Chapter - I

INTRODUCTION

This introductory chapter on “Some Aspects Of Transportation Problem” discusses briefly the transportation problems, their origin and development, the work carried out by various authors in this field and finally gives a brief summary of the work carried out by the author of the present thesis. This chapter is divided into three sections. Section I is concerned with the origin and development of transportation problem. Section II gives the related work done by various authors in the field of transportation. Finally, Section III outlines the work presented in the subsequent chapters of this thesis.

Section – I

Origin And Development Of Operations Research

The ambiguous term “Operations Research” was first coined by Mc Closky and Trefthan in the United Kingdom, a little more than three decades ago. Its initial development began during World War II in military context. The need to solve systematically and scientifically, strategic and tactical problems concerning the various military projects led to the birth of Operations Research. The overwhelming success of the military teams during the war attracted the attention
of industrial managers who were seeking solutions to their complex executive type problems. The rage caught on with various economists, business concerns and the government; and Operations Research was heralded as the panacea of many decision making processes. Operations Research is an interdisciplinary field comprising elements of mathematics, economics, computer science and engineering; and is concerned with the allocation of scarce resources, be they human, man made or natural. For example, capital, man power, raw materials, finished goods, time, machine capacities, warehouse limitations, market potentials and the like. Operations Research provides a rational and systematic way to handle decision problems.

Man’s longing for perfection finds expression in the theory of optimization, which encompasses the quantitative study of optima and methods for finding them. Leibniz based his coined word “optimum” on the Latin “optimus” meaning “best”. Since optimization involves selection of the most desirable decision in a very complex real life environment, it has obvious applications in the practical world of production, trade and politics, where sometimes small changes in efficiency spell the difference between success and failure. Although many phases of optimization theory have been known to mathematicians for centuries, the tedious and voluminous computations required prevented their political application. The development of rapid, inexpensive, automatic computers in the
middle of the twentieth century has not only made these older methods attractive, but also encouraged much new research on optimization.

In design, construction and maintenance of any system or project, one has to take many technological and managerial decisions at several stages. The ultimate goal of all such decisions is to either minimize the effort required or maximize the desired benefit, such as profits, quantities produced or other measures of effectiveness. Since the effort required or benefit desired in any practical situation can be expressed as a function of certain decision variables, optimization can be defined as the process of finding the conditions that give the maximum or minimum value of the utility function, called the objective. The decision variables may be independent of one another, or they may be related through one or more restraining conditions (or constraints), which may arise from a variety of sources, such as government, marketing, business, production, storage, raw materials or legal restrictions.

A mathematical program is an optimization problem in which the objective and the constraints (finite in number) are given as mathematical functions and functional relationships. A mathematical program thus has the form:

Optimize : \( z = f(x_1, x_2, x_3, \ldots, x_n) \) subject to \( g_i(x_1, x_2, x_3, \ldots, x_n) \leq, \geq, = b_i, \quad i = 1,2,3,\ldots,m.\)
Each of the m constraint relationships involves one of the three signs $\leq, \geq, =$. The function $z = f(x_1, x_2, x_3, \ldots, x_n)$ is called the objective function. The function $g_i$’s and the constants $b_i$'s are assumed to be known. It is usually assumed that the n decision variables $x_j$’s: $j = 1, 2, 3, \ldots, n$ are all non-negative integral values only, the problem is called an integer programming problem.

A mathematical program is linear if $f(x_1, x_2, x_3, \ldots, x_n)$ and each $g_i (x_1, x_2, x_3, \ldots, x_n)$, $i = 1, 2, 3, \ldots, m$ are linear in each of the variables. Linear programming is undoubtedly the most widely studied aspect of mathematical programming, since many practical situations, when approximated, lead to linear programming models.

Mathematically, a linear programming problem may be described as:

$$\max \text{imize} (\min \text{imize}) z = \sum_{j=1}^{n} c_j x_j$$

subject to

$$\sum_{j=1}^{n} a_{ij} x_j \leq b_i, i = 1, 2, 3, \ldots, m$$

$$\sum_{j=1}^{n} a_{ij} x_j \geq b_i, i = 1, 2, 3, \ldots, m$$

$$x_j \geq 0, j = 1, 2, 3, \ldots, n$$

where $a_{ij}$, $b_i$, $c_j$'s are assumed to be known constants. The values of the decision variables $x_j$’s which optimize the objective function subject to the constraints form an optimal solution.
The most systematic, powerful and widely used procedure for solving linear programming problems is the simplex method developed by George Dantzig in 1947 [31]. It is an algebraic iterative procedure to solve exactly any linear programming problem in a finite number of steps, or give an indication that there is an unbounded solution.

