CHAPTER I

INTRODUCTION

Let X be a non-empty set and V(X) be a vector space of complex valued functions on X, under the pointwise operations of addition and scalar multiplication. Let T : X \rightarrow X be a mapping such that the composite f \circ T of f with T is in V(X) whenever f is in V(X). Then T induces a linear transformation C_T on V defined by C_T f = f \circ T for every f in V(X). If V(X) is a topological vector space and C_T is bounded, then we call it a composition operator on V(X). We are interested in the case in which V(X) is a Hilbert space. Such a study can be carried out either when V is the L^2-space of a measure or when V is a functional Hilbert space. In this chapter we present a survey of the study of composition operators in these two contexts.

Before taking up the survey, we wish to observe that examples of composition operators abound and present some principal examples known.

(i) Translation operator on L^2(R), induced by T(x) = x + a, where 'a' is a fixed real number is a composition operator defined by A_f(x) = f(x + a). In this example, Lebesgue measure is used.
(ii) Left shift on $l^2$ induced by $T(n) = n+1$, $n \in \mathbb{N}$ is a composition operator.

(iii) Similarly, right shift on $l^2$ induced by $T(n) = n-1$, $n \in \mathbb{N}$, $n \neq 1$ is also a composition operator.

(iv) Bilateral shift on $l^2$ is a composition operator induced by $T(n) = n-1$, $n \in \mathbb{Z}$.

In these three shifts counting measure is used.

(v) Weighted shift: In examples (ii), (iii) and (iv), instead of counting measure, if suitable measures are assigned to the integers, we get weighted shifts i.e. shifts on a weighted sequence space.

(vi) The unitary operator on $L^2(x)$ induced by an ergodic transformation $T$ on $X$ given by $Af(x) = f(T(x))$ is a composition operator.

(vii) Any orthogonal projection on a Hilbert space: Let $X$ be the set of atoms representing the space, $X_1$, the subset representing the range and $T$ be an identity on $X_1$.

(viii) Let $(e_n)_{n=-\infty}^{\infty}$ be an orthonormal basis of $l^2$ and let $S$ be the symmetry $Se_n = e_{-n}$. This has at least two natural representations as a composition operator.
(a) \( X = \) integers with counting measure, 
\[ T(n) = -n, \quad L^2 = L^2(x) \]

(b) \( X = (-\pi, \pi) \) with normalised Lebesgue measure, 
\[ T(x) = -x. \quad L^2(x) \text{ has an orthonormal basis} \]
\[ t_n(x) = e^{inx} \text{ and } C_T t_n(x) = t_n \circ T(x) = t_n(-x) = e^{-inx} = t_n(x) \text{ and so } C_T \text{ is unitarily equivalent to } S. \]

The study of composition operators on Hilbert spaces is suggested by recent developments in two diverse areas of Mathematics. The work on composition operators on a \( L^2 \) space was inspired by the works of Von Neumann (\cite{99}, \cite{100}) and Koopman (\cite{84}, \cite{85}). The study of composition operators on functional Hilbert spaces appeared implicitly in the work of Konaigs \cite{83}, who gave a solution to Schroeder's functional equation as early as 1884 and in the work of Littlewood \cite{93} in 1925. Recently the composition operators on functional Hilbert spaces such as \( H^2 \) of the disc have been studied explicitly to a considerable extent. We shall present a survey of the study of composition operators on these two categories of spaces in this chapter.

**COMPOSITION OPERATORS ON FUNCTIONAL HILBERT SPACES**

A functional Hilbert space \( H \) is a Hilbert space of complex valued functions on a non-empty set \( X \) such that
the operations of addition and scalar multiplication are defined pointwise and each evaluation functional \( f \mapsto f(z) \) is bounded. By the Riesz representation theorem, corresponding to each point \( x \) of \( X \) there is a vector \( k_x \in H \) such that for \( f \) in \( H \),

\[
f(x) = \langle f, k_x \rangle
\]

Caughran and Schwartz [34] observed that an operator \( A \) on a functional Hilbert space \( H \) is a composition operator if and only if the set \( \{ k_x, x \in X \} \) is invariant under \( A^* \). In this case the inducing function \( T \) is determined by

\[
A^*k_x = k_{T(x)}
\]

Let the kernel function \( K \) on \( H \) be defined as follows:

\[
K(x, y) = \langle k_y, k_x \rangle
\]

