^2 (V^4 + M^2 k^2)^2 - V^4 a^2 b_1] \theta = -Ra^2 (V^4 + M^2 k^2) \theta - 2mM^2 T^{1/2} k^3 a b_1 V^2 \theta
+ m^2 k^2 (V^4 - a^2 b_1^2) \theta - m^2 Ra^2 k^2 V^2 \theta
$$

(2.199)

The main stream solution must satisfy the boundary conditions

$$
\theta_0 = W_0 = 0, \text{ at } z = \frac{1}{2}
$$

for all types of boundaries. The above conditions are equivalent to

$$
\theta_0 = V^2 \theta_0 = 0, \text{ at } z = \frac{1}{2}.
$$

Then the even solution of (2.199) can be taken as

$$
\theta_0 = \cos (2n + 1) \pi z, \ (n = 0, 1, 2, \ldots).
$$

In order to get the minimum Rayleigh number $R$ we must take the smallest $n$. Hence we take $n = 0$ to get

$$
\theta_0 = \cos \pi z
$$

(2.200)

for all types of boundary. If we substitute (2.200) into equation (2.199) we obtain

$$
R \left[ x_1 (1 + x_1)^2 + x_1^2 a^2 b_1 \sin^2 \psi + m^2 \sin^2 \psi x_1^2 (1 + x_1) \right]
= (1 + x_1)^5 + 2M^2 \sin^2 \psi x_1 (1 + x_1)^3 + M^4 \sin^4 \psi x_1^2 (1 + x_1)
+ T_1 b_1^2 x_1 (1 + x_1)^2 + 2mM^2 T_1^{1/2} b_1^3 \sin^3 \psi x_1^2 (1 + x_1)
+ m^2 \sin^2 \psi x_1 (1 + x_1)^4 + m^2 T_1 b_1^2 \sin^2 \psi x_1^2 (1 + x_1)
$$

(2.201)
where $x, M, T$ and $R$ are already defined in Section 2.4.

For large $M$, we can write

$$
\frac{R}{M^2} = (1+x) \sin^2 \psi + \frac{b^2 \tau (1+x)^2}{x^2 \sin^2 \psi} + 2m \tau \nabla \sin \psi (1+x) + m^2 \tau b^2 (1+x) \quad (2.202)
$$

where $\tau = \frac{I}{M^4}$.

The minimum Rayleigh number, $R_c$, is obtained by solving

$$
\frac{\partial}{\partial x_1} \left( \frac{R}{M^2} \right) = 0 \quad \text{and} \quad \frac{\partial}{\partial \sin \psi} \left( \frac{R}{M^2} \right) = 0 \quad \text{simultaneously}.
$$

These give respectively,

$$
\frac{1}{2} x^2 \sin^4 \psi = b^2 (1-x^2-m^2 x^2 \sin^2 \psi)^{1/2} \sin \psi (1+x) + m \tau b x^2 \sin \psi (1+x) \quad (2.203)
$$

Assuming $\tau$ and $\psi$ small, we have $b = f$. Then (2.204) has the positive root given by

$$
\sin^4 \psi = \frac{f^2 \tau (1+x)}{x_1} \quad (2.205)
$$

Since $\psi$ is small we get from the above
\[
\Psi = \pm f \frac{1/2}{T} \frac{1/4}{(\frac{1+x_1}{x_1})^{1/4}}. \tag{2.206}
\]

Using (2.205) and (2.206) in (2.203) and using the fact that \( \Psi \) is small we have

\[
x_1^2 \frac{f^2 T(1+x_1)}{x_1} = f^2 T(1-x_1^2) + O\left(\frac{3}{2} \frac{1}{T}\right) + O\left(\frac{5}{4} \frac{1}{T}\right)
\]

Since \( T \) is small, we have from the above

\[
x_{1c} = \frac{1}{2}. \tag{2.207}
\]

Substituting \( x_1 = \frac{1}{2} \) in (2.206), we get

\[
\Psi_c = -(3f^2 T)^{1/4}
\]

(Negative sign is taken to get the minimum \( R \)).

From (2.202), after using (2.207) and (2.208), we obtain in the leading order

\[
R_{1c} = 3(3f^2 T)^{1/2} M_1 = 3f(3T_1)^{1/2} \tag{2.209}
\]

When \( f = 1 \), that is, when \( \phi = 90^\circ \), the direction of the magnetic field and that of rotations are at right angles and hence the above value of \( R_{1c} \) is identical to (2.187) of Model II. Since we are going to consider second order terms, let us take
\[ x_1 = x_0 + x'_1 \]

and
\[ \psi = \psi_0 + \psi_1. \]

Substituting for \( x_1 \) and \( \psi \) as given above in (2.203) and (2.204) and after using the leading order values \( x_{1c} = x_0 \) and \( \psi_c = \psi_0 \) we obtain the following equations in \( x'_1 \) and \( \psi_1 \):

\[
4f^2 \tau x'_1 + \psi_0^3 \psi_1 = \frac{3}{2} fg \tau \psi_0 - \frac{m \ell}{2} \psi_0^{1/2} \] \tag{2.210}

\[
2f^2 \tau x'_1 + 2\psi_0^3 \psi_1 = \frac{3}{2} fg \tau \psi_0 - \frac{m \ell}{2} \psi_0^{1/2} \] \tag{2.211}

The above two equations yield
\[
\psi_1 = \frac{2f^2 \tau}{\psi_0^{1/2}} x'_1 \] \tag{2.212}

Substituting for \( \psi_1 \) in (2.210) we get
\[
x'_1 = - \frac{(3\tau)^{1/4}}{4} \left( \frac{g^2}{f} \right)^{1/2} + \frac{m}{12} (3f) \frac{1/2}{(3\tau)} \frac{1/4}{(\tau)} , \] \tag{2.213}

Then (2.212) gives
\[
\psi_1 = \frac{(\sqrt{3} \ g - mf)}{6} \tau^{1/2}. \] \tag{2.214}