A very important application of linear programming is the one when a homogenous commodity available at different warehouses is to be transported to various markets and depots at minimum transportation costs/time. Such problems are called transportation problems. Pioneer work was done by Hitchcock [42] and Koopman [53] on these problems. Hitchcock formulated the transportation problem, and later Koopman exploited its special matrix structure which allowed the development of a solution procedure which is computationally more efficient, but essentially follows the exact steps of the simplex method. A transportation problem may be described as follows: Let there be m sources $S_1$, $S_2$, …$S_m$ with $a_i$; $i=1,2,3,…,m$ units of supply of a particular commodity and n destinations $D_1$, $D_2$, …,$D_n$ having $b_j$; $j=1,2,3,…,n$ units of demand respectively. It is assumed that $a_i$, $b_j > 0$. Let $c_{ij}$ be the unit cost of transportation from source $S_i$ to destination $D_j$. Since there is only one commodity, a destination can receive the demand from one or more sources. The problem is to determine the feasible shipping pattern from sources to destinations that minimizes the total
transportation cost. The basic assumption here is that the transportation cost of a given route is directly proportional to the number of units transported. The definition of “unit of transportation” will vary depending on the commodity transported. Let $x_{ij}$ be the number of units transported from source $i$ to destination $j$. Assuming that the total supply equals the total demand, that is $\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j$, the mathematical formulation of the standard transportation problem is:

$$\begin{align*}
\text{minimize} \quad & z = \sum_{i \in I} \sum_{j \in J} c_{ij} x_{ij} \\
\text{subject to} \quad & \sum_{j \in J} x_{ij} = a_i, \quad i \in I \\
& \sum_{i \in I} x_{ij} = b_j, \quad j \in J \\
& x_{ij} \geq 0; \quad i \in I, \; j \in J
\end{align*}$$

where $I = \{1,2,3,\ldots,m\}$ and $J = \{1,2,3,\ldots,n\}$. However, in real life, it is not necessarily true that the total supply equals the total demand. In such situations, source and / or destinations constraints are inequations as opposed to the usual equations. Such unbalanced transportation problems can be studied by developing equivalent standard transportation problems.

The special structure of this class of problems which led to simple and efficient techniques for solving them, arises due to the following facts:

(1) Each variable $x_{ij}$ occurs in just two constraints and
(2) The coefficient of each variable \(x_{ij}\) is either 1 or 0.

Thus the \((m+n) \times mn\) matrix of coefficients is of the special form

\[
\begin{bmatrix}
1_n & 0 & 0 & \ldots & 0 \\
0 & 1_n & 0 & \ldots & 0 \\
\ldots & \ldots & \ldots & \ldots & \ldots \\
0 & 0 & \ldots & \ldots & 1_n \\
1_n & 1_n & \ldots & \ldots & 1_n
\end{bmatrix}
\]

where \(1_n\) is an \(n\) row vector of all 1's and \(I_n\) is an \(n \times n\) identity matrix. Also \(0\) stands for an \(n\) row vector of all zeroes.

Transportation is a burning problem in our day to day life today. Millions of people travel everyday by air, water, rail or land. Extensive research is being carried out to improve the overall efficiency of such transport systems, which is going to benefit millions of people all over the world.

In practice, many problems occur which can not be cast as linear programming problems, and thus, non linear mathematical programming came in to picture. But there is no single algorithm available to solve all non linear programming problems. The solution technique depends on the characteristics of the particular problem. A special type of non linear programming problems is the time minimizing transportation problem where the objective function is non–linear but all the constraints are linear. A time \(t_{ij}\) is associated with each route \((i,j)\)
and it is required to minimize the time taken to complete the transportation from various sources to the destinations. Thus the objective function in this problem is

$$\max_{(i,j)} \left\{ t_{ij} / x_{ij} > 0 \right\}$$

which is required to be minimized. So a time minimizing transportation problem is mathematically formulated as:

\[
\begin{align*}
\text{minimize} & \quad \max_{(i,j)} \left\{ t_{ij} / x_{ij} > 0 \right\} \\
\text{subject to} & \\
\sum_{j \in J} x_{ij} = a_i, & i \in I \\
\sum_{i \in I} x_{ij} = b_j, & j \in J \\
x_{ij} & \geq 0, i \in I, j \in J
\end{align*}
\]

