CHAPTER IV

THE HYDROMAGNETIC VISCOUS FINITELY CONDUCTING BAROCLINIC INSTABILITY OF A ZONAL SHEAR FLOW WITH OR WITHOUT HALL CURRENT
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4.1 Introduction

The phenomenon and theory of baroclinic instability occupy a central position in meteorological fluid mechanics. The pioneering work of Charney (1947) and Eady (1949) demonstrated that the large-scale cyclone waves in the atmosphere could be explained in terms of the instability of a baroclinic zonal current to infinitesimal wave disturbances. Observations of the atmospheres of major planets of the solar system reveal a marked preference for the type of symmetry displayed by the symmetric instability (which occurs when the Richardson number $R_i$ is less than unity) over the conventional baroclinic instability. These symmetric instabilities play an equally important role in the dynamics of these atmospheres. Such a role is possible only if there is some range of conditions under which the symmetric instabilities have the largest growth rates when $0.25 < R_i < 0.95$ [Eliassen and Kleinschmidt (1957); Charney (1973)], so that they will dominate the other types of instabilities that may occur.
In an attempt to determine if there is some range of conditions under which the symmetric instabilities will dominate Stone (1966) extended Eady's model to include moderate values of $R_i$. This model neglects dissipative and curvature effects and uses the Boussinesq approximation for compressibility effects. This model is too simplified one to apply directly to the atmospheres of major planets. Stone (1967) suggested a model in which the motions in the Jovian atmosphere are caused by differential heating. He gave the interpretation of cloud bands as baroclinic instabilities in light of the theoretical model first studied by Eady (1949) and elaborated by Stone (1966). Gierasch and Stone (1968) have developed the attractive hypothesis that symmetric instabilities of a zonal shear flow may account for some features of Jupiter's atmosphere, chiefly its symmetric cloud bands and its equatorial jet. In order to maintain the zonal momentum of the equatorial jet, Stone's interesting results showed that the symmetric instabilities provide a mechanism for this provided that $1/4 < R_i < 1/3$. On the other hand, Hide (1970) argues from physical considerations that this cannot be possible whatever be the value of $R_i$. 
The first salient feature of Stone's analysis which merit further attention is the fact that for an inviscid fluid the critical wavelength for the onset of instability is zero. In other words a new length scale is required in this neighbourhood. This is obtained by introducing the viscosity $\gamma$ and conductivity $\kappa$ and combining appropriate powers of $(\gamma/f)^{1/2}$ and $H$, where $(1/2)f$ is the angular velocity of the system and $H$ is the depth of the atmosphere. In a similar problem McIntyre (1970) set the boundaries at infinity and used the only possible length scale, $(\gamma/f)^{1/2}$. He showed that in the absence of horizontal shear in the basic flow, instability sets in when $R_{ic} = \frac{(1+\sigma)^2}{4\sigma} > 1$ (when $\sigma \neq 1$), where $\sigma$ is the Prandtl number. The destabilization of certain classically stable modes by the introduction of viscosity and thermal conductivity is an interesting feature of the solution [Acheson and Hide (1973)], but at this stage the most important feature is that yet another length scale is required to discuss the onset of instability. Walton (1975) showed that the appropriate length scale for marginal instability is $O((\frac{\gamma}{H^2f})^{1/3})$ and further instability sets in at a finite wave length when the Richardson number given by McIntyre is less by a term of $O((\frac{\gamma}{H^2f})^{2/3})$.

The second important feature is that a linear stability
analysis was considered by Stone to the basic zonal flow
when $\Delta = R_{ic} - R_{i}$ is no longer small. These values of
$R_{i}$ leads to a complicated mathematical problem in which
a continuous spectrum of modes are simultaneously unstable.
In fact the basic zonal flow is destroyed within a few
rotations of the planet. In other words instability is
set up as $R_{i}$ decreases from a stable value through the
critical value. The flow after the onset of instability
is likely to be non-linear and the linear theory is likely
to be inadequate. Walton further considered nonlinear
analysis similar to that of Stuart (1958, 1960) about the
point of onset of monotonic instability for $\Delta \ll 1$, to
find out whether the flow evolves from the unstable form
of a uniform zonal shear to a new stable form or becomes
catastrophically unstable and turbulent. He found out
that the amplitude of the disturbance tends to a constant
value thus implying that there is an exchange of stabili­
ties at $\Delta = 0$ and further for $\Delta > 0$ there exists a
stable solution of the problem when the meridional velo­
city $v$, is non-zero and such that $v \to 0$ as $\Delta \to 0$.
His theory also provides some evidence that the flow is
stable to nonlinear as well as linear disturbances when
$R_{i}$ has supercritical values. In all, his model is an
acceptable approximation to the conditions of Jupiter then,
at least when the Richardson number is very near the critical value, symmetric instabilities do not convect zonal momentum equatorwards and therefore will not be able to support an equatorial jet.

A magnetic field is a normal accompaniment of any cosmic body that is both fluid (wholly or in partially) and rotating. It is the mere existence of these fields which provide the initial motivation for the various investigations. It follows that the determination of magnetic field strength and configuration at levels ranging from above the visible surface to well below the surface, together with appropriate measurements of electrical conductivity are investigations that can be carried out with the dynamics of the atmospheres of major planets. Possible complications arise because the electrical conductivity of the lower reaches of these atmosphere might be sufficiently large for magnetohydrodynamic processes to occur.

A comprehensive review of certain major observational features in terms of basic fluid dynamical processes is the heart of the subject of discussion for quite sometime. The interpretation in terms of the basic hydrodynamical processes of certain prominent observational features exhibited by the atmospheres of major planets lead to accurate
information about the internal structure of these bodies. The features are (i) the Great Red Spot and the many other, less persistent and generally smaller, spots on Jupiter's visible surface (ii) the banded appearance, and complicated and striking variation of rotation rate with latitude, especially the equatorial jets, of the visible surfaces of Jupiter and Saturn and (iii) Jupiter's magnetic field, with its characteristic period.

4.2 Author's Contribution

In the light of these above investigations we have studied the hydromagnetic viscous linear finitely conducting baroclinic instability of a zonal shear flow. As already pointed out earlier Hall effects may become important for certain situations and hence we have also included its effect for varying values of the parameter $m$.

4.3 The Linear Stability Problem

The basic model consists of a Boussinesq adiabatic electrically conducting fluid in the presence of a magnetic field contained between two horizontal planes at $z^* = 0, H$. The lower boundary is at rest but the upper one moves with a constant velocity in keeping with the velocity of the
zonal current. The co-ordinate system is the Cartesian system \((x^*, y^*, z^*)\), where \(x^*, y^*, z^*\) measure distances along the zonal, meridional and vertical directions respectively. The motion takes place on an \(f\)-plane rotating about a vertical axis with angular velocity \(\frac{1}{2} f\). Then the conservation equations are

\[
\frac{\partial u^*}{\partial t^*} + \frac{\partial v^*}{\partial y^*} + \frac{\partial w^*}{\partial z^*} = 0 \tag{4.1}
\]

\[
\frac{\partial u^*}{\partial t^*} = f v^* - \frac{1}{\rho^*} \frac{\partial p^*}{\partial x^*} + \gamma \nabla^2 u^* 
+ \frac{1}{\rho^*} \left[ B_z^* \left( \frac{\partial H_x^*}{\partial z^*} - \frac{\partial H_z^*}{\partial x^*} \right) - B_y^* \left( \frac{\partial H_y^*}{\partial x^*} - \frac{\partial H_x^*}{\partial y^*} \right) \right] \tag{4.2}
\]

\[
\frac{\partial v^*}{\partial t^*} = -f u^* - \frac{1}{\rho^*} \frac{\partial p^*}{\partial y^*} + \gamma \nabla^2 v^* 
+ \frac{1}{\rho^*} \left[ B_x^* \left( \frac{\partial H_y^*}{\partial x^*} - \frac{\partial H_x^*}{\partial y^*} \right) - B_z^* \left( \frac{\partial H_z^*}{\partial y^*} - \frac{\partial H_y^*}{\partial z^*} \right) \right] \tag{4.3}
\]

\[
\frac{\partial w^*}{\partial t^*} = -\frac{1}{\rho^*} \frac{\partial p^*}{\partial z^*} + \alpha g \theta^* 
+ \frac{1}{\rho^*} \left[ B_y^* \left( \frac{\partial H_z^*}{\partial y^*} - \frac{\partial H_y^*}{\partial z^*} \right) - B_x^* \left( \frac{\partial H_x^*}{\partial z^*} - \frac{\partial H_z^*}{\partial x^*} \right) \right] \tag{4.4}
\]
\[
\frac{d\theta^*}{dt^*} = \kappa \nabla^2 \theta^* \tag{4.5}
\]