From (2.202), to second order we obtain
\[
\frac{R_1}{M_1} = \frac{1}{2\psi_0^2} \left( 3\psi_0^4 + 9f^2\tau \right) + \frac{9fg\tau}{\psi_0} + 3m \frac{1/2}{\tau} f \psi_0
\]

Substituting for \( \psi_0 \), we have

\[
\frac{R_1}{M_1} = 3(3f^2\tau)^{1/2} 5/4 \tau^{-3} \frac{\sqrt{f}(\sqrt{3}g+mf)}{T} \]

(or) \[ R_{lc} = 3f(3T^1)^{1/2} 5/4 \tau^{-3} \frac{(\sqrt{3}g+mf)}{M_1} \]

Therefore in the second order for non-zero \( f \) and \( g \) we have found

\[
x_{lc} = \frac{1}{2} \left[ 1 - \frac{1}{2} \left( 3T \right)^{1/4} \left( g^2/f \right)^{1/2} + \frac{m}{6} \left( 3f \right)^{1/2} (3T)^{1/4} \right]
\]

\[
\psi_c = -(3f^2\tau)^{1/4} + \frac{(\sqrt{3}g-mf)}{6} \tau^{1/2}
\]

and \( R_{lc} \) as given in (2.215).

From the above results we see that the effect of Hall current is found in the second order in the values of \( x_{lc} \), \( \psi_c \) and \( R_{lc} \).

The Case \( k = 0 \).

This is the case of rolls with axes parallel to \( \hat{B}_0 \). Here we have \( l = a \) (that is, \( \psi = 0 \)) and \( b_1 = f \). For this
case from (2.201) we obtain

\[ R_1 = \frac{(1+x_1)^3}{x_1} + f^2 T_1. \]

Minimization of \( R_1 \) with respect to \( x_1 \), yields

\[ x_{1c} = \frac{1}{2} \] (2.218)

(or)

\[ a_c = \frac{\pi}{\sqrt{2}} = \ell_c \]

and

\[ R_{1c} = f^2 T_1 \] (2.219)

The Case \( b_1 = 0 \)

It is the case of rolls with axes parallel to \( \hat{\omega}_2 \). Here we must have

\[ \ell = a g, \; k = -af, \; f = \sin \psi. \]

Then from (2.201) we can write

\[ R_1 = \frac{(1+x_1)^5 + 2f^2 M_1 x_1 (1+x_1)^3 + f^4 M_1^4 x_1^2 (1+x_1) + m^2 f^2 x_1 (1+x_1)^4}{x_1 (1+x_1)^2 + f^2 M_1^2 x_1^2 + f^2 m^2 x_1^2 (1+x_1)} \]

\[ \cdots \] (2.220)

Since we want to find the minimum \( R_1 \),

\[ \frac{\partial R_1}{\partial x_1} = 0, \]
which gives

\[(1+x_1)^7-(3-2f^2m^2)x_1(1+x_1)^6-(6f^2m^2-f^4m^4)x_1^2(1+x_1)^5
+2f^2M_1^2x_1(1+x_1)^5-\left\{7f^2M_1^2-3f^4m^2M_1^2\right\}x_1^2(1+x_1)^4
-3f^4m^4x_1^3(1+x_1)^4-\left\{6f^4M_1^4m^2+2f^4m^2M_1^2\right\}x_1^3(1+x_1)^3
+f^4M_1^4x_1^2(1+x_1)^3-5f^4M_1^4x_1^3(1+x_1)^2-f^6M_1^6x_1^4=0\]  \hspace{1cm} (2.221)

This is a seventh degree equation in \(x_1\) and in order to determine the roots, first let us determine the smaller roots by assuming \(x_1\) to be small, that is, \(1+x_1 \approx 1\). Then (2.221) approximate to

\[f^6M_1^6x_1^4+\left\{(6m^2+5)f^4M_1^4+2f^4m^2M_1^2+3f^4m^4\right\}x_1^3
+(6f^2m^2-f^4m^4+7f^2M_1^2-3f^4m^2M_1^2-f^4M_1^4)x_1^2
+(3-2f^2m^2-2f^2M_1^2)x_1-1=0\]  \hspace{1cm} (2.222)

For large \(M_1\), we get a valid positive root given by

\[x_{lc} = \frac{1}{fM_1}\]

(or) \[a_c = \pi\left(\frac{1}{fM_1}\right)^{1/2}, \hspace{0.5cm} \ell_c = \pi\left(\frac{2\pi}{fM}\right)^{1/2}, \hspace{0.5cm} k_c = -\pi\left(\frac{f\pi}{M}\right)^{1/2}\].

There is no effect of Hall current in the leading \(x_1\) and therefore we proceed to find the second order terms by taking
Using (2.223) in (2.222) after solving we obtain

\[ x_1' = -\frac{3}{2} \left(1+2m^2\right) \frac{1}{2M_1^2} \]

To second order

\[ x_{1c} = \frac{1}{fM_1} - \frac{3}{2} \left(1+2m^2\right) \frac{1}{2f^2M_1^2} \]

Substituting for \( x_1 \) in (2.220), we obtain the following value of \( R_{lc} \) to second order

\[ R_{lc} = f^2M_1^2+2fM_1. \]

Therefore by assuming \( x_1 \) small, for large \( M_1 \), the critical wave number is affected by the presence of Hall current in its second order terms.

Assuming \( x_1 \) large, we have \( 1+x_1 \approx x_1 \). Then (2.221) yields the following cubic equation for \( x_1 \), namely

\[ 2(f^2m^2+1)^2x_1^3+(6f^4M_1^4m^2-f^4M_1^2m^2+5f^2M_1^2)x_1^2+4f^4M_1^4x_1+f^6M_1^6 = 0. \]

For large \( M_1 \), the signs of all the terms are positive and as such there is no positive root of the equation (2.224).