It is assumed that

1. The carriers have sufficient capacity to carry goods from an origin to a destination in a single trip.

2. They start simultaneously from their respective origins.

3. The time $t_{ij}$ is independent of the quantity $x_{ij}$.

Another class of non linear programming problems is an indefinite quadratic transportation problem where the objective function is a product of two linear functions. This gives more insight in to the situation than the optimization of each criterion. Many authors have contributed in the field of indefinite quadratic
transportation problem. Another interesting class of non-linear programming problems is that of a fractional functions programming where the objective function is a ratio of two linear functions. Optimization of a ratio of criteria often describes some kind of an efficiency measure for a system.
Section – II

Review Of The Literature

The classical transportation problem is one of the many well-structured problems in operations research that has been extensively used in literature. Other examples are the travelling salesman and shortest route problems. A transportation problem refers to a class of linear programming problems that involves selection of most economical shipping routes for transfer of a uniform commodity from a number of sources to a number of destinations. In case of an unbalanced transportation problem, the total availability is not equal to the total demand, thus some of the source and/or destination constraints are satisfied as inequalities. In transportation problem, the amount to be sent from each origin, the amount to be received at each destination, and the cost per unit shipped from any origin to any destination are specified. In literature, much effort has been concentrated on transportation problems with equality constraints. As transportation problem is easy to solve, “what–if” questions can be asked and answered relatively rapidly, both at strategic level and at the tactical level. For example- what if the capacity of a factory is substantially increased, how will this affect the overall production and transportation technique? Increasing factory capacity is a strategic decision and presumably an increase in factory capacity reduces the variable costs involved in production and transportation. Hence this
variable cost reduction should be set against the capital required to increase the factory capacity (a typical investment decision, balance cost savings against capital investment). Thus in real life, however, most problems have mixed constraints accommodating many applications that include job scheduling, production inventory, production distribution, allocation problems and investment analysis. There are different types of transportation problems and the simplest of them that is now standard in literature was first presented by Hitchcock [42] in 1941 along with a constructive solution and later independently by Koopman [53] in 1947. Koopman began to spearhead research on the potentialities of linear programs for the study of problems in economics. His historic paper “Optimum Utilization Of The Transportation Systems” was based on his war time experience. Because of this and the work done earlier by Hitchcock, the classical case is often referred as the Hitchcock Koopman’s transportation problem. Kantorovich [44] published a paper on a continuous version of the problem and later with Gavurin, an applied study of the capacitated transportation problem (Kantorovich and Gavurin [45] in 1949). F. Glover, D. Klingman and G. Terry Ross [35] in 1974 considered a constrained transportation problem, i.e. a transportation problem with an additional linear constraint; and shown how it can be transformed in to a larger equivalent standard transportation problem. The arbitrary linear constraint has been transformed in to an equivalent bounded partial sum of variables involving a
single node constraint. Then such problems have been reformulated into standard transportation problems by extending the works of Wagner [77] and Charnes [26] who reformulated transportation problems with bounded partial sums into standard transportation problems. Klingman and Russell [52] have developed an efficient procedure for solving transportation problems with additional linear constraints. Their method exploits the topological properties of basis trees within a generalized upper bounding framework. Srinivasan and Thompson [68] proposed implicit enumeration and branch and bound algorithms for problems in which each destination is supplied by a single source and later Murthy [56] considered the same problem and provided an efficient algorithm. G.M Appa [4] in 1973 considered all the 81 variants of the transportation problem obtained by taking all possible combinations. But Appa did not deal with the case where availability and requirement constraints are of mixed type. Brigden [25] in 1974 was the first author to have considered the mixed transportation problem.