The modified induction equation expressed in terms of its components are

\[
\frac{\partial H_x^*}{\partial t^*} = \eta \nabla_x^2 H_x^* + \frac{\partial}{\partial y^*} (u^* H_y^* - v^* H_x^*) - \frac{\partial}{\partial z^*} (w^* H_x^* - u^* H_z^*) - \frac{1}{en_e} \left[ \nabla_x^* \times \{(\nabla_x^* \times \vec{H}^*) \times \vec{H}^* \} \right]_x \tag{4.6}
\]

\[
\frac{\partial H_y^*}{\partial t^*} = \eta \nabla_y^2 H_y^* + \frac{\partial}{\partial z^*} (v^* H_z^* - w^* H_y^*) - \frac{\partial}{\partial x^*} (u^* H_y^* - v^* H_x^*) - \frac{1}{en_e} \left[ \nabla_y^* \times \{(\nabla_y^* \times \vec{H}^*) \times \vec{H}^* \} \right]_y \tag{4.7}
\]

\[
\frac{\partial H_z^*}{\partial t^*} = \eta \nabla_z^2 H_z^* + \frac{\partial}{\partial x^*} (w^* H_x^* - u^* H_z^*) - \frac{\partial}{\partial y^*} (v^* H_z^* - w^* H_y^*) - \frac{1}{en_e} \left[ \nabla_z^* \times \{(\nabla_z^* \times \vec{H}^*) \times \vec{H}^* \} \right]_z \tag{4.7a}
\]

\[
\frac{\partial H_x^*}{\partial x^*} + \frac{\partial H_y^*}{\partial y^*} + \frac{\partial H_z^*}{\partial z^*} = 0 \tag{4.8}
\]

where \((u^*, v^*, w^*)\) are the components of velocity and \((H_x^*, H_y^*, H_z^*)\) the components of magnetic field in the
(x*, y*, z*) co-ordinates, p* the pressure, ρ* the fluid density, θ* the temperature, ν the kinematic viscosity, κ the thermometric conductivity, η(= 1/(μσc)) the magnetic diffusivity and t* measures time.

The basic flow is assumed to consist of a zonal wind of magnitude U with constant vertical shear and a temperature field with constant vertical stratification (∂θ*/∂z*) related to U by the thermal wind equation. We assume that

$$\frac{\partial}{\partial z^*} >> \frac{\partial}{\partial x^*}, \frac{\partial}{\partial y^*} \quad \text{and} \quad w^* << u^*, v^* \quad (4.9)$$

The boundary conditions to be satisfied are

$$u^* = v^* = w^* = θ^* = 0 \quad \text{at} \quad z^* = 0, H \quad (4.10)$$

In addition to the above, normal component of the magnetic induction must be continuous. Tangential component of the magnetic field must be continuous in the case of insulating boundaries and normal component of the current must be continuous in the case of perfectly conducting boundaries.

We employ dimensionless units by writing
(x*, y*, z*) = [ (\frac{U}{1})x, (\frac{U}{1})y, Hz ]

(u*, v*, w*) = (Uu, Uv, fHw)

\[ \theta^* = H(\frac{\partial \phi}{\partial z^*} \partial t^* = f^{-1} t \]  \hspace{1cm} (4.11)

\[ p^* = \alpha \delta^{*} g H^2 (\frac{\partial \theta}{\partial z^*} p \]

(H*, H*, H*) = H*(H_x, H_y, H_z)

where \[ |H^*| = H_0. \]

**Horizontal Magnetic Field**

Take \[ (u^*, 0, 0) \] and \[ (H^*, H^*, 0) \] where

H* and H* are constants. So we can take

\[ B_x^* = \mu H_x^* = \mu H_0 \cos \phi \]
\[ B_y^* = \mu H_y^* = \mu H_0 \sin \phi \]

where \[ \phi \] is some constant.

Then the equations (4.1) to (4.8) after taking into account the effect of Hall current can be written in the following non-dimensional form

\[ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \]  \hspace{1cm} (4.12)
\[
\begin{align*}
\frac{du}{dt} &= v-R_i \frac{\partial p}{\partial x} + E \frac{\partial^2 u}{\partial z^2} - S^2 \sin \phi \left( \frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} \right) \quad (4.13) \\
\frac{dv}{dt} &= -u-R_i \frac{\partial p}{\partial y} + E \frac{\partial^2 v}{\partial z^2} + S^2 \cos \phi \left( \frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} \right) \quad (4.14) \\
\frac{\partial p}{\partial z} &= \theta - \frac{S^2}{R} \left( \sin \phi \frac{\partial H_y}{\partial z} + \cos \phi \frac{\partial H_x}{\partial z} \right) \quad (4.15) \\
\frac{d\Theta}{dt} &= \frac{E}{\sigma} \frac{\partial^2 \Theta}{\partial z^2} \quad (4.16) \\
\frac{\partial H_x}{\partial t} &= E \eta \frac{\partial^2 H_x}{\partial z^2} - z \frac{\partial H_x}{\partial x} + \sin \phi \frac{\partial u}{\partial x} - \cos \phi \frac{\partial v}{\partial y} - \cos \phi \frac{\partial w}{\partial z} \\
& \quad + m(\sin \phi \frac{\partial^2 H_y}{\partial y \partial z} + \cos \phi \frac{\partial^2 H_y}{\partial z \partial x}) \quad (4.17) \\
\frac{\partial H_y}{\partial t} &= E \eta \frac{\partial^2 H_y}{\partial z^2} - z \frac{\partial H_y}{\partial x} - \sin \phi \frac{\partial u}{\partial x} + \cos \phi \frac{\partial v}{\partial y} - \sin \phi \frac{\partial w}{\partial z} \\
& \quad - m(\sin \phi \frac{\partial^2 H_x}{\partial x \partial z} + \cos \phi \frac{\partial^2 H_x}{\partial x \partial z}) \quad (4.18) \\
\frac{\partial H_z}{\partial t} &= E \eta \frac{\partial^2 H_z}{\partial z^2} - z \frac{\partial H_z}{\partial x} + \epsilon \left( \cos \phi \frac{\partial w}{\partial x} + \sin \phi \frac{\partial w}{\partial y} \right) \\
& \quad - \mu \left[ \cos \phi \left( \frac{\partial^2 H_y}{\partial x^2} - \frac{\partial^2 H_x}{\partial y \partial x} \right) + \sin \phi \left( \frac{\partial^2 H_y}{\partial y^2} - \frac{\partial^2 H_x}{\partial y^2} \right) \right] \quad (4.19) \\
\text{and} \quad \frac{\partial H_x}{\partial x} + \frac{\partial H_y}{\partial y} + \frac{\partial H_z}{\partial z} &= 0 \quad (4.20)
\end{align*}
\]
While writing the above equations we have used the approximation (4.9), where

\[
R_i = \frac{\alpha g H^2 (\frac{\partial \Theta_0}{\partial z^*})}{U^2}
\]  
(Richardson number)

\[
E = \frac{\gamma}{H^2 f}
\]  
(Ekman number)

\[
E_\eta = \frac{\eta}{H^2 f}
\]  
(Magnetic Ekman number)

\[
\sigma = \frac{\nu}{\kappa}
\]  
(Prandtl number)

\[
S^2 = \frac{\mu H^2}{\rho U^2}
\]  
(Interaction parameter)

\[
m = \frac{H_0}{\epsilon \text{e} U H}
\]  
(Hall parameter)

\[
\epsilon = \frac{H f}{U}
\]  
(aspect ratio)