This shows that the critical wave number does not become large with increasing \( M_1 \). Thus, for large \( M_1 \) and \( x_1=O(1) \),
we shall adopt the following method. That is first eliminate $T$ between (2.203) and (2.204). From (2.203) we get

$$\frac{1}{T} = \frac{b_1^2 T (1-x_1^2-m_2 x_1^2 \sin^2 \psi)-x_1^2 \sin^4 \psi}{2mb_1 x_1^2 \sin^3 \psi}, \text{ (b}_1 \neq 0) \quad (2.225)$$

Using (2.225) in (2.204) and after simplifications we obtain

$$T = -\frac{fx_1^2 \sin^4 \psi}{b_1^2 [(1+x_1) \{ f \cos 2\psi-fx_1 (1+2 \cos^2 \psi)+(1-x_1) g \sin 2\psi \}]}$$

$$-m^2 fx_1^2 \sin^2 \psi]$$

$$\cdots \quad (2.226)$$

Use this value of $T$ in (2.225) to get

$$T^{1/2} = \frac{\sin \psi [fx_1 (1+x_1)+f(\cos^2 \psi+2-m^2 fx_1^2 \sin^2 \psi-(1-x_1^2) g \sin \psi \cos \psi]}{mb_1 [(1+x_1) \{ f \cos 2\psi-fx_1 (1+2 \cos^2 \psi)+(1-x_1) g \sin 2\psi \}]}$$

$$-m^2 fx_1^2 \sin^2 \psi]$$

$$\cdots \quad (2.227)$$

Eliminating $T$ between (2.226) and (2.227) finally we obtain

$$[f \{ x_1 (1+\cos^2 \psi)-\cos^2 \psi \}-(1-x_1) g \sin \psi \cos \psi]^2$$

$$= m^2 fx_1^2 \frac{1-x_1}{1+x_1} \sin^2 \psi \quad (2.228)$$

Then from (2.202), we get after substituting for $T$ and $T^{1/2}$
For continuity of \( R_1 \) we must have the above R.H.S equal to \( f^2 \). This gives

\[
\frac{R_1}{M_1^2} = - \frac{f(1+x_1)^2 \sin^2 \psi}{(1+x_1)(f \{ \cos 2\psi - x_1(1+2 \cos^2 \psi) \} + (1-x_1)g \sin 2\psi) - m^2 f x_1^2 \sin^2 \psi}
\]

... (2.229)

When \( \phi = 90^\circ \) (\( f = 1 \), \( g = 0 \)), we get from (2.228) and (2.230)

\[
\{ x_1(1+\cos^2 \psi) - \cos^2 \psi \}^2 = m^2 x_1^2 \frac{1-x_1}{1+x_1} \sin^2 \psi
\]

... (2.231)

\[
\cos^2 \psi = \frac{m^2 x_1^2}{(m^2-3)x_1^2 - 2x_1 + 1}
\]

Eliminating \( \Psi \) between (2.231) and (2.232) we get the following equation in \( x_1 \)

\[
(m^4 - 3m^2 + 9)x_1^4 + (12-3m^2)x_1^3 - 2x_1^2 + 4(m^2-1)x_1 + (1-m^2) = 0
\]

... (2.233)

By giving different values for the Hall parameter \( m \), in (2.233) we can find the corresponding value of \( x_1 \) for a given \( m \). Once \( x_1 \) and \( m \) are known we can find \( \Psi \).
from (2.232) and then \( \tau \) can be found using (2.226). The matching values of the modes for various \( m \) are given in Table 7.

Let \( m = KM_1 \) and \( K = K_0M_1^{\beta-1} \) so that \( m = K_0M_1^\beta \). Then from (2.201) for large \( M_1 \), we have the expression for \( \frac{R_1}{M_1^2} \), to the leading order,

\[
\frac{R_1}{M_1^2} = \frac{x_1^2(1+x_1) \sin^2 \psi \{ KM_1 \frac{1}{2} b_1 + \sin \psi \}^2 + \tau b_1^2 x_1(1+x_1)^2}{x_1^2 \{ 1 + K^2(1+x_1) \} \sin^2 \psi}.
\]

(2.234)

Since \( \frac{R_1}{M_1^2} \) can be treated as a function of \( x_1 \) and \( \sin \psi \) minimization with respect to them yield

\[
\frac{\partial}{\partial x_1} \left( \frac{R_1}{M_1^2} \right) = 0, \quad \frac{\partial}{\partial \sin \psi} \left( \frac{R_1}{M_1^2} \right) = 0.
\]

The above equations are given respectively by

\[
x_1^2(KM_1 \frac{1}{2} b_1 + \sin \psi)^2 \sin^2 \psi - \tau b_1^2(1-x_1^2) - \tau b_1^2 K^2(1+x_1)^2 = 0
\]

(2.235)

and

\[
\sin \psi [x_1 \sin^3 \psi (KM_1 \frac{1}{2} b_1 + \sin \psi) \{ KM_1 \frac{1}{2} (g-f \tan \psi) + 1 \} + \tau b_1(1+x_1) \sin \psi (g-f \tan \psi) - b_1^2 \tau (1+x_1)] = 0
\]

(2.236)
Assuming $K_1 \ll 1$ and $\sin \psi \neq 0$, from (2.236) we get

$$\sin^4 \psi = \frac{\tau b_1^2(1+x_1)}{\chi_1}.$$  \hspace{1cm} (2.237)

Using the expression for $\sin^4 \psi$ in (2.235) we obtain after using the approximation

$$(2-K^2)x_1 - (1+K^2) = 0$$  \hspace{1cm} (2.238)

(or)

$$x_{1c} = \frac{1+K^2}{2-K^2}.$$  \hspace{1cm} (or)

Taking $\psi$ small, we have $b_1 = f = \sin \phi$. Along with this we use (2.238) in (2.237) to yield

$$\sin \psi_c = \frac{-3f^2}{\chi_1} 1/4.$$  \hspace{1cm} (2.239)

(The negative sign is taken to get the minimum $R$)

The critical Rayleigh number in this case is given by

$$R_{1c} = \frac{2\sqrt{\tau}f(1+x_1)^{3/2}}{\sqrt{x_1} \left(1+K^2(1+x_1)\right)} M_1^2.$$  \hspace{1cm} (2.240)

Using (2.238), we obtain

$$1+x_1 = \frac{3}{2-K^2} ,$$

$$1+K^2(1+x_1) = \frac{2(1+K^2)}{2-K^2}.$$  \hspace{1cm} (2.241)
When we substitute from (2.241) in (2.240) we get

$$R_{1c} = \frac{3f \sqrt{3\tau}}{(1+K^2)^{3/2}} M_1^2. \quad (2.242)$$

The above results are true, in general for any $\beta$ and we give the corresponding values of $x_{1c}$, $\sin \Psi_c$ and $R_{1c}$ from (2.238), (2.239) and (2.242) for various $\beta$ as follows.