In the classical transportation problem, the cost of transportation is directly proportional to the number of units of the commodity transported. But in real world situations, when a commodity is transported, a fixed cost is incurred in the objective function. The fixed cost may represent the cost of renting a vehicle, landing fees at an airport, set up cost for machines etc. Fixed charge problems arise in situations that involve the planning of several interdependent
activities some or all of which have set – up charges independent of the activity level as long as it is positive. These problems arise in many integer and non linear programming applications. Many of these problems are network problems with fixed charges attached to subsets of the arcs. Examples include the well known network expansion problems, plant location problems, process selection problems plus a wide variety of related investment and distribution problems. The decision “to invest or not invest”, “to build or not build”, “to ship or not ship”, can be modelled by imposing fixed charges on appropriate arcs of the network. The fixed charge transportation problem was originally discussed by W. Hirsch and G.B. Dantzig [41] in 1954. He established that the feasible region of the general fixed charge problem is a bounded convex set, that its optimal solution will be at an extreme point of this set, and that a local minimum solution is not necessarily a global minimum. After that several procedures for solving fixed charge transportation problems were developed. Sandrock [59] gave a simplex algorithm for solving a fixed charge transportation problem. Sandrock made an effort to introduce fixed charge formulated as a step function of the load assigned to each source. The problem analyzed by him constituted the so called source induced fixed charge transportation problem, which has a completely different formulation compared to the step fixed charge transportation problem. The step fixed charge transportation problem is a variation of the fixed charge transportation problem where the fixed cost is in the form of a step function.
dependent on the load in a given route. While the value of the objective function $z$ in the fixed charge transportation problem is a step function, the introduction of the step fixed cost in the step fixed charge transportation problem results in the objective function $z$ being itself a step function with many more steps. Kowalski and Benjamin [54] presented a computationally simple heuristic algorithm for solving small step fixed charge transportation problems. Adlakha et.al. [1] proposed more–for–less algorithm for fixed charge transportation problem. The more–for–less phenomenon in distribution problems occurs when it is possible to ship more total goods for less total cost, while shipping the same quantity or more from each origin and to each destination. This paradox occurs often in fixed charge transportation problems and further analysis could bring significant reduction in costs. The more–for–less phenomenon for fixed charge transportation problems had received minimal attention in the literature despite the fact that analytical algorithms such as branch and bound are limited to small problems due to excessive computational effort. They developed a simple heuristic algorithm to identify the demand destinations and the supply points to ship more for less in fixed charge transportation problems. Basu et.al. [17] proposed an algorithm for finding the optimal solution of solid fixed charge transportation problem. Fixed charge transportation problems have been studied by Arora et.al.[6], Thirwani [74] and many others. Veena Adlakha et.al. [2] provided a branching method for the fixed charge transportation problem.
Starting with a linear formulation of the problem, a method is developed which converges to the optimal solution by sequentially separating the fixed costs and finding a direction to improve the value of the linear formulation. The method was based on the computation of lower bound and upper bound embedded within a branching process. The iterative procedure continually tightens the lower and upper bounds as it progresses. The optimal solution is achieved when the two bounds are matched. There is a wide scope of capacitated transportation problem with bounds on rim conditions. It can be used extensively in telecommunication networks, production–distribution system, rail and urban road system when there is a limited capacity of resources such as vehicles, docks, equipment capacity, location shipping etc. These are bounded variable transportation problems. Many researchers like A. K. Bit et.al [24], Dahiya and Verma [30] have contributed in this field. In the present thesis, an algorithm for solving a capacitated fixed charge transportation problem with bounds on rim conditions is presented [12].