The basic zonal flow is taken as follows.

\[
u_o = z, v_o = w_o = 0, \Theta_o = z - \frac{y}{R_i}, p_o = \frac{1}{2} z^2 - \frac{yz}{R_i}
\]

When deviations are taken from the steady state we may expand the total solution as

\[
u = \nu_o + u_1 + \ldots, \text{etc}
\]  \hspace{1cm} (4.21)
The first order equations are obtained by substituting (4.21) into (4.12) to (4.20). As we are interested in discussing the symmetric instabilities of the zonal flow, the perturbed flow is assumed to be independent of \( x \). Let the linearized problem have normal-mode solutions of the form

\[ u_1 = u_1(z) \exp(ily+wt) \text{ etc.}, \] (4.22)

Then the equations (4.12) to (4.20) can be written as

\[ ilv_1 + Dw_1 = 0 \] (4.23)
\[ D_\omega u_1 = -w_1 + v_1 + S^2 \sin \phi ilh_x \] (4.24)
\[ D_\omega v_1 = -u_1 - ilR_1 p_1 - S^2 \cos \phi ilh_x \] (4.25)
\[ D_\phi p_1 = \Theta_1 - \frac{S^2}{R_1} (\sin \phi Dh_y + \cos \phi Dh_x) \] (4.26)
\[ D_\sigma \Theta_1 = \frac{v_1}{R_1} - w_1 \] (4.27)
\[ D_\eta h_x = \sin \phi ilu_1 + m \sin \phi ilDh_y \] (4.28)
\[ D_\eta h_y = -\sin \phi Dw_1 - m \sin \phi ilDh_x \] (4.29)
\[ D_\eta h_z + m \sin \phi \ell^2 h_x = 0 \] (4.30)
\[ ilh_y + D h_z = 0 \] (4.31)

where

\[ D \equiv \frac{d}{dz}, \quad D_\omega = \omega - ED^2, \quad D_\sigma \equiv \omega - \frac{E}{\sigma} D^2 \quad \text{and} \quad D_\eta \equiv \omega - E_\eta D^2 \]
Equations (4.23) and (4.29) can be solved to obtain equations for \( h_x \) and \( h_y \) as follows.

\[
(D^2 - m^2 \sin^2 \phi \ l^2 \mathbf{D}^2) h_x = \sin \phi \ i l D_\eta u_1 - m \sin^2 \phi \ i l D^2 w_1 \tag{4.32}
\]

\[
(D^2 - m^2 \sin^2 \phi \ l^2 \mathbf{D}^2) h_y = -\sin \phi \ D_\eta D_{w_1} + m \sin^2 \phi \ l^2 D_u \tag{4.33}
\]

Eliminating \( h_x \) between (4.24) and (4.32) we get

\[
\Delta_\eta u_1 = (D^2 - m^2 \sin^2 \phi \ l^2 \mathbf{D}^2)(v_1 - w_1) + m S^2 \sin^3 \phi \ l^2 D^2 w_1 \tag{4.34}
\]

where

\[
\Delta_\eta = (D^2 - m^2 \sin^2 \phi \ l^2 \mathbf{D}^2)D_\omega + m S^2 \sin^2 \phi \ l^2 D_\eta \tag{4.35}
\]

Eliminating \( p_1, \Theta_1, u_1 \) and \( h_y \) by using equations (4.25), (4.26), (4.27), (4.33) and (4.34) we obtain the following equation in \( v_1 \) and \( w_1 \).

\[
\begin{align*}
[&\Delta_\eta D_\omega + D^2 - m^2 \sin^2 \phi \ l^2 \mathbf{D}^2](D^2 - m^2 \sin^2 \phi \ l^2 \mathbf{D}^2) D_\sigma D v_1 \\
= & (D^2 - m^2 \sin^2 \phi \ l^2 \mathbf{D}^2) D_\sigma D w_1 - m S^2 \sin^3 \phi \ l^2 (D^2 - m^2 \sin^2 \phi \ l^2 \mathbf{D}^2) D_\sigma D^3 w_1 \\
& - i l \Delta_\eta (D^2 - m^2 \sin^2 \phi \ l^2 \mathbf{D}^2) v_1 + i l R_1 \Delta_\eta (D^2 - m^2 \sin^2 \phi \ l^2 \mathbf{D}^2) w_1 \\
& - i l S^2 \sin^2 \phi \ D_\sigma \Delta_\eta D_\eta D^2 w_1 + m S^2 \sin^3 \phi \ i l^3 D_\sigma (D^2 - m^2 \sin^2 \phi \ l^2 \mathbf{D}^2) D^2 v_1 \\
& - m S^2 \sin^3 \phi \ i l^3 D_\sigma (D^2 - m^2 \sin^2 \phi \ l^2 \mathbf{D}^2) D^2 w_1 + m^2 S^4 \sin^6 \phi \ i l^5 D_\sigma D^4 w_1 \\
& \ldots \tag{4.36}
\end{align*}
\]
We use equation (4.23) to eliminate \( v_1 \) from (4.36).

This gives rise to a single equation in \( w_1 \)

\[
(D^2 - m^2 \sin^2 \phi \cdot \ell^2 D^2) D_\sigma \left[ \Delta_\eta D_\omega + D^2 - m^2 \sin^2 \phi \cdot \ell^2 D^2 \right] D^2 w_1 \\
= -i \ell (D^2 - m^2 \sin^2 \phi \cdot \ell^2 D^2)^2 D_\sigma D w_1 - i \ell \Delta_\eta (D^2 - m^2 \sin^2 \phi \cdot \ell^2 D^2) D w_1 \\
+ \ell^2 R_1 \Delta_\eta (D^2 - m^2 \sin^2 \phi \cdot \ell^2 D^2) w_1 - \ell^2 S^2 \sin^2 \phi D_\sigma \Delta_\eta D_\eta D^2 w_1 \\
+ 2m S^2 \sin^3 \phi i \ell^3 D_\sigma (D^2 - m^2 \sin^2 \phi \cdot \ell^2 D^2) D^3 w_1 \\
- m S^2 \sin^3 \phi \ell^4 D_\sigma (D^2 - m^2 \sin^2 \phi \cdot \ell^2 D^2) D^2 w_1 + m^2 S^4 \sin^6 \phi \ell^6 D_\sigma D^4 w_1 \\
= 0 \tag{4.37}
\]

When \( m = 0 \), the hydromagnetic equation without Hall effect reduces to

\[
[(D_\eta D_\omega + \ell^2 S^2 \sin^2 \phi)^2 + D^2] D_\sigma D^2 w_1 + i \ell [D_\eta D_\sigma + D_\eta D_\omega + \ell^2 S^2 \sin^2 \phi] D_\eta D w_1 \\
- \ell^2 R_1 (D_\eta D_\omega + \ell^2 S^2 \sin^2 \phi) D_\eta w_1 = 0 \tag{4.38}
\]

Equation (4.38) along with the boundary conditions (4.10) defines the eigen value problem for the complex frequency \( \omega \).

The basic flow is then unstable to axisymmetric disturbances if \( \text{Real } \omega > 0 \) and we are particularly interested here in determining the neutral-stability curve \( \text{Real } \omega = 0 \).