Let \( \tilde{K}(x_1, x_2, \ldots, x_n) \) be the matrix with \( K(x_i, x_j) \) at the \((i, j)\)th place. Then it has been proved that a map \( T \) of \( X \) into itself induces a composition operator on \( X \) if and only if there is a constant \( M \) such that

\[
\tilde{K}(Tx_1, \ldots, Tx_n) \leq M\tilde{K}(x_1, \ldots, x_n)
\]

for every finite subset \( \{x_1, x_2, \ldots, x_n\} \) of \( X \). In this case \( \|C_T\|^2 \) is the best possible constant.
Now we present some of the works of composition operators on the special functional Hilbert space $H^2(D)$, where $D$ is the open unit disc in the complex plane. $H^2(D)$ is defined as

$$H^2(D) = \{ f : f(z) = \sum_{n=0}^{\infty} a_n z^n \text{ for } z \in D \text{ and } \sum_{n=0}^{\infty} |a_n|^2 < \infty \}$$

The study of composition operators on $H^2(D)$ was initiated by Nordgren [101] and Ryff [127] exploiting the earlier results of Littlewood [93] and Rogosinski [122], [123]. Ryff proved that if $T$ is an analytic function mapping $D$ into itself, then $T$ induces a composition operator $C_T$ on $H^2(D)$ and

$$||C_T|| \leq \left( \frac{1 + |T(0)|}{1 - |T(0)|} \right)^{-\frac{1}{2}}$$

Schwartzs [128] also gave a similar result and characterised the composition operators on $H^2(D)$ as follows: An operator $A$ on $H^2(D)$ is a composition operator if and only if $Ae_n = (Ae_1)^n$ for $n = 0, 1, 2, \ldots$, where $e_n(z) = z^n$ for $n = 0, 1, \ldots$, form a basis for $H^2(D)$. He also observed that a non-zero operator $A$ on $H^2(D)$ is a composition operator if $A(fg) = (Af)(Ag)$ whenever $f$ and $g$ are bounded. He further established that a composition operator on
$H^2(D)$ is invertible if and only if $T$ is a Mobius transformation of $D$ onto $D$. Cima, Thomson and Wogen have proved that a composition operator on $H^2(D)$ is Fredholm if and only if it is invertible. They have also obtained a necessary and sufficient condition for the range of $C_T$ to be closed. The condition is that $d \lambda T^\wedge d\lambda$ is bounded away from zero, where $T^\wedge$ denotes the extension of $T$ to the closed disc $D$.

Caughran has obtained several conditions for the range of $C_T$ to be dense. If $G$ denotes the range of $T$, then it is a simply connected open subset of $D$. The space $H^2(G)$ can be defined in the same manner as $H^2(D)$. If $C_T$ has a dense range, then $T$ is an injection. But the converse is not true. Caughran has cited an example for the above and proved that if $T$ is an injection and $T(D)$ is a Carathéodory domain, then $C_T$ has dense range.

Compactness of composition operators on $H^2(D)$ has given rise to several outstanding results. Schwartz proved that if $C_T$ is compact, then $|T^\wedge| < 1$ (a.e.). He gave an example to show that the above condition is not sufficient. Shapiro and Taylor proved that a composition operator $C_T$ is Hilbert-Schmidt if and only if $1/1-|T^\wedge| \in L^1(\lambda)$. They have also found a necessary condition for a composition operator to be trace class. They further proved
that if \( T \) has an angular derivative at one point of the unit circle, then \( C_T \) is not compact. It has been proved by Caughman and Schwartz \([34]\) that if \( C_T^n \) is compact for some \( n \in \mathbb{N} \), then \( T \) has a fixed point in \( D \).

The spectra of composition operators on \( H^2(D) \) has been dealt with extensively by many Mathematicians. Nordgren \([101]\) took up the case when \( T \) is an inner function and has a fixed point in \( D \). In this case it has been proved that the spectrum of \( C_T \) is the closed unit disc. Nordgren also observed that if \( T \) is a Mobius transformation of \( D \) onto \( D \), then the behaviour of \( T \) depends on its fixed points. Here three cases arise. If \( T \) has one fixed point inside the unit circle and one outside the unit circle, then the spectrum of \( C_T \) is the closure of \( \{e^{in\theta}, n=0,1,\ldots\} \). If it has two fixed points on the unit circle, then the spectrum of \( C_T \) is an annulus centered at zero whose inner and outer radii are determined. Finally, if \( T \) has one fixed point on the unit circle, then the spectrum of \( C_T \) is the unit circle.