**Case (i):** $0 < \beta < 1$.

In this case $K \ll 1$ and hence we obtain $x_{1c} \simeq \frac{1}{2}$, $\sin \Psi_c = -(3f^2 \tau)^{1/4}$, $R_{1c} = 3f(3\tau)^{1/2} M_1^2$. The above results are identical to (2.207), (2.208) and (2.209).

**Case (ii):** $\beta = 1$.

This implies that $K = K_0$ and so we get

$$x_{1c} = \frac{1+K_0^2}{2-K_0^2},$$

$$\sin \Psi_c = -(\frac{3f^2 \tau}{1+K_0^2})^{1/4}, \quad (2.243)$$

$$R_{1c} = \frac{3f \sqrt{3\tau}}{(1+K_0^2)^{3/2}} M_1^2.$$

We note that the expression for $x_{1c}$ is valid if $K_0^2 < 2$. 54815
For $K^2 > 2$, we have to consider the case corresponding to large $K$ discussed in the following section.

**Case (iii): $\beta > 1$.**

For this range of values of $\beta, K$ is much greater than unity and hence we rewrite (2.235) and (2.236) for large $K$. They are

\[
M_1^2 x_1^2 \sin^2 \psi - (1+x_1)^2 = 0 \tag{2.244}
\]

\[
K^2 M_1^2 g x_1 \sin^3 \psi - b_1 (1+x_1) = 0 \tag{2.245}
\]

From (2.244) we obtain

\[
\sin^2 \psi = \frac{(1+x_1)^2}{M_1^2 x_1^2} \tag{2.246}
\]

This implies that for large $M_1, \psi$ is small and hence we can take $b_1 = f = \sin \phi$.

Eliminating $\psi$ between (2.245) and (2.246) we get

\[
\frac{(1+x_1)^2}{x_1^2} = \frac{fM_1}{K^2 g}.
\]

This yields

\[
x_{ic} = \frac{K_o}{\sqrt{(f/g) \cdot (3/2)-\beta}} - K_o
\]


For $1 < \beta \leq 3/2$, \( x_{1c} = K_0 \sqrt{(g/f)} M_1^{\beta-(3/2)} \quad (f \neq 0) \)

Further \( \sin \psi_c = \frac{f}{gK^2M_1} \),

\[ R_{1c} = M_1^2 T f^2. \]

For \( \beta \geq 3/2 \), the value of \( x_{1c} \) will be negative and hence we have to consider the case of \( \psi = 0 \) (or) \( k = 0 \). This is exactly the case of rolls with axes parallel to \( \hat{S}_o \) for which \( x_{1c} \) and \( R_{1c} \) are given by (2.218) and (2.219).

When \( M_1^{1/2} = O(1) \) and \( \sin \psi \neq 0 \), the equation (2.236) reduces to for \( \psi \) small,

\[ K M_1^{1/2} x_1 \sin^4 \psi (K M_1^{1/2} g+1) - fT(1+x_1) \sin \psi = 0, \]

that is,

\[ \sin^3 \psi = \frac{fT(1+x_1)}{K K_1 M_1^{1/2} x_1} \quad (2.247) \]

where \( K_1 = K M_1^{1/2} g+1 \).

Similarly from (2.235) we obtain

\[ K^2 M_1^2 T f^2 x_1^2 \sin^2 \psi = T f^2 (1-x_1^2) + T f^2 k^2 (1+x_1)^2 \]

(or) \[ \sin^2 \psi = \frac{(1+x_1)[(1-x_1)+k^2(1+x_1)]}{K^2 M_1^2 x_1^2} \quad (2.248) \]
Eliminating $\psi$ between (2.247) and (2.248) we get

$$f^2 \kappa^4 \tau^4 \xi^4 = \tau K_1^2 (1+\xi_1)^{3} \left[ (1+K^2) + (K^2-1)\xi_1 \right]$$  \hspace{1cm} (2.249)

where $\tau_0 = \frac{1}{2}$, $M_1 = O(1)$.

For $\xi_1$ small, the relevant root of (2.249) is given by

$$f^2 \kappa^4 \tau_0 \xi_1^4 = \tau K_1^2 (1+K^2)^3$$

(or)

$$\xi_1 = \frac{\sqrt{K_1} (1+K^2)}{\sqrt{f \kappa \tau_0}} \tau^{1/4} \left( f \neq 0 \right)$$  \hspace{1cm} (2.250)

When we substitute for $\xi_1$ using (2.250) in (2.247), we get

$$\sin \psi = \frac{\sqrt{f}}{\sqrt{K_1} (1+K^2)} \frac{1}{\tau^{1/4}}$$  \hspace{1cm} (2.251)

Equations (2.250) and (2.251) lead to the following value of the critical Rayleigh number from (2.234)

$$R_{1c} = \frac{K_1^2 \tau_0 f^2}{1+K^2} \frac{M_1^2}{\kappa^2}$$  \hspace{1cm} (2.252)

We give below the corresponding values of $\xi_{1c}$, $\sin \psi_c$ and $R_{1c}$ for different $\beta$.