The time of transportation might be a significant factor in several transportation problems. The time minimizing or Bottleneck transportation problem is a special case of a transportation problem in which time is associated with each shipping route. Rather than minimizing cost, the objective is to minimize time to transport all supply to the destination. In a bottleneck transportation problem, a set of supplies and a set of demands are specified such
that the total supply is equal to the total demand. There is a transportation time associated between each supply point and each demand point. It is required to find a feasible distribution (of the supplies) which minimizes the maximum transportation time associated between a supply point and a demand point such that the distribution between the two points is positive. In addition to this, one may wish to find from among all optimal solutions to the bottleneck transportation problem, a solution which minimizes the total distribution that requires the maximum time. Hammer [39] introduced the problem and an algorithm in English literature. In 1971 Szwarc [72] noted that the problem was originally proposed ten years earlier by a Russian named Bar Sow and that several related papers were published in East European Journals. He also corrects some errors in Hammer’s original paper. Garfinkel and Rao [34] presents a primal approach and a threshold algorithm which generates a sequence of improving lower bounds on the maximum time. Hammer [40] noted the work of Szwarc and Garfinkel and Rao in correcting his original paper. In 1978 Sharma and Swarup [65] and Bhatia et.al. [19] had given iterative methods for the solution of time – minimizing transportation problem. The procedure involves finite iterations and is based on moving from a basic feasible solution to another till the last solution is arrived at. Bhatia et.al. [20] present an algorithm for the solution of time minimizing multi- index transportation problem having three indices which is known as solid transportation problem. In 1977, Sharma and
Swarup [64] presented an iterative method for solving a time minimizing multi-dimensional transportation problem having three indices. In 1978, a transportation technique for time minimization in fractional functional programming problem was developed by the same authors[65]. In 2010 Sharma, Dahiya and Verma [67] studied capacitated two stage time minimization transportation problem in which the total availability of a homogenous product at various sources is more than the total requirement of the same at destinations. In this problem, transportation takes place in two stages such that the minimum requirement of the destinations is satisfied in the first stage and the surplus amount is transported in the second stage. Each time the transportation from sources to destinations is done parallel and the capacity on each route remains fixed i.e. the total amount transported in both the stages can not exceed its upper bound. In each stage, the objective is to minimize the shipment time and the overall goal is to find a solution that minimizes the sum of first and second stage times. In 2011, Pandian and Natarajan [58] proposed a new method namely blocking method for solving bottleneck – cost transportation problems which is very different from other existing methods. Then, another method namely Blocking zero point method was proposed for finding all efficient solutions to a Bottleneck Cost transportation problem which is based on Zero Point method. These methods provide the necessary decision support to the user while handling time oriented logistic problems. By blocking zero point method, a sequence of
optimal solution to a bottleneck – cost transportation problem is obtained for a sequence of various time in a time interval. This method provides a set of transportation schedules to bottleneck – cost transportation problems which helps the decision makers to select an appropriate transportation schedule, depending on his financial position and the extent of bottleneck that they can afford. The blocking zero point method enables a decision maker to evaluate the economical activities and make the correct managerial decisions. In the present thesis, a technique for minimizing time in a capacitated transportation problem with bounds on rim conditions is presented [13]. The procedure involves finite iterations and is based on moving from one extreme point to another extreme point till the optimal solution is reached.