In order to determine the relevant scale let us take \( w_1 = e^{i \lambda z} \) as the solution of (4.38). Then (4.38) can be
For the monotonically mode the neutral stability curve is given by \( \omega = 0 \). Letting \( \omega = 0 \) in (4.39) and rearranging the terms we get.

\[
E^2 \lambda^2 (\lambda^4 + \lambda^2 M^2)^2 + \lambda^6 + \left\{ (1 + \sigma) \lambda^4 + \sigma \lambda^2 M^2 \right\} l \lambda + l^2 R_1 \sigma (\lambda^4 + \lambda^2 M^2) = 0
\]

(4.40)

where

\[
M^2 = \frac{S^2 \sin^2 \phi}{E E_\eta}.
\]

Equation (4.40) is of tenth degree in \( \lambda \) and we shall solve this equation in the limit \( E \rightarrow 0 \) by assuming suitable asymptotic expansion in powers of \( E^\alpha \) as follows. The determination of appropriate value for \( \alpha \) gives us the vertical scale length of the boundary layer. Let us take

\[
l = E^{-\alpha} (l_0 + E^\alpha l_1 + E^{2\alpha} l_2 + ...) \]

\[
\lambda = E^{-\alpha} (\lambda_0 + E^\alpha \lambda_1 + E^{2\alpha} \lambda_2 + ...)
\]

(4.41)

\[
R_1 = R_0 + E^\alpha R_1 + E^{2\alpha} R_2 + ...
\]

\[
M = M_0 E^\beta.
\]
Assuming $\ell^2 M^2 \ll \lambda^4$, that is $\alpha + \beta > 0$, equation (4.40) takes the form

$$E^2 \lambda^{10} + \lambda^6 + (1+\sigma)\ell \lambda^5 + \ell^2 R_1 \sigma \lambda^4 = 0 \quad (4.42)$$

that is,

$$E^2 \lambda^6 + \lambda^2 + (1+\sigma)\ell \lambda^1 + \ell^2 R_1 \sigma = 0 \quad (4.43)$$

Substituting (4.41) into (4.43) and assuming that $\alpha < 1/2$ we have

$$\lambda_0^2 + (1+\sigma)\ell_0 \lambda_0 + \sigma R_0 \ell_0^2 = 0 \quad (4.44)$$

The boundary conditions $w_1 = 0$ at $z = 0,1$ are satisfied if

$$\lambda_1 - \lambda_2 = 2n\pi$$

where $n$ is an integer. Using the expansion (4.41) the above condition can be written as

$$E^{-\alpha}[(\lambda_{10} + E^{\alpha} \lambda_{11} + \ldots) - (\lambda_{20} + E^{\alpha} \lambda_{21} + \ldots)] = 2n\pi$$

Then we must have

$$\lambda_{10} - \lambda_{20} = 0, \quad \lambda_{11} - \lambda_{21} = 2n\pi \quad (4.45)$$

The condition (4.45) imply that, the equation (4.44) has equal roots. The condition yields $R_1$ to the leading order
We see that leading $R_0$ is not affected due to the presence of magnetic field. Continuing the expansion in (4.42) we find that for $\alpha < 2/5$ we get

$$
\lambda^1_1 [6\lambda^5_0 + 5(1+\sigma)\lambda^4_0 + 4\sigma R_0 \lambda^2_0 \lambda^3_0] + \lambda^1_1 [(1+\sigma)\lambda^5_0 + 2\sigma R_0 \lambda^4_0 \lambda^1_0] + \sigma R_1 \lambda^2_0 \lambda^4_0 = 0
$$

(4.47)

Using (4.44) and (4.46) in (4.47), we get

$$
R_1 = 0.
$$

Proceeding further with the next order terms gives $R_2$ non-zero with $\alpha = 1/3$ and $\beta = 0$. For the above values of $\alpha$ and $\beta$, (4.42) reduces to

$$
\lambda^2_1 (1+\sigma)\lambda_1 \lambda^6_0 + \sigma R_0 \lambda^2_1 + \sigma R_2 \lambda^2_0 \lambda^2_0 + \frac{4M^2 (1-\sigma)}{(1+\sigma)^3} = 0
$$

Applying the condition (4.45), we have

$$
\lambda^2_1 [(1+\sigma)^2 - 4\sigma R_0] - 4\sigma R_2 \lambda^2_0 - 4\lambda^6_0 - \frac{16M^2 (1-\sigma)}{(1+\sigma)^3} = 4n^2 \pi^2
$$

(4.48)

Using (4.46) in (4.48), we get
\[ R_2 = -\frac{(1+\sigma)^2}{4\sigma} \left( \frac{6 n^2 \pi^2}{\lambda_0^2} \right) - \frac{M^2(1-\sigma)}{\sigma(1+\sigma)} \frac{1}{\lambda_0^2} \]  
(4.49)

and \[ \lambda_1 = \frac{1}{2} \left[ -(1+\sigma) \lambda_1 + 2n\pi \right] . \]

Since \( R_2 < 0 \), \( R_1 \) is minimum when \( R_2 \) is a maximum, and this occurs when \( \lambda \) satisfies the equation

\[ \lambda_0^6 = \frac{1}{2} n^2 \pi^2 + \frac{2M^2(1-\sigma)}{(1+\sigma)^3} \]  
(or) \[ \lambda_0^2 = \left[ \frac{1}{2} n^2 \pi^2 + \frac{2M^2(1-\sigma)}{(1+\sigma)^3} \right]^{1/3} \]  
(4.50)

Maximum \( R_2 \) is given by

\[ R_{2c} = -\frac{(1+\sigma)^2}{4\sigma} \left[ \frac{3}{2} n^2 \pi^2 + \frac{6M^2(1-\sigma)}{(1+\sigma)^3} \right] \frac{1}{\lambda_0^2} \]  
(4.51)

where \( \lambda_0^2 \) is given by (4.50). Therefore we find that the effect of magnetic field is to reduce the Richardson number \( R_1 \). In particular this effect is nullified for \( \sigma = 1 \).

As the order of \( M \) is increased to \( E^{-\alpha/2} \) with \( \alpha < 2/5 \), we find that the magnetic field affects Richardson number to the order \( E^\alpha \) and yields the following equation for \( R_1 \).
\[ \lambda_0^4 R_1 + l_0 M_o^2 \lambda_0 + l_0^2 R_o M_o^2 = 0 \]

(or)

\[ R_1 = -\frac{l_0 M_o^2}{\lambda_0^4} (\lambda_0 + l_0 R_o) \]

Substituting for \( \lambda_0 \) and \( R_o \) using (4.46) in the above equation we get

\[ R_1 = -\frac{4M_o^2(1-\alpha)}{\sigma(1+\sigma)^3 l_0^2} \]

(4.52)

When the order of \( M \) is further increased so that

\[ \lambda^2 M^2 >> \lambda^4 \], that is, \( \alpha + \beta < 0 \)

equation (4.40) reduces to

\[ E_2^2 M^2 l \lambda^2 + \sigma \lambda + \sigma R l = 0 \]

(4.53)

Assuming the expansions as given in (4.41) for \( \alpha < 1/2 \) and \( \beta < -1/2 \) with the condition \( 1+\beta = \alpha \) we obtain

\[ l_0 M_o^2 \lambda_o^2 + \sigma \lambda_o + \sigma R_o l_o = 0 \]

The application of boundary conditions \( w_1 = 0 \) at \( z = 0,1 \) yields (4.45) which shows that the root of the above equation are equal. Then we must have
Considering terms of next order in (4.40) for values of $a < \frac{2}{\sqrt{5}}$, we find that $R^1$ is given by

\begin{equation}
R^1 = \frac{\alpha}{4l_0^2 M_0^2} \tag{4.54}
\end{equation}

\begin{equation}
\lambda_0 = -\frac{\alpha}{2l_0^2 M_0^2} \tag{4.55}
\end{equation}

When the order of $M$ is such that $O(A^2) = O(\ell^2 M^2)$, $\lambda$ satisfies a fourth degree equation and hence $R^1$ cannot be determined by the above method.

From the above discussion we find that the effect of magnetic field is found in the value of $R^2$ when $\ell^2 M^2 \ll \lambda^4$. But by taking $M = O(E^{-a/2})$, when $\ell^2 M^2 \ll \lambda^4$ the effect of magnetic field is found in $R^1$ itself as evident from the expression in (4.52). When $\ell^2 M^2 \gg \lambda^4$, the effect is found in the leading $R^1$ as seen from its value in (4.54).