**Case (i):** When $0 < \beta < 1$, $K$ is less than unity and so we have
\[ x_{1c} = \frac{\tau^{1/4}}{\sqrt{f} K \tau_0}, \quad (f \neq 0) \]

\[ \sin \psi = \sqrt{f} \frac{\tau^{1/4}}{T}, \]

\[ R_{1c} = K^2 \tau_0 f^2 M_1^2. \]

**Case (ii):** When \( \beta = 1, K = K_0 = 0(1) \) and hence

\[ x_1 = \frac{(K_0 \tau_0 g + 1)^{1/2} (1 + K_0^2)^{3/4}}{\sqrt{f} K_0 \tau_0}, \quad f \neq 0 \]

\[ \sin \psi = \frac{\sqrt{f} \tau^{1/4}}{(K_0 \tau_0 g + 1)^{1/2} (1 + K_0^2)^{1/4}}, \]

\[ R_{1c} = \frac{K_0^2 \tau_0 f^2}{1 + K_0^2} M_1^2. \]

**Case (iii):** When \( \beta > 1, K \) is much greater than unity and hence we obtain

\[ x_1 = \frac{Kg^{1/2}}{\sqrt{f} M_1}, \quad (f \neq 0) \]

\[ \sin \psi = \frac{\sqrt{f}}{K V M_1 g}, \quad (g \neq 0) \]

\[ R_{1c} = \tau_0 f^2 M_1^2. \]
When $\hat{B}_o$ and $\hat{\lambda}$ are parallel

In this case $\phi = 0$, so we have $f = 0$, $g = 1$ and $b_1 = \sin \psi$. Then from (2.234) we have,

$$R_{\frac{1}{2}} \frac{1}{M_1^2} = \frac{(K M_1^2 \tau + 1)^2 (1 + x_1)}{\{1 + k^2 (1 + x_1)\}^2} \sin^2 \psi + \frac{\tau(1 + x_1)^2}{x_1 \{1 + k^2 (1 + x_1)\}}$$

(2.253)

As usual minimizing $R_{\frac{1}{2}} \frac{1}{M_1^2}$ with respect to $x_1$ and $\sin \psi$ yields

$$x_1^2(K M_1^2 \tau + 1)^2 \sin^2 \psi - \tau[(1 - x_1^2) + k^2 (1 + x_1)^2] = 0$$

(2.254)

$$x_1^2(K M_1^2 \tau + 1)^2 (1 + x_1) \sin \psi = 0$$

(2.255)

From (2.255) we get

$$\psi_c = 0$$

Substituting $\psi = 0$ in (2.254) we get the following value of $x_1$

$$x_{1c} = \frac{1 + k^2}{1 - k^2}, \ (k^2 < 1)$$

(2.256)

Then we can write the value of $R_{\frac{1}{2}} \frac{1}{M_1^2}$ from (2.253) as

$$R_{\frac{1}{2}} \frac{1}{M_1^2} = \frac{\tau(1 + x_1)^2}{x_1 \{1 + k^2 (1 + x_1)\}}$$
or
\[ R_{1c} = \frac{4 \mathcal{T}}{(1+K^2)^2} M_1^2 \]

For various ranges of \( \beta \), the values of \( x_{1c} \) and \( R_{1c} \) are as follows.

When \( 0 < \beta < 1 \), (that is, when \( K \) is less than unity)

\[ x_{1c} = 1, \quad R_{1c} = 4\mathcal{T}M_1^2 \]

When \( \beta = 1 \), (that is, when \( K = K_0 = O(1) \))

\[ x_{1c} = \frac{1+K_0^2}{1-K_0^2}, \quad (K_0^2 < 1) \]

\[ R_{1c} = \frac{4\mathcal{T}}{(1+K_0^2)^2} M_1^2. \]

The expression for \( x_{1c} \) is valid only when \( K_0^2 < 1 \). If \( K_0^2 \geq 1 \) we must take large values of \( K \) as mentioned in the succeeding case.

When \( \beta > 1 \), \( K \) becomes large and this yields negative value for \( x_1 \). Hence for large \( K \), we take \( \Psi = 0 \) that is \( k = 0 \), which corresponds to the case discussed earlier with critical values given in (2.218) and (2.219).

When \( \hat{\mathbf{B}}_0 \) and \( \mathbf{\Omega} \) are at right angles

Here \( \phi = 90^\circ, f = 1, g = 0 \) and \( b_1 = \cos \Psi \).
Using the above values in (2.234) we get

\[
\frac{R_1}{M_1^2} = (KM_1^1 \frac{1}{2} \cos \psi + \sin \psi)^2 \frac{1+x_1}{1+K^2(1+x_1)} + \frac{(1+x_1)^2}{x_1 \{1+K^2(1+x_1)\}}
\]

Minimizing \( \frac{R_1}{M_1^2} \) with respect to \( x_1 \) and \( \sin \psi \) we obtain

\[
K^2M_1^2x_1 \sin^4 \psi + (1+x_1) = 0 \quad (2.257)
\]

\[
K^2M_1^2x_1^2 \sin^2 \psi = (1-x_1^2)+K^2(1+x_1)^2
\]

Equation (2.257) does not give rise to a positive root and hence we have the other possibility \( \psi = 0 \). This implies that \( k = 0 \) and for which we have already obtained the values of \( x_{1c} \) and \( R_{1c} \) given in (2.218) and (2.219).

2.7 Conclusion

For free insulating boundaries, when the rotation parameter \( T \) lies in the range \( O(M^{8/3}) \leq T \leq O(M^6) \), the presence of Hall current is to increase the Rayleigh number and critical wave number. On the other hand, for other types of boundaries the critical values are increased when \( T \) lies in the range \( O(M^4) \leq T \leq O(M^6) \) by the effect of Hall current. Further it is found that as the Hall parameter \( m \) increases both the critical wave number and critical Rayleigh number increases.
When \( m = O(M^S), \ (s > 0) \) and \( T = O(M^4+\beta) \), in the case of free insulating boundaries, the effect of Hall current is found in the leading order for the magnetic field dominant case, that is, in the region determined by \(-4 < \beta < -1, \ 1/3 < s < 2 \) and \( 2s + \beta < 0 \). In the other regions there is no effect of Hall current on the critical mode. In particular, for free insulating boundaries and free perfectly conducting boundaries when \( m = O(M) \), the effect of Hall current is found to the leading order for values of \( T \leq O(M^2) \). When \( T > O(M^2) \), however there is no effect of Hall current on the critical mode.