Deddens \([42]\) studied the composition operators given by \( T(z) = az + \beta \) where \( |a| + |\beta| < 1 \) and found spectra in these cases. Caughman and Schwartz \([34]\) showed that if \( C_T^n \) is compact for some \( n \in \mathbb{N} \), then \( T \) has a fixed point \( 'a' \) and in this case the spectrum of \( C_T \) is the
closure of the set consisting of 0,1 and the powers of $T^+(a)$. Kamowitz \cite{78} also found the spectra of composition operators when $T$ is analytic on $\overline{D}$.

Lubin \cite{94} dealt with the isometries induced by composition operators and proved that isometries are bilateral shifts and the subcollection of these isometries generates a reflexive algebra.

Apart from $H^2(D)$, composition operators have been studied on various spaces including the Banach space $H^p(D)$, the Bergman space $B(D)$ and the Hilbert space $H^2(\pi^+)$. Swanton \cite{163} concentrated on composition operators on $H^2$ of multiply connected domains. He employed the technique of harmonic majorants in the study of composition operators. He has also characterised the compact composition operators on $B(D)$, the uniform algebra of bounded analytic functions on a domain $D$.

Later Singh \cite{137} initiated the study of composition operators on $H^2(\pi^+)$, where $\pi^+$ is the upper half plane and $H^2(\pi^+)$ consists of all those functions $f$ which are holomorphic in $\pi^+$ and

$$||f||^2 = \sup_{y>0} \int_{-\infty}^{\infty} |f(x+iy)|^2 \, dx < \infty$$
It is shown that even though every holomorphic function from $D$ into itself induces a composition operator on $\mathcal{H}^2(D)$, this is not true in the case of $\pi^+$. In $|149|$, Singh and Sharma have characterised the holomorphic functions which induces composition operators on $\mathcal{H}^2(\pi^+)$. They have also characterised invertible composition operators on $\mathcal{H}^2(\pi^+)$ and non-compact composition operators $|150|$.

David Boyd $|20|$ has studied these operators on the Bergman space. He characterised compact, non-compact and Hilbert-Schmidt composition operators on $\mathcal{B}(D)$.

This concludes the review of some of the known results about composition operators on functional Hilbert spaces.

**COMPOSITION OPERATORS ON $L^2$-SPACES**

The study of operators induced by measurable and measure preserving transformations was taken up by Von Neumann $|99|$ as early as 1931 and Birkhoff $|18|$. Halmos $|60|$ made an extensive study of the ergodic transformations and compiled the available results. Earlier works were concentrated mainly on the operators induced by invertible measure preserving transformations.
Chokai \cite{35} studied the unitary composition operators on $L^2(X,\mathcal{B},\lambda)$ induced by measure preserving transformations, where $(X,\mathcal{B},\lambda)$ is a totally finite separable measure space. He proved that for a non-ergodic transformation with discrete spectrum, there exists an equivalent nonconjugate transformation. In \cite{36}, he took up the problem of when an unitary operator on an abstract Hilbert space $H$ is induced by a measure preserving transformation $T$ on $[0,1]$ when $H$ is represented as $L^2[0,1]$. This was done in two stages. First two distinct characterisations were obtained for these subsets of $H$ which could serve as the set of characteristic functions of measurable sets in a representation of $H$ as $L^2[0,1]$. Then for each of these characterisations necessary and sufficient conditions were obtained for a unitary operator to be induced by a measure preserving transformation. In \cite{37}, he continued his earlier work and obtained necessary and sufficient conditions for a unitary operator to be induced by an invertible, measurable, non-singular transformation $T$ in some representation of $H$ as $L^2[0,1]$.

Later composition operators on $L^2(\lambda)$ which are induced by general non-singular measurable transformations from $X$ into itself were taken up. Ridge \cite{118} and Singh \cite{134} have found out the condition which makes the composition transformation $C_T$ induced by a measurable non-singular
transformation $T$, a bounded operator on $L^2(\lambda)$. The condition is that there exists a constant $M > 0$ such that $\lambda T^{-1}(E) \leq M\lambda(E)$ for every measurable set $E$. So $C_T$ is a bounded operator implies that the measurable transformation is non-singular and hence the induced measure $\lambda T^{-1}$ is absolutely continuous with respect to the measure $\lambda$.