The problem without dissipative effects is defined by
setting \( E = E_\eta = 0 \) in (4.38), which yields

\[
\left\{ \omega^4 + (1 + 2 S^2 \sin^2 \phi) \omega^2 + \frac{l^4 S^4 \sin^4 \phi}{\pi^2} \right\} D^2 w_1
\]

\[
+ (2i \omega^2 + i l^2 S^2 \sin^2 \phi) D w_1 - l^2 R (\omega^2 + l^2 S^2 \sin^2 \phi) w_1 = 0 \quad (4.56)
\]

The characteristic equation has got two roots, say \( \alpha_1 \) and \( \alpha_2 \) and application of boundary conditions leads to the result

\[
(\alpha_1 - \alpha_2)^2 = 4n^2 \pi^2.
\]

This reduces to

\[
\left\{ (\omega^2 + l^2 S^2 \sin^2 \phi)^2 + \omega^2 \right\}^2 + \frac{l^2 R}{n \pi^2} (\omega^2 + l^2 S^2 \sin^2 \phi) \left\{ (\omega^2 + l^2 S^2 \sin^2 \phi)^2 + \omega^2 \right\}
\]

\[- \frac{l^2}{4n^2 \pi^2} (2 \omega^2 + l^2 S^2 \sin^2 \phi)^2 = 0 \quad (4.57)
\]

Letting \( \omega^2 = x \) and \( l^2 S^2 \sin^2 \phi = s \), the above equation may be written as

\[
P(x) \equiv x^4 + \left\{ 2(2s+1) + \frac{l^2 R_i}{n \pi^2} \right\} x^3 + \left\{ (2s+1)^2 + 2s^2 + \frac{l^2 R_i}{n \pi^2} (3s+1) - \frac{l^2}{n^2 \pi^2} \right\} x^2
\]

\[+ \left\{ 2s^2 (s+1) + \frac{l^2 R_i s}{n^2 \pi^2} (3s+1) - \frac{l^2 s}{n^2 \pi^2} \right\} x
\]

\[+ \left\{ \frac{l^2 R_i s^3}{n^2 \pi^2} - \frac{l^2 s^2}{4n^2 \pi^2} \right\} = 0 \quad (4.58)
\]

If \( P(0) < 0 \), that is, when
the equation (4.58) has at least a pair of real roots differing in sign. A positive root in $x$ corresponds to 

$\omega = \pm \sqrt{x}$, which implies a growing mode.

For smaller values of $S^2$, taking $\omega^2 = x S^2 \sin^2 \phi$, the condition (4.57) leads to

$$[1- \frac{\ell^2}{n^2 \pi^2} (1-R_i)] x_i^2 + \frac{\ell^4}{n^2 \pi^2} (R_i-1) x_i - \frac{\ell^6}{4 n^2 \pi^2} = 0 \quad (4.59)$$

We will consider the following cases.

Case (i): $R_i > 1$.

For this range of values of $R_i$, the product of the roots of the equation (4.59) is negative and this implies the existence of a positive root and hence it is a case of instability.

Case (ii): $R_i < 1$.

Here we must consider three possibilities.

(a) When $\ell^2 = \frac{n^2 \pi^2}{1-R_i}$, equation (4.59) yields

$$x_i = -\frac{\ell^2}{4(1-R_i)} < 0.$$ Hence the flow is stable for $R_i < 1$ and $\ell^2 = (n^2 \pi^2)/1-R_i$. 

\[54815\]
(b) When $\xi^2 < \frac{n^2\pi^2}{1-R_i}$, then the product of the roots of equation (4.59) is negative. This shows that there is one positive root and hence instability occurs in this case.

(c) When $\xi^2 > \frac{n^2\pi^2}{1-R_i}$, the product of the roots of equation (4.59) is positive and this implies that either both the roots are real and positive or the roots are complex with real part $> 0$. Here again instability sets in for $R_i < 1$ and $\xi^2 > \frac{n^2\pi^2}{1-R_i}$.

From the above discussion we conclude that in the presence of weak magnetic field instability occurs whatever may be the value of the Richardson number $R_i$, except when $R_i < 1$ and $\xi^2 = \frac{n^2\pi^2}{1-R_i}$ for which the flow is stable.

Effect of Hall Current

Now we shall investigate the effect of Hall current on the stability problem. Assuming $w_1 = e^{i\lambda z}$ as the solution and letting $\omega = 0$ in (4.37) we get after some simplifications...
\[
\{ (\lambda^2 + m_\eta^2 \ell^2) \lambda^2 + \ell^2 M^2 \} (\lambda^2 + m_\eta^2 \ell^2) E^2 \lambda^6 + \lambda^4 (\lambda^2 + m_\eta^2 \ell^2) E^2 \\
+ \ell \lambda^3 (\lambda^2 + m_\eta^2 \ell^2) ^2 + 2m_\eta \omega M^2 E \ell^3 \lambda^3 (\lambda^2 + m_\eta^2 \ell^2) \\
+ \sigma \ell \lambda \{ (\lambda^2 + m_\eta^2 \ell^2) \lambda^2 + \ell^2 M^2 \} (\lambda^2 + m_\eta^2 \ell^2) \\
+ \sigma \ell^2 R_i \{ (\lambda^2 + m_\eta^2 \ell^2) \lambda^2 + \ell^2 M^2 \} (\lambda^2 + m_\eta^2 \ell^2) \\
+ E^2 \ell^2 M^2 \lambda^4 \{ (\lambda^2 + m_\eta^2 \ell^2) \lambda^2 + \ell^2 M^2 \} \\
+ m_\eta M^2 \ell^4 E \lambda^2 (\lambda^2 + m_\eta^2 \ell^2) + m_\eta M^2 E^2 \ell^6 \lambda^2 = 0
\]

(4.60)

where \( m_\eta = \frac{m \sin \phi}{E_\eta} \) and \( M^2 = \frac{S^2 \sin^2 \phi}{E E_\eta} \).

Case (i):

Let \( \ell^2 M^2 \ll \lambda^4 \), that is \( \alpha + \beta > 0 \).

Taking the usual expansions for \( \ell, \lambda, R_i \) we have from (4.60) for \( \alpha < 1/2 \) and \( \beta > -1/2 \) in the leading order after using boundary conditions

\[
(\lambda_0^2 + m_0^2 \ell_0^2) ^2 \left( \lambda_0^2 + (1+\sigma) \ell_0 \lambda_0 + \sigma \ell_0 \ell_0 \right) = 0.
\]

Since \( \lambda_0^2 + m_0^2 \ell_0^2 \neq 0 \) is not possible, we get

\[
\lambda_0^2 + (1+\sigma) \ell_0 \lambda_0 + \sigma \ell_0 \ell_0 = 0
\]

which gives values for \( R_0 \) and \( \lambda_0 \) as found in (4.46) for the case \( m = 0 \).
Collecting terms of next order in (4.60) we obtain after taking $p = 0$ and equating terms in $E^{-2\alpha}$ with those in $E^{-6\alpha}$, that is by taking $\alpha = \frac{1}{3}$

$$
\lambda_1^2 + (1+\sigma)\lambda_1 + \lambda_o^6 + \sigma R_0 l_1^2 + \sigma R_2 l_o^2 + \frac{4M^2(1-\sigma)}{(1+\sigma)\left\{(1+\sigma)^2 + 4m^2\right\}} = 0
$$

The condition in (4.45) yields

$$
l_1^2 \left\{(1+\sigma)^2 - 4\sigma R_o\right\} - 4\sigma R_2 l_o^2 - 4\lambda_o^6 - \frac{16M^2(1-\sigma)}{(1+\sigma)\left\{(1+\sigma)^2 + 4m^2\right\}} = 4n^2\pi^2
$$

Since $(1+\sigma)^2 - 4\sigma R_o = 0$, we get

$$
R_2 = -\left[ \frac{(\lambda_o^6 + n^2\pi^2)}{\sigma} + \frac{4M^2(1-\sigma)}{\sigma(1+\sigma)\left\{(1+\sigma)^2 + 4m^2\right\}} \right] \frac{1}{\lambda_o^2}
$$