In 1976, V. Srinivasan and G. L. Thompson [69] gave algorithms for attacking optimally the total cost, bottleneck time and the bottleneck shipment simultaneously in a transportation problem. They gave an algorithm to determine all efficient cost – time solution pairs. The algorithms are based on the cost operator theory of parametric programming for the transportation problem developed by the same authors [68]. In between the two extremes of minimization of cost and minimization of time, there exists situations (such as police, fire or ambulance services) where the decision maker does not mind a partial trade off on cost to attain a certain degree of time advantage. A technique for obtaining
such trade–off relationships was developed by H.L. Bhatia, K. Swarup and M.C. Puri [22] in 1976. It starts with a solution with minimum time and leads ultimately to a solution giving minimum cost of transportation. An enumerative technique is developed to obtain successive time–cost commodity in pipe-line trade off relationships in a transportation problem. The time–cost minimum commodity in the pipe–line trade off relationships are also obtained in the process. The procedure leads ultimately to the minimum time (with minimum commodity in the pipeline at this time) of transportation of goods from sources to the destinations. The algorithm also gives the minimum time at a given cost for which the transportation schedule is known. In this procedure, the extreme point solutions of the ordinary cost transportation problem are enumerated until the minimum time of transportation with minimum pipeline is achieved. Basu et al. [18] developed an algorithm for the optimum time cost trade off in a fixed charge linear transportation problem giving same priority to cost as well as time. Satyaprakash [60] considered a transportation problem with two objectives – one primary and the other secondary. The primary objective is to minimize the total cost of transportation and the secondary objective is to minimize the duration of transportation. This problem has been reduced to a goal programming type problem which readily lends itself to solution by the standard transportation method. Sometimes, there may exist emergency situations such as fire services, ambulance services, police services etc when the time of transportation is more
important than cost of transportation. If the total flow in a transportation problem with bounds on rim conditions is also specified, the resulting problem makes the transportation problem more realistic. Moreover, if the total capacity of each route is also specified then optimal solution of such problems is of greater importance which gives rise to capacitated transportation problems. Khurana et.al.[49] studied time cost trade off in an indefinite quadratic transportation problem with restricted flow. In the present work, an algorithm is developed to obtain time cost trade off pairs in a capacitated fixed charge transportation problem with bounds on rim conditions [14]. Then an algorithm for solving a capacitated fixed charge bi – criterion indefinite quadratic transportation problem with restricted flow is presented [8]. The algorithm is developed by forming a related fixed charge indefinite quadratic transportation problem and it is shown that to each basic feasible solution called corner feasible solution to related transportation problem , there is a corresponding feasible solution to the given restricted flow problem. It is also shown that the efficient time- cost trade off pairs to the given problem are derivable from this related problem . An algorithm to find time -cost trade off pairs in a fractional capacitated transportation problem with restricted flow is also developed by the same authors [11]. Here the objective function is a ratio of two linear functions consisting of variable costs and profits respectively.
In the standard transportation problem where the objective is to minimize the total cost of transporting a homogenous product from various supply points to various destinations, the total flow is $\sum a_i = \sum b_j$. Necessitated by situations where due to some emergency, say, there may be a need to keep reserve stocks at the supply points or there may be a shortfall in the level of production or there may be an extra demand in the markets, the total flow in the system needs to be either curtailed or enhanced. Thus there is an additional flow constraint in the system. This additional constraint breaks the transportation structure of the problem. It gives rise to restricted flow problems where one wishes to keep reserve stocks at the origins for emergencies, thereby restricting the total transportation flow to a known specified level say $P \left( \min\left(\sum a_i, \sum b_j\right) \right)$. S. Khanna and Puri [46] solved a transportation problem with mixed constraints and specified transportation flow. Khanna, Bakshi and Arora [47] proposed a method to solve time minimization transportation problem with restricted flow. Thirwani et.al.[73] proposed an algorithm for solving fixed charge bi-criterion transportation problem with restricted flow in 1997. This motivated the authors to develop methods for solving restricted flow in non-linear capacitated transportation problem and in indefinite quadratic transportation problem [8,9]. An algorithm to study restricted flow in fractional capacitated transportation
A situation parallel to the one of restricted flow is that of enhanced flow. At times, situations may arise when there is an extra demand in the market for the commodities. In such cases, the factories are compelled to increase their production. As a result, the total flow in the system is enhanced by the amount of extra demand. Such a situation gives rise to enhanced flow problems. The enhanced flow is a specified quantity $P \left( > \max \left( \sum_i a_i, \sum_j b_j \right) \right)$. Khurana et al. [48] studied restricted flow and enhanced flow in sum of a linear and linear fractional transportation problem. An algorithm to solve fixed charge bi-criterion indefinite quadratic transportation problem with enhanced flow was also developed by the same authors in 2011 [50].

W. Szwarc [71] in 1971 observed an unusual phenomenon called paradox that arises when there is a solution of a transportation problem involving lesser cost than the optimal cost and is available by shipping larger quantities of goods over the original optimal routes. So a paradox arises in the sense that more is being shipped at a lesser cost. He developed the transportation scheme method for finding a paradoxical pair. As his approach was confined to a particular set of routes only, he was unable to identify the exhaustive set of existing paradoxical pairs. In 2000, Arora et al. [5] developed a method to find existing paradoxical range of flows and the best paradoxical pair. Khurana et al. [51] studied paradox
in indefinite quadratic transportation problem. K. Dahiya and V. Verma [29] discussed paradox in fixed charge capacitated transportation problem where the objective function is the sum of two linear fractional functions consisting of variable costs and fixed charges respectively. They established the sufficient condition for the existence of a paradox and also obtained the paradoxical range of flow and the best paradoxical pair. This motivated the authors to establish a sufficient condition for the existence of a paradox in a fractional capacitated transportation problem [15]. In the present thesis, paradoxical range of flow is obtained for any given flow in which the corresponding objective function value is less than the optimum value of the given transportation problem. If a paradox exist, one would obviously be interested in the best paradoxical pair. The present work finds the best paradoxical pair in a fractional capacitated transportation problem. Joshi and Gupta [43] developed a heuristic for finding the initial basic feasible solution for linear plus linear fractional transportation problem and also established the sufficient condition for the existence of a paradoxical solution in 2010.