The discussion is confined to free perfectly conducting boundaries in Model II (in which the magnetic field is horizontal and angular velocity is vertical). For \( \ell = 0 \) the Hall current is found to affect the critical mode in the leading order for values of \( T \) in the range \( O(M^4) \leq T \leq O(M^6) \). For \( T \geq O(M^6) \), the preferred mode is \( \ell = 0 \) and we find the effect of Hall current in the second order terms of \( k_c \) and \( R_c \).

For Model III (magnetic field and angular velocity horizontal) the results obtained show that if \( \phi \neq 0 \) and \( T \ll 1 \), the critical wave number, the angle made by the roll with the magnetic field and the critical Rayleigh number are all affected by Hall current in their second
order terms. In the case of rolls with axes parallel to the direction of angular velocity \( \omega \), for large magnetic field the critical wave number \( x_{\perp c} \) is small and of order \( M_1^{-1} \) while the effect of Hall current is found to decrease its value by a quantity proportional to \( M_1^{-2} \). For large \( M_1 \) and \( x_1 = O(1) \) the analysis carried out shows that the values of \( x_{\perp c}, \Psi \) and \( T \) depend on the Hall parameter \( m \) and we have obtained their values for \( \phi = 90^\circ \) and for various values of \( m \) as given in Table 7.

When \( m = K_o M_1^{\beta} \), that is in the case of large \( m \), discussions carried out for various cases and for different range of values of \( T \), in general shows that Hall current affect the values of \( x_{\perp c}, \Psi_c \) and \( R_c \). When the directions of the magnetic field and angular velocity are parallel for \( \beta > 1 \) the results are identical to the case of rolls with axes parallel to \( \vec{B}_0 \) (when \( k = 0 \)). Again when the directions of the magnetic field and angular velocity are perpendicular, the minimization of \( R_1/M_1^2 \) does not yield a positive root and the other possibility \( \Psi = 0 \) (or \( k = 0 \)) does not lead to any modified values of the critical wave number and Rayleigh number.
**Free Insulating Boundaries**

<table>
<thead>
<tr>
<th>Case</th>
<th>Range of $T$</th>
<th>$a_c$</th>
<th>$R_c$</th>
</tr>
</thead>
<tbody>
<tr>
<td>(i)</td>
<td>$\beta \leq -\frac{4}{3}$</td>
<td>$T \leq K(\pi^4M^8)^{1/3}$</td>
<td>$(\frac{1}{2} \pi^4M^2)^{1/6}$</td>
</tr>
<tr>
<td>(ii)</td>
<td>$-\frac{4}{3} \leq \beta \leq 2$</td>
<td>$K(\pi^4M^8)^{1/3} \leq T \leq \frac{27\pi^2}{4(1+\sqrt{1+m^2})} M^6$</td>
<td>$O(1)$</td>
</tr>
<tr>
<td>(iii)</td>
<td>$\beta \geq 2$</td>
<td>$T \geq \frac{27\pi^2}{4(1+\sqrt{1+m^2})} M^6$</td>
<td>$(\frac{1}{2} \pi^2T)^{1/6}$</td>
</tr>
</tbody>
</table>

Table showing the critical values of Rayleigh number and the ranges of $T$ when $m = O(1)$

**TABLE 1**
### Free Insulating Boundaries

<table>
<thead>
<tr>
<th>Case</th>
<th>Range of T</th>
<th>$a_c$</th>
<th>$R_c$</th>
</tr>
</thead>
<tbody>
<tr>
<td>(i) $\beta &gt; -2$</td>
<td>$T &lt; \frac{m_0^2 \pi^2 M^2}{m_0^2}$</td>
<td>$\left(\frac{1}{2m_0^2} \pi^4 M^2\right)^{1/6}$</td>
<td>$3\left(\frac{1}{2m_0^2} \pi^4 M^2\right)^{2/3}$</td>
</tr>
<tr>
<td>(ii) $\beta = -2$</td>
<td>$T = \frac{(m_0 \sqrt{K+1})^2}{m_0^2} \pi^2 M^2$</td>
<td>$\left(\frac{m_0 \sqrt{K+1}}{2m_0^2} \pi^4 M^2\right)^{1/6}$</td>
<td>$3\left[\frac{(m_0 \sqrt{K+1})^2}{2m_0^2} \pi^4 M^2\right]^{2/3}$</td>
</tr>
<tr>
<td>(iii) $\beta &gt; -2$</td>
<td>$T &gt; \frac{(m_0 \sqrt{K+1})^2}{m_0^2} \pi^2 M^2$</td>
<td>$\left(\frac{1}{2} \pi^2 T\right)^{1/6}$</td>
<td>$3\left(\frac{1}{2} \pi^2 T\right)^{2/3}$</td>
</tr>
</tbody>
</table>

Table showing the critical values of Rayleigh number and the ranges of T when $m = 0(M)$