Substituting for $l_o$ using (4.46), we can write

$$
R_2 = -\left[ \frac{(1+\sigma)^2}{4\sigma} (\lambda_o^6 + n^2\pi^2) - \frac{M^2(1-\sigma^2)}{\sigma\left\{(1+\sigma)^2 + 4m^2\right\}} \right] \frac{1}{\lambda_o^2}
$$

We can show that $R_2$ has a maximum as a function of $\lambda_o$ and
\[ \lambda_0^2 = \left[ \frac{2M^2(1-\sigma)}{(1+\sigma)\{(1+\sigma)^2+4\sigma\}} + \frac{n^2\pi^2}{2} \right]^{1/3} \]  

and \[ \text{max } R_{2c} = -\frac{(1+\sigma)^2}{4\sigma} \left[ \frac{3}{2} n^2\pi^2 + \frac{6M^2(1-\sigma)}{(1+\sigma)\{(1+\sigma)^2+4\sigma\}} \right] \frac{1}{\lambda_0^2} \]

This clearly shows that Hall current increases the Richardson number corresponding to neutral stability. In particular for large \( m \),

\[ R_{2c} \propto -\frac{3(1+\sigma)^2}{8\sigma} \left[ n^2\pi^2 + \frac{M^2(1-\sigma)}{(1+\sigma)m_\eta^2} \right] \frac{1}{\lambda_0^2} \]

and \[ \lambda_0^2 = \left[ \frac{M^2(1-\sigma)}{2(1+\sigma)m_\eta^2} + \frac{n^2\pi^2}{2} \right]^{1/3}. \]

Case (ii):

When \( \ell^2M^2 \gg \lambda^4 \), that is, when \( \alpha+\beta < 0 \), assuming the expansions for \( \ell, \lambda, R_i \) we have from (4.60) for \( \alpha < \frac{1}{2}, \beta < -\frac{1}{2} \) with the condition \( \alpha = 1+\beta \) the following equation for \( \lambda_0 \)

\[ M^2\ell_0\lambda_0^2 + \sigma\lambda_0 + \sigma\ell_0R_0 = 0. \]

This gives identical values for \( R_0 \) and \( \lambda_0 \) already found in (4.54). Considering terms of second order in (4.60)
for values of $\alpha < 1/3$ with $1+\beta = \alpha$ we get similar value for $R_i$ given in (4.55). This means the presence of Hall current has no influence on the critical Richardson number.

When $O(\lambda^2 M^2) = O(\lambda^4)$, $\lambda$ satisfies a higher degree equation and hence $R_i$ cannot be found by the method followed in the above cases.

**Oscillatory Motions**

Changing $\omega$ into $i\omega$ in (4.39) and separating the real and imaginary parts we get

$$A \omega^4 + B \omega^2 + C = 0$$

$$A' \omega^4 + B' \omega^2 + C' = 0$$

where

$$A = \left\{ E(2+\sigma^{-1})+2E_\eta \right\} \lambda^2$$

$$B = -\left\{ \frac{2E^2E_\eta}{\sigma} + (E+E_\eta)^2 \frac{E}{\sigma} + 2EE_\eta(E+E_\eta) \right\} \lambda^6$$

$$+ 2 \lambda^2 S^2 \sin^2 \phi \left\{ E(1+\sigma^{-1})+E_\eta \right\} \lambda^2 + \left( \frac{E}{\sigma} + 2E_\eta \right) \lambda^2$$

$$+ \left\{ E(1+\sigma^{-1})+4E_\eta \right\} \lambda + \lambda^2 R_i(E+2E_\eta)$$
Consider the case electrical resistivity being negligible ($E_\eta = 0$). Then from (4.64) we get

\[
A = (2+\sigma^{-1}) E \lambda^2 \\
B = -\left[ \frac{E^3}{\sigma} \lambda^6 + 2E(1+\sigma^{-1}) \lambda^2 S^2 \sin^2 \phi \lambda^2 + \frac{E}{\sigma} \lambda^2 + E(1+\sigma^{-1})l \lambda + E E_\eta^2 R_i \right] \\
C = \lambda^6 S^4 \sin^4 \phi \frac{E}{\sigma} \lambda^2
\]
A' = \lambda^2

B' = -\left\{ E^2(2\sigma^{-1}+1)\lambda^6+2\lambda^2S^2\sin^2\phi \lambda^2 + \lambda^2+2\lambda^2+2R_i \right\}

C' = 2\lambda^2S^2\sin^2\phi \frac{E^2}{\sigma} \lambda^6+\lambda^4S^4\sin^4\phi \lambda^2+\lambda^3S^2\sin^2\phi \lambda+\lambda^4R_iS^2\sin^2\phi

Eliminating \omega between the equation (4.63) we get

\left( CA' - C'A \right)^2 = (BC' - B'C)(AB' - A'B) \quad (4.66)

Using the values given in (4.65) we obtain

CA' - C'A = -E\lambda^2S^2\sin^2\phi \lambda^2 \left[ 2\sigma^{-1}(2+\sigma^{-1})E^2\lambda^6+2\lambda^2S^2\sin^2\phi \lambda^2 \right]

+ (2+\sigma^{-1})\lambda^2 + (2+\sigma^{-1})\lambda^2R_i

BC' - B'C = -E [ \frac{E^2}{\sigma} \lambda^6+2(1+\sigma^{-1})\lambda^2S^2\sin^2\phi \lambda^2 + \lambda^2 + (1+\sigma^{-1})\lambda^2 + \lambda^2R_i ] \lambda

\times \lambda^2S^2\sin^2\phi \left[ \frac{E^2}{\sigma} \lambda^6+\lambda^2S^2\sin^2\phi \lambda^2+\lambda^2+\lambda^2R_i \right]

+ \lambda^4S^4\sin^4\phi \frac{E^2}{\sigma} \lambda^2 \left\{ E^2(2\sigma^{-1}+1)\lambda^6+2\lambda^2S^2\sin^2\phi \lambda^2 \right\}

+ \lambda^2+2\lambda^2+\lambda^2R_i \}

AB' - A'B = -(2+\sigma^{-1})E \lambda^2 \left\{ (2\sigma^{-1}+1)E^2\lambda^6+2\lambda^2S^2\sin^2\phi \lambda^2+\lambda^2+2\lambda^2+\lambda^2R_i \right\}

+ E\lambda^2 \left\{ \frac{E^2\lambda^6}{\sigma} + 2(1+\sigma^{-1})\lambda^2S^2\sin^2\phi \lambda^2 + \lambda^2 \right\}

+(1+\sigma^{-1})\lambda^2 + \lambda^2R_i \}
When we use the above expressions in (4.66), after some simplification we have

\[ 4\phi^4 s^4 \sin^4 \phi \lambda^4 (E^2 \lambda^6 + \lambda^2 + \mathcal{L} \lambda) \]
\[ + \mathcal{L}^2 s^2 \sin^2 \phi \lambda^2 \left[ 2 \left( \frac{E^2}{\mathcal{L}^2} \lambda^6 + \mathcal{L}^2 + \mathcal{L}^2 R_i \right) \left( \frac{E^2 \lambda^6}{\mathcal{L}^2} + \frac{\lambda^2}{\mathcal{L}^2} + (1 + \sigma^{-1}) \mathcal{L} \lambda + \mathcal{L}^2 R_i \right) \right] \]
\[ + \left\{ 2(1 + \sigma^{-1}) \frac{E^2 \lambda^6}{\mathcal{L}^2} + 2 \lambda^2 + (3 + \sigma^{-1}) \mathcal{L} \lambda + (1 + \sigma^{-1}) \mathcal{L}^2 R_i \right\} \times \]
\[ x \left\{ 2(1 + \sigma^{-1}) \frac{E^2 \lambda^6}{\mathcal{L}^2} + (3 + \sigma^{-1}) \mathcal{L} \lambda + (3 + \sigma^{-1}) \mathcal{L}^2 R_i \right\} \]
\[ - \left\{ 2(1 + \sigma^{-1}) \frac{E^2 \lambda^6}{\mathcal{L}^2} + (2 + \sigma^{-1}) \mathcal{L} \lambda + (2 + \sigma^{-1}) \mathcal{L}^2 R_i \right\} \]
\[ + \left\{ 2(1 + \sigma^{-1}) \frac{E^2 \lambda^6}{\mathcal{L}^2} + 2 \lambda^2 + (3 + \sigma^{-1}) \mathcal{L} \lambda + (1 + \sigma^{-1}) \mathcal{L}^2 R_i \right\} \times \]
\[ x \left\{ \frac{2E^2 \lambda^6}{\mathcal{L}^2} + \mathcal{L} \lambda + \mathcal{L}^2 R_i \right\} \left\{ \frac{E^2 \lambda^6}{\mathcal{L}^2} + \frac{\lambda^2}{\mathcal{L}^2} + (1 + \sigma^{-1}) \mathcal{L} \lambda + \mathcal{L}^2 R_i \right\} = 0 \]
\[ \ldots \quad (4.67) \]