If $f_0$ denotes the Radon-Nikodym derivative of the measure $\lambda T^{-1}$ with respect to the measure $\lambda$, then it has been shown that for a composition operator $C_T$ on $L^2(\lambda)$, $||C_T||^2 = ||f_0||_\infty$, where $||.||_\infty$ indicates the essential sup. norm.

Improving upon a result of Von Neumann [100], Ridge [120] characterised composition operators on $L^2(\lambda)$. He proved that, if $\lambda$ is a sigma-finite measure on the Borel subsets of a standard Borel space and $A$ is an operator on $L^2(\lambda)$ such that $A(fg) = (Af)(Ag)$, whenever $f, g$ and $fg$ are in $L^2(\lambda)$, then $A$ is a composition operator. The invertibility of $C_T$ implies the invertibility of $T$ and the inverse of $C_T$ is induced by any inverse of $T$. But the invertibility of $T$ does not imply the invertibility of $C_T$. Examples are given by Singh [138]. In fact, he has proved that $C_T$ is invertible if and only if $T$ is invertible and $C_T^{-1}$ is in $B(L^2(\lambda))$, where $T^{-1}$ is an inverse of $T$.

Ridge also obtained that a self-adjoint composition operator on $L^2(\lambda)$ is a partial isometry. Singh
characterised hermitian composition operators and proved that every hermitian composition operator on $L^2(\lambda)$ is an isometry \(^1\)\(^{136}\). It is also shown in \(^1\)\(^{118}\) that no injective composition operator with dense range is compact on $L^2(\lambda)$, when the measure $\lambda$ is non-atomic. Later Singh \(^1\)\(^{136}\) removed the conditions in Ridge's theorem and proved that no composition operator on $L^2(\lambda)$ is compact when $\lambda$ is non-atomic. Compactness of composition operators on $L^2(\lambda)$ has also been dealt with by Singh and Kumar \(^1\)\(^{148}\). They have determined the spaces which admit and which do not admit compact composition operators.

Singh has also characterised quasinormal composition operators on $L^2(\lambda)$ and proved that $C_T^*C_T = M_{f_0}$, where $f_0$ is the Radon-Nikodym derivative of the measure $\lambda T^{-1}$ with respect to the measure $\lambda$. Singh and Kumar \(^1\)\(^{146}\) have characterised composition operators with closed ranges and also composition operators with dense ranges. Further they characterised all invertible, normal, unitary, isometric and co-isometric composition operators on $L^2(\lambda)$. Kumar \(^1\)\(^{87}\) has proved that if $X = [0,1]$ and $\lambda$ is the Lebesque measure, then $C_T$ on $L^2[0,1]$ is Fredholm if and only if it is invertible. He has also discussed partial isometric and hyponormal composition operators on $L^2(\lambda)$.

Singh and Komal have studied the composition operators on $l^2$ extensively. In \(^1\)\(^{144}\), they have characterised
all composition operators on $\ell^2$ and proved that $C_T$ is invertible if and only if $T$ is invertible. They further proved that normal, unitary, invertible and isometric composition operators on $\ell^2$ coincide and co-isometric, partial isometric and surjective composition operators coincide. In $|145|$ spectra of composition operators are discussed and the spectra of surjective composition operators are obtained in all cases. It has also been proved that every composition operator on $\ell^2$ has an invariant subspace. This result becomes all the more important due to the fact that no composition operator on $\ell^2$ is compact. The reducing subspaces of $C_T$ have also been given. Binormal and centered composition operators on $\ell^2$ have also been studied in $|82|$.

Singh and Gupta ($|140|$, $|141|$) have made a thorough study of composition operators on weighted sequence spaces. They have computed the adjoint of a composition operator and characterised normal, unitary, invertible, Fredholm, isometry and co-isometry composition operators on the weighted sequence spaces. In $|143|$, Singh, Gupta and Kumar have explored compact and Hilbert-Schmidt composition operators on the weighted sequence spaces. In $|56|$, it has been proved that paranormal and quasi-hyponormal composition operators coincide on $\ell^2_p$ if it is a finite measure space.
Very little is done about the spectra of composition operators on $L^2(\lambda)$. Ridge [119] took up the study of the spectra of composition operators on $L^2(\lambda)$ and announced some results about the symmetry of the spectra of composition operators. He proved that point spectrum, approximate point spectrum and spectrum has circular symmetry about zero except on the unit circle where they form union of subgroups.