**TABLE 2**
<table>
<thead>
<tr>
<th>Case</th>
<th>Range of T</th>
<th>m</th>
<th>ac</th>
<th>Rc</th>
</tr>
</thead>
<tbody>
<tr>
<td>(i)</td>
<td>$T \leq \frac{9}{8(m+1)^2} \left(\frac{1}{2} \pi^4 M^8\right)^{1/3}$</td>
<td>-</td>
<td>$\left(\frac{1}{2} \pi^4 M^8\right)^{1/6}$</td>
<td>$\pi^2 M^2$</td>
</tr>
<tr>
<td>(ii)</td>
<td>$0 \leq \beta \leq 0$</td>
<td>$\frac{9}{8} \left(\frac{1}{2} \pi^4 M^8\right)^{1/3} \leq T \leq .091 M^4$</td>
<td>0</td>
<td>$\pi (\frac{M^4}{T})^{1/4}$</td>
</tr>
<tr>
<td></td>
<td>$\frac{1}{2} \left(\frac{1}{2} \pi^4 M^8\right)^{1/3} \leq T \leq .079 M^4$</td>
<td>.5</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>$\frac{9}{32} \left(\frac{1}{2} \pi^4 M^8\right)^{1/3} \leq T \leq .056 M^4$</td>
<td>1</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>$\frac{9}{128} \left(\frac{1}{2} \pi^4 M^8\right)^{1/3} \leq T \leq .016 M^4$</td>
<td>3</td>
<td></td>
<td></td>
</tr>
<tr>
<td>(iib)</td>
<td>$0 \leq \beta \leq 2$</td>
<td>$.091 M^4 \leq T \leq .00052 M^6$</td>
<td>0</td>
<td>4.5</td>
</tr>
<tr>
<td></td>
<td>$.079 M^4 \leq T \leq .00033 M^6$</td>
<td>.5</td>
<td>4.9</td>
<td>125.7 TM$^{-2}$</td>
</tr>
<tr>
<td></td>
<td>$.056 M^4 \leq T \leq .00012 M^6$</td>
<td>1</td>
<td>5.5</td>
<td>174.9 TM$^{-2}$</td>
</tr>
<tr>
<td></td>
<td>$.016 M^4 \leq T \leq .00003 M^6$</td>
<td>3</td>
<td>8.9</td>
<td>612.6 TM$^{-2}$</td>
</tr>
<tr>
<td>(iii)</td>
<td>$\beta \geq 2$</td>
<td>$T \geq .00052 M^6$</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$T \geq .00033 M^6$</td>
<td>.5</td>
<td>$\left(\frac{1}{2} \pi^2 T\right)^{1/6}$</td>
<td>$3\left(\frac{1}{2} \pi^2 T\right)^{2/3}$</td>
</tr>
<tr>
<td></td>
<td>$T \geq .00012 M^6$</td>
<td>1</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>$T \geq .00003 M^6$</td>
<td>3</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table showing the critical values of Rayleigh number and the ranges of T when $m = O(1)$

**TABLE 3**
Free Perfectly Conducting Boundaries

<table>
<thead>
<tr>
<th>Case</th>
<th>Range of $T$</th>
<th>$a_c$</th>
<th>$R_c$</th>
</tr>
</thead>
<tbody>
<tr>
<td>(i) $\beta &lt; -2$</td>
<td>$T &lt; \frac{m_o}{m^2} - \frac{2}{M}$</td>
<td>$\left(\frac{\pi^2}{2m_o} M^2\right)^{1/6}$</td>
<td>$3\left(\frac{\pi^2}{2m_o} M^2\right)^{2/3}$</td>
</tr>
<tr>
<td>(ii) $\beta = -2$</td>
<td>$T = \frac{\left(\frac{m_o\tau_o}{m^2} + 1\right)^{1/2}}{M^2}$</td>
<td>$\left(\frac{\pi^2}{2m_o} \frac{1}{M^2}\right)^{1/6}$</td>
<td>$3\left[\frac{\left(\frac{m_o\tau_o}{m^2} + 1\right)^{1/2}}{2m_o} \frac{1}{\pi^2 M^2}\right]^{2/3}$</td>
</tr>
<tr>
<td>(iii) $\beta &gt; -2$</td>
<td>$T &gt; \frac{\left(\frac{m_o\tau_o}{m^2} + 1\right)^{1/2}}{M^2}$</td>
<td>$\left(\frac{1}{2} \pi^2 T\right)^{1/6}$</td>
<td>$3\left(\frac{1}{2} \pi^2 T\right)^{2/3}$</td>
</tr>
</tbody>
</table>

Table showing the critical values of Rayleigh number and the ranges of $T$ when $m = O(M)$