Taking \( S \) small, that is letting \( S^2 = O(E^\beta), \ (\beta > 0) \) in (4.67) and considering the order of the various terms we find for \( \alpha < 1/2, \ \beta > 0 \) and \( 2\alpha < \beta \)

\[ 2\lambda_0^2 + (3 + \sigma^{-1}) \mathcal{L}_0 \lambda_0 + (1 + \sigma^{-1}) \mathcal{L}_0^2 R_0 = 0 \quad (4.68) \]

This equation has equal roots if

\[ R_0 = \frac{(3\sigma + 1)^2}{8\sigma(\sigma + 1)} \]

\[ \lambda_0 = -\frac{(3\sigma + 1)}{4\sigma} \mathcal{L}_0 \quad (4.69) \]
Collecting terms of next order with the condition \( a < 2/5 \) and \( \phi = 3\alpha \), we have

\[
\ell_0^2 s^2 \sin^2 \phi \lambda_0^2 \left[ 2 \left( \frac{\lambda_0^2}{\sigma} + (1+\sigma^{-1}) \ell_0^2 \lambda_0^2 + \ell_0^2 R_0 \right) - (2+\sigma^{-1}) \lambda_0^2 (\ell_0^2 \lambda_0^2 + \ell_0^2 R_0) \right] + \left[ \frac{\lambda_0^2}{\sigma} + (1+\sigma^{-1}) \ell_0^2 \lambda_0^2 + \ell_0^2 R_0 \right] \lambda_1 \left[ 4\lambda_0^2 + (3+\sigma^{-1}) \ell_0^2 \right] + \ell_1 \left[ (3+\sigma^{-1}) \lambda_0^2 + 2(1+\sigma^{-1}) \ell_0^2 R_0 \right] + (1+\sigma^{-1}) \ell_0^2 R_1 = 0 \quad (4.70)
\]

Using equation (4.69), we find

\[
4\lambda_0^2 + (3+\sigma^{-1}) \ell_0^2 = 0
\]

\[
(3+\sigma^{-1}) \lambda_0^2 + 2(1+\sigma^{-1}) \ell_0^2 R_0 = 0
\]

\[
\ell_0^2 \lambda_0^2 + \ell_0^2 R_0 = \frac{(3\sigma+1)(\sigma-1)}{8\sigma(\sigma+1)} \ell_0^2 \quad (4.71)
\]

\[
\frac{\lambda_0^2}{\sigma} + (1+\sigma^{-1}) \ell_0^2 \lambda_0^2 + \ell_0^2 R_0 = \frac{(3\sigma+1)(2\sigma+1)(\sigma-1)^2}{16\sigma^3(\sigma+1)} \ell_0^2
\]

Using these results in (4.70) we get

\[
R_1 = \frac{\ell_0^2 \sin^2 \phi (3\sigma+1)^2 (2\sigma+1)}{8\sigma^2 (\sigma^2-1)} \ell_0^2 \quad (4.72)
\]

This shows that the value of \( R_1 \) is increased by the presence of a small magnetic field when \( \sigma \neq 1 \). For \( \sigma = 1 \)
and \( \alpha < \frac{1}{2} \), equation (4.67) takes the form

\[
4 \ell^4 \sin^4 \phi \lambda^4 (\lambda^2 + \ell \lambda) \\
+ \ell^2 \sin^2 \phi \lambda^2 [10(\lambda^2 + 2 \ell \lambda + \ell^2 R) - 3](\ell \lambda + \ell^2 R) \\
+ 2(\lambda^2 + 2 \ell \lambda + \ell^2 R)^2(\ell \lambda + \ell^2 R) = 0
\]

which allows \( \lambda = -\ell \) and \( R = 1 \) as an exact solution.

The case when \( E_\eta > E \)

We consider the case when magnetic diffusion dominates over the viscous diffusion.

Let \( E_\eta = O(\epsilon^{1-p}) = K E^{1-p} \) (say) \( p > 0 \). Using the expansions for \( \lambda, \ell, \) and \( R_1 \) we find the leading terms in \( A, B, C, A', B', C' \). For \( p > 1, \alpha < 1/2 \) and \( 2\alpha > 2-p \) from the equation (4.66), we get in the leading order

\[
\lambda_0^2 + 2\lambda_0 \ell_0 + \ell_0^2 R_0 = 0
\]

For equal roots we must have \( R_0 = 1 \) and \( \lambda_0 = -\ell_0 \). Considering second order terms in (4.66), we get for \( \alpha < 1/3 \) and \( \alpha-p < 0 \)

\[
R_1 = 0.
\]

Collecting terms of next order in (4.66), we find for \( p > 1 \)

\( \alpha < 1/4, 2\alpha-p < 0, 2\alpha > 2-p \)
\[
\lambda_1^2(15\lambda_0^4+20\lambda_0^3l_0+6\lambda_0^2l_0^2R_0) + \lambda_1l_1(10\lambda_0^4+8\lambda_0^3l_0R_0) \\
+\lambda_2(5\lambda_0^5+10\lambda_0^4l_0+4\lambda_0^3l_0^2R_0)+l_2(2\lambda_0^5+2\lambda_0^4l_0R_0) \\
+l_0^2\lambda_0^4R_2 + l_1\lambda_0^4R_0 = 0
\]

Since $\lambda_0 = -l_0$ and $R_0 = 1$, this finally reduces to

\[
\lambda_1^2 + 2l_1\lambda_1+l_1^2+l_0^2R_2 = 0
\]

The condition $(\lambda_1-\lambda_2)^2 = 4n^2\pi^2$ yields

\[
R_2 = -\frac{n^2\pi^2}{l_0^2}
\]

Finally $R_1$ is given by

\[
R_1 = 1-E^{2a} \frac{n^2\pi^2}{l_0^2}
\]  \hspace{1cm} (4.73)

We conclude that the effect of magnetic field is suppressed by large resistivity.

4.3 Magnetic Field Vertical

Here we take $H^* = (0,0,H_z^*)$ and write the basic equations similar to the previous case and finally obtain the equation for $w_1$ given by

\[
\Delta^I_\eta D^I_\eta D_\sigma D_\omega D^2w_1 = -D^2_\eta D_\sigma D_\omega D^2w_1 - ilD^2_\eta D_\omega Dw_1 \\
-il\Delta^I_\eta D^I_\eta Dw_1 + l_0^2R_1\Delta^I_\eta D^I_\eta w_1
\]
\[ +S^2 D_\sigma \Delta'_{\eta} D_\eta D^4 w_1 - 2m S^2 D_\sigma D'_\eta D^6 w_1 \]
\[ -i \ell m S^2 D_\sigma D'_\eta D^5 w_1 - m^2 S^4 D_\sigma D^{10} w_1 \]  
\[ (4.74) \]

where
\[ D'_\eta \equiv (\omega - E_{\eta} D^2)^2 + m^2 D^4 \]
\[ \Delta'_{\eta} \equiv [(\omega - E_{\eta} D^2)^2 + m^2 D^4] (\omega - E D^2) - S^2 (\omega - E_{\eta} D^2) D^2 \]

and the operators $D, D_\sigma, D_\omega$ and $D_\eta$ are already defined.