Singh and Veluchamy [156], completely determined the spectrum of a normal composition operator when the number 1 is not in the essential range of $f_\omega$. Further, they obtained the weyl spectrum of a normal composition operator. In [155], they obtained the adjoint of a composition operator on $L^2(\lambda)$ and the necessary and sufficient condition for an operator on $L^2(\lambda)$ to be a composition operator is discussed when the underlying measure $\lambda$ is atomic. In [158] they studied Fredholm, essentially unitary and essentially normal composition operators on $L^2(\lambda)$ when the underlying measure-space is non-atomic.

Singh and Dharmadhikari [159] characterised compact and Fredholm composite multiplication operators.

Whitley [176] showed that $C_T$ is normal iff $T$ is measure preserving and $T^{-1}(\mathbb{E})$ is essentially all of $\mathbb{E}$; $C_T$ is quasinormal iff $T$ is measure preserving. Whitley and Harrington [67] examined several questions of semi-normality for composition operators and obtained characterisations for $C_T$ hyponormal, $C_T$ quasinormal and $C_T$ hyponormal. They also showed that, if $f_0 \geq f_0 \circ T$ then $C_T$ is hyponormal. Harrington [68] proved that on a non-atomic measure space $C_T$ is Fredholm iff $C_T$ is invertible.

Lambert [88] established criteria for hyponormality for weighted composition operators and showed that there are hyponormal weighted composition operators on infinite measure spaces which are not scalar multiplies of isometries. In [89], he characterised subnormal $C_T$ in terms of moment sequence. In [91], he showed how a subnormal composition operator may be extended to a normal composition operator. Further, he obtained a minimal normal composition operator extension of an arbitrary subnormal composition operator.

Lambert and Hoover [71] developed characterisations of essentially normal composition operators. Lambert
and Embry Wardrop [53], with a combination of measure theoretic and operator theoretic techniques, obtained criteria for $C_T$ to be centered.

Campbell and Dibrell [43] showed that the sufficient condition established by Harrington and Whitley for hyponormality of a composition operator is actually sufficient for all powers to be hyponormal.

In the present thesis, some further properties of composition operators on $L^2$ of a sigma finite measure space are explored. Composition operators belonging to some specific larger classes of operators are widely studied. This thesis comprises of six chapters. The first chapter contains a survey of the results on composition operators as a base to make the study more meaningful.

In chapter II, paranormal composition operators are discussed. Some preliminary results are quoted in the beginning. Different characterisations of $M$-paranormal composition operators are determined. Examples are included. It is shown that paranormal composition operators form a class larger than the class of hyponormal composition operators. Many properties of paranormal composition operators are also discussed. Finally, criteria for $C_T^*$ to be paranormal is obtained.
Chapter III deals with composition operators of class \( (M,k), k \geq 2 \). The chapter begins with the characterisation of such an operator. Examples of such operators are provided. A necessary condition for a composition operator to have dense range is developed and is used to discuss when \( C_T \) of \( (M,k) \) class becomes hyponormal. Further \( C_T^* \) of class \( (M,k) \) is discussed. The chapter ends with a result on partial isometry \( C_T \) of class \( (M,k) \).

Chapter IV is devoted to study composition operators of \( (N,k) \) class \( k \geq 2 \) and normaloid composition operators. Criteria for such operators are established and examples are included. Finally, \( C_T^* \) of \( (N,k) \) class is briefly discussed.

In chapter V, a detailed study of binormal composition operators on \( L^2(\lambda) \) is attempted. Several related characterisations of binormality for \( C_T \) on \( L^2(\lambda) \) are obtained. Examples cited show that binormal \( C_T \) need not be an injection and there exist binormal \( C_T \) which is a normaloid but not paranormal.

Chapter VI deals with miscellaneous results on composition operators. In section 1, ascents of quasi-hyponormal \( C_T \), \( C_T \) of class \( (M,k) \), \( C_T \) of class \( (N,k) \) and normaloid \( C_T \) are obtained. In section 2, sufficient
conditions for the product of quasinormal $C_T$ to be quasinormal are obtained. Further, two sets of necessary and sufficient conditions for the product of normal composition operators to be normal are also obtained. In section 3, methods of computing $C_T^*$ in some special cases are discussed.