**TABLE 4**
Rigid Insulating Boundaries

<table>
<thead>
<tr>
<th>Case</th>
<th>Range of ( T )</th>
<th>( m )</th>
<th>( a_c )</th>
<th>( R_c )</th>
</tr>
</thead>
<tbody>
<tr>
<td>(i)</td>
<td>( \beta \leq -\frac{4}{3} )</td>
<td>( T \leq \frac{9}{8(m+1)} \left( \frac{1}{2} \pi^4 M^8 \right)^{1/3} )</td>
<td>-</td>
<td>( \left( \frac{1}{2} \pi^4 M^2 \right)^{1/6} )</td>
</tr>
<tr>
<td>(ii)</td>
<td>( -\frac{4}{3} \leq \beta \leq 0 )</td>
<td>( \frac{9}{8} \left( \frac{1}{2} \pi^4 M^8 \right)^{1/3} \leq T \leq 0.25 M^4 )</td>
<td>0</td>
<td>( \pi^4 \left( \frac{M^4}{T} \right)^{1/4} )</td>
</tr>
<tr>
<td></td>
<td>( \frac{1}{2} \left( \frac{1}{2} \pi^4 M^8 \right)^{1/3} \leq T \leq 0.225 M^4 )</td>
<td>.5</td>
<td>( \pi^4 \left( \frac{M^4}{T} \right)^{1/4} )</td>
<td>( \pi^2 M^2 )</td>
</tr>
<tr>
<td></td>
<td>( \frac{9}{32} \left( \frac{1}{2} \pi^4 M^8 \right)^{1/3} \leq T \leq 0.172 M^4 )</td>
<td>1</td>
<td>( \pi^4 \left( \frac{M^4}{T} \right)^{1/4} )</td>
<td>( \pi^2 M^2 )</td>
</tr>
<tr>
<td></td>
<td>( \frac{9}{128} \left( \frac{1}{2} \pi^4 M^8 \right)^{1/3} \leq T \leq 0.058 M^4 )</td>
<td>3</td>
<td>( \pi^4 \left( \frac{M^4}{T} \right)^{1/4} )</td>
<td>( \pi^2 M^2 )</td>
</tr>
<tr>
<td>(iib)</td>
<td>( 0 \leq \beta \leq 2 )</td>
<td>( .25 M^4 \leq T \leq .011 M^6 )</td>
<td>0</td>
<td>3.14</td>
</tr>
<tr>
<td></td>
<td>( .223 M^4 \leq T \leq .008 M^6 )</td>
<td>.5</td>
<td>3.32</td>
<td>44.28 ( TM^{-2} )</td>
</tr>
<tr>
<td></td>
<td>( .172 M^4 \leq T \leq .003 M^6 )</td>
<td>1</td>
<td>3.74</td>
<td>57.56 ( TM^{-2} )</td>
</tr>
<tr>
<td></td>
<td>( .058 M^4 \leq T \leq .00013 M^6 )</td>
<td>3</td>
<td>5.59</td>
<td>171.11 ( TM^{-2} )</td>
</tr>
<tr>
<td>(iii)</td>
<td>( \beta \geq 2 )</td>
<td>( T \geq .011 M^6 )</td>
<td>0</td>
<td>( \left( \frac{1}{2} \pi^2 T \right)^{1/6} )</td>
</tr>
<tr>
<td></td>
<td>( T \geq .008 M^6 )</td>
<td>.5</td>
<td>( \left( \frac{1}{2} \pi^2 T \right)^{1/6} )</td>
<td>( 3 \left( \frac{1}{2} \pi^2 T \right)^{2/3} )</td>
</tr>
<tr>
<td></td>
<td>( T \geq .003 M^6 )</td>
<td>1</td>
<td>( \left( \frac{1}{2} \pi^2 T \right)^{1/6} )</td>
<td>( 3 \left( \frac{1}{2} \pi^2 T \right)^{2/3} )</td>
</tr>
<tr>
<td></td>
<td>( T \geq .00013 M^6 )</td>
<td>3</td>
<td>( \left( \frac{1}{2} \pi^2 T \right)^{1/6} )</td>
<td>( 3 \left( \frac{1}{2} \pi^2 T \right)^{2/3} )</td>
</tr>
</tbody>
</table>

Table showing the critical values of Rayleigh number and the ranges of \( T \) when \( m = O(1) \)

**TABLE 5**
### Rigid Perfectly Conducting Boundaries

<table>
<thead>
<tr>
<th>Case</th>
<th>Range of T</th>
<th>m</th>
<th>$a_c$</th>
<th>$R_c$</th>
</tr>
</thead>
<tbody>
<tr>
<td>(i)</td>
<td>$T \leq \frac{9}{8(m+1)^2}(8\pi^4 M^8)^{1/3}$</td>
<td>-</td>
<td>$(8\pi^4 M^2)^{1/6}$</td>
<td>$4\pi^2 M^2$</td>
</tr>
<tr>
<td>(ii)</td>
<td>$\frac{9}{8}(8\pi^4 M^8)^{1/3} \leq T \leq .056 M^4$</td>
<td>0</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>$\frac{1}{2}(8\pi^4 M^8)^{1/3} \leq T \leq .051 M^4$</td>
<td>.5</td>
<td>$2\pi(M^4)^{1/4}$</td>
<td>$4\pi^2 M^2$</td>
</tr>
<tr>
<td></td>
<td>$\frac{9}{32}(8\pi^4 M^8)^{1/3} \leq T \leq .040 M^4$</td>
<td>1</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>$\frac{9}{128}(8\pi^4 M^8)^{1/3} \leq T \leq .015 M^4$</td>
<td>3</td>
<td></td>
<td></td>
</tr>
<tr>
<td>(iiib)</td>
<td>$0 \leq \beta \leq 2$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>$.056 M^4 \leq T \leq .00012 M^6$</td>
<td>0</td>
<td>7.2</td>
<td>175.7 $TM^{-2}$</td>
</tr>
<tr>
<td></td>
<td>$.051 M^4 \leq T \leq .00009 M^6$</td>
<td>.5</td>
<td>7.7</td>
<td>194.8 $TM^{-2}$</td>
</tr>
<tr>
<td></td>
<td>$.040 M^4 \leq T \leq .00004 M^6$</td>
<td>1</td>
<td>8.4</td>
<td>247.3 $TM^{-2}$</td>
</tr>
<tr>
<td></td>
<td>$.015 M^4 \leq T \leq .000002 M^6$</td>
<td>3</td>
<td>12.1</td>
<td>647.8 $TM^{-2}$</td>
</tr>
<tr>
<td>(iii)</td>
<td>$\beta \geq 2$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>$T \geq .00012 M^6$</td>
<td>0</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>$T \geq .00009 M^6$</td>
<td>.5</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>$T \geq .00004 M^6$</td>
<td>1</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>$T \geq .000002 M^6$</td>
<td>3</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table showing the critical values of Rayleigh number and the ranges of T when $m = O(1)$

**TABLE 6**
<table>
<thead>
<tr>
<th>m</th>
<th>$\Psi_{\psi=90^\circ}$ (in degrees)</th>
<th>$x_{lc}$</th>
<th>$\phi=90^\circ$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>45.00</td>
<td>.3333</td>
<td>.0625</td>
</tr>
<tr>
<td>1/4</td>
<td>49.44</td>
<td>.3310</td>
<td>.0844</td>
</tr>
<tr>
<td>1/2</td>
<td>39.28</td>
<td>.3288</td>
<td>.0412</td>
</tr>
<tr>
<td>1</td>
<td>35.20</td>
<td>.3204</td>
<td>.0292</td>
</tr>
<tr>
<td>2</td>
<td>30.00</td>
<td>.3022</td>
<td>.0179</td>
</tr>
</tbody>
</table>

MODEL III The Matching Values of the Modes

TABLE 7