Putting $m = 0$ in the equation (4.74), we get
\[ \begin{align*}
[D_\eta D_\sigma - S^2 D^2] D_\eta D_\sigma D_\omega D^2 w_1 &= -D^2_\eta D_\sigma D^2 w_1 - i \ell D^2_\eta D_\sigma D w_1 \\
- i \ell (D_\eta D_\sigma - S^2 D^2) D_\eta D w_1 + \ell^2 R_1 (D_\eta D_\sigma - S^2 D^2) D_\eta w_1 \\
+ S^2 (D_\eta D_\sigma - S^2 D^2) D_\sigma D^4 w_1 
\end{align*} \]
\[ (4.75) \]

Taking $\omega_1 = e^{i \lambda z}$ as the solution of the equation (4.75), we can write (4.75) in the form given below.
\[ \begin{align*}
[(\omega + E_{\eta} \lambda^2)(\omega + E \lambda^2) + S^2 \lambda^2] & (\omega + E_{\eta} \lambda^2)(\omega + \frac{E}{\sigma} \lambda^2)(\omega + E \lambda^2) \lambda^2 \\
+ (\omega + E_{\eta} \lambda^2)^2 & (\omega + \frac{E}{\sigma} \lambda^2) \lambda^2 + (\omega + E_{\eta} \lambda^2)^2 (\omega + \frac{E}{\sigma} \lambda^2) \ell \lambda \\
+ \left\{ (\omega + E_{\eta} \lambda^2)(\omega + E \lambda^2) + S^2 \lambda^2 \right\} (\omega + E_{\eta} \lambda^2) \ell \lambda \\
+ \ell^2 R_1 \left\{ (\omega + E_{\eta} \lambda^2)(\omega + E \lambda^2) + S^2 \lambda^2 \right\} (\omega + E_{\eta} \lambda^2)
\end{align*} \]
\[ +S^2 \left\{ (\omega + E_\eta \lambda^2)(\omega + E \lambda^2) + S^2 \lambda^2 \right\} \left( \omega + \frac{E}{\sigma} \lambda^2 \right) \lambda^4 = 0 \] 

(4.76)

For neutral stability \( \omega = 0 \), then the above equation yields after some steps

\[ E^2 (\lambda^2 + M^2)^2 \lambda^4 + \lambda^4 + (1+\sigma) \lambda^3 + \sigma M^2 \lambda^2 + \sigma I^2 R_1 (\lambda^2 + M^2) = 0, \] 

(4.77)

where \( M^2 = \frac{S^2}{EE_\eta} \).

Here we restrict our discussion to \( \lambda^2 \gg M^2 \). Under this approximation, equation (4.77) can be written as

\[ E^2 \lambda^6 + \lambda^2 + (1+\sigma) \lambda + \sigma I^2 R_1 = 0 \]

The above equation is exactly the same as the equation (4.43) of the magnetic field horizontal case and similar discussions lead to same results (4.46) for \( R_1 \) and \( I_0 \) (for \( \alpha < 1/2 \)).

For \( \alpha < 2/5 \), the second order terms in (4.77) yield \( R_1 = 0 \).

Continuing the expansion in (4.77) for \( \alpha = 1/3 \) and \( M = O(1) \) we obtain

\[ R_2 = -\frac{(1+\sigma)^2}{4\sigma \lambda_0^2} (\lambda_o^6 + n^2 \pi^2) - \frac{M^2(1-\sigma^2)}{4\sigma \lambda_0^2} \]

and 
\[ \lambda_1 = \frac{1}{\alpha^2} \left[-(1+\sigma) I_1 \pm 2n\pi \right]. \]
It is easily seen that $R_2$ has a maximum as a function of $\lambda_0$, when $\lambda_0$ is the positive real root of the equation.

$$\lambda_0^6 = \frac{1}{2} n_2^2 \pi^2 + \frac{M_2^2(1-\sigma)}{2(1+\sigma)}$$

(or)

$$\lambda_0^2 = \left[ \frac{1}{2} n_2^2 \pi^2 + \frac{M_2^2(1-\sigma)}{2(1+\sigma)} \right]^{1/3}$$

(4.78)

The maximum $R_2$ being

$$R_{2c} = -\frac{3(1+\sigma)^2}{8\sigma} \left[ n_2^2 \pi^2 + \frac{M_2^2(1-\sigma)}{(1+\sigma)} \right] \frac{1}{\lambda_0^2}$$

(4.79)

where $\lambda_0^2$ is given by (4.78).

Comparing (4.79) with (4.51) we find that the value of $R_2$ is altered in this case but the effect of magnetic field is again to reduce the value of $R_i$ and there is no effect of magnetic field if $\sigma = 1$.

Oscillatory motions can be discussed for the case of vertical magnetic field also. We find that in this case when $E_\eta > E$, proceeding in the manner described in horizontal case under the assumption $E_\eta = 0$, the values of $R_i$ and the wave number $\lambda$ are unaffected to the leading order as given in (4.69). The corresponding first order corrections are also unchanged.
4.4 Conclusion

The effect of magnetic field and Hall current on the baroclinic symmetric instability of a zonal shear flow may be summarized as follows. It is found that in the case of horizontal magnetic field when $M = O(1)$, the Richardson number is reduced when the scale length $\lambda$ in the vertical direction and the wave number $l$ of the disturbance are both of order $E^{-1/3}$. However when Hall current effects are added the Richardson number is increased. In particular we find that for $\sigma = 1$ the effect of magnetic field and Hall current disappear up to second order. As the strength of the magnetic field is increased such that $M = O(E^{-\alpha/2})$, with $\alpha < 2/5$ we find $R_1 = R_0 + E^\alpha R_1$, where $\alpha(1+\sigma)^3 l_0^2$. Therefore the value of $R_1$ is reduced and we find for $\sigma = 1$, $R_1$ is again zero. If the magnetic field strength reaches a value of order $M^{-(1-\alpha)}$, where $0 < \alpha < 2/5$, $R_1$ is significantly affected by the presence of magnetic field and the presence of Hall current has no influence on the critical Richardson number.

When we consider the problem without dissipative effects, $R_1 < \frac{1-4n^2 \pi^2 l^2 s^4 \sin^4 \phi}{4 l_s^2 s^2 \sin^2 \phi}$ implies the existence of a growing mode, provided $1-4n^2 \pi^2 l^2 s^4 \sin^4 \phi > 0$, that is $\phi$
is small. But in the presence of a weak magnetic field instability occurs whatever may be the value of \( R_i \), except when \( R_i < 1 \) and \( \mathcal{J}^2 = \frac{n^2 \pi^2}{1-R_i} \) for which the flow is stable.

For oscillatory motions when \( E_\eta = 0 \), corresponding to marginal stability we find that for \( \alpha < 1/2, \beta > 0 \) and \( 2\alpha < \beta \) the leading value of \( R_i \) is not affected by the presence of magnetic field. But when \( S^2 \) is small of order \( E^\beta (\beta > 0) \), with \( \alpha < 2/5 \) and \( \beta = 3\alpha \), we find the value of the Richardson number is increased by the presence of a small magnetic field when \( \sigma \neq 1 \). For \( \sigma = 1 \) and \( \alpha < 1/2 \) we obtain \( \lambda = -\ell \) and \( R = l \) as an exact solution. When the magnetic diffusion dominates over the viscous diffusion, that is when \( E_\eta > E \) the effect of magnetic field tends to vanish because of large magnetic diffusion.

The analysis carried out when the magnetic field is vertical with \( \alpha = 1/3 \) and \( M^2 = O(1) \) shows that the leading value of \( R_i \) is identical to the case of horizontal magnetic field and \( R_2 \) is proportional to \( M^2 \), but the multiplying factor is different. The effect of magnetic field is to reduce the value of \( R_i \) and again there is no effect if \( \sigma = 1 \) upto second order. In the case of oscillatory motions under the assumption \( E_\eta > E \) we get identical results as obtained in the case of horizontal magnetic field for \( E_\eta = 0 \).