The traditional transportation problem involves two indices on the variables and the costs. In 1955 E.D. Schell [61] was the first to introduce the concept of multi-dimensional transportation problem. In such a problem, there are various types of products which are to be distributed between a set of supply
points and destinations. The equation then give rise to the conditions on the amount of the various types of combination that is available and required. Alternatively, there may be a single product which has to be transported by different modes such as road, rail, sea, canal, air etc. Similarly a multi – index formulation will be required if intermediate depots are to be used. Transportation problems in three dimensions are called solid transportation problems. The solid transportation problem was first studied by K.B. Haley [36-38]. The multi – index transportation problem was solved by Deepak Bammi [16] in 1978. His algorithm is an extension of the MODI method. C. Das and S. Misra [32] had studied a three axial sum transportation problem with mixed type of constraints similarly to the mixed problem considered by Brigden [25] and D. Klingman and R. Russel [52]. The problem of minimizing the total time of transportation in a solid transportation problem had been studied by Bhatia, Swarup and Puri [21] in 1976. The time – cost trade off in the solid transportation problem was discussed by H. L. Bhatia and M.C. Puri in 1977 [23]. In 1994, M. Basu, B.B. Pal and A. Kundu [17] presented an algorithm for finding the optimum solution of solid fixed charge transportation problem. Arora et.al [6] studied multi – index fixed charge bi- criterion transportation problem in 2001. In 2004, three dimensional fixed charge bi- criterion indefinite quadratic transportation problem was studied by S.R. Arora and A. Khurana [7].
In 1970, Swarup [70] developed a technique similar to transportation technique in linear programming to minimize a locally indefinite quadratic function conditions for local optimality have been obtained. Sharma and Swarup [63] had developed the same concepts for multi-dimensional transportation problem. Indefinite quadratic transportation problem where the objective function to be optimized is the product of two linear functions gives more insight into the situation than the optimization of each criterion. Khurana et. al. [49-51] have contributed a lot in the field of indefinite quadratic transportation problem. In section I of chapter III of the present work, an algorithm is suggested for solving a capacitated fixed charge bi-criterion indefinite quadratic transportation problem. The authors also presents an algorithm to find the efficient cost-time trade off pairs in a capacitated fixed charge bi-criterion indefinite quadratic transportation problem with restricted flow [8].

Transportation problems with nonlinear objective function was studied by Sharma [62]. Non linear programs finds its application in a variety of real world problems such as stock cutting problem, resource allocation problems, routing problem for ships and planes, cargo-loading problem, inventory problem and many other problems. Dorina Moanta [33] presented a double sum model in which the objective function is the ratio of two positive linear functions in a three
dimensional transportation problem. He explained how to obtain optimum with simplex method. Dahiya and Verma [29] studied paradox in a non linear capacitated transportation problem. In section II of chapter III of the present work, restricted flow in a non linear capacitated transportation problem with bounds on rim conditions is studied [9]. Transportation problem with fractional functional objective function is studied by many authors [15]. Optimization of a ratio of criteria often describes some kind of an efficiency measure for a system. In 1962, Charnes and Cooper [27] presented programming with linear fractional functionals. They presented a complete analysis and explicit solution for the problem of linear fractional programming with internal programming constraints whose matrix is of full row rank. They showed that a linear programming problem with a linear fractional objective function could be solved by solving at most two ordinary linear programming problems. In addition to this, they showed that where it is known as priori that the denominator of the objective function has a unique sign in the feasible region, only one problem need to be solved. In 1971, Almogy and Levin [3] studied a class of fractional programming problems where a sum of linear or concave convex fractional functions on closed and bounded polyhedral sets is maximized. Verma and Puri [76] studied paradox in a linear fractional transportation problem in 1991. Section I of chapter IV of the present thesis studies the constraint of restricted flow in fractional capacitated transportation problem with bounds on total
availabilities at sources and total destination requirements. Section II of chapter IV of the present work shows the optimality criterion for the solution of fractional capacitated transportation problem. Further, condition for the existence of a paradox in a fractional capacitated transportation problem is established.
Section – III

Summary of the thesis

This section gives a brief summary of the research work carried out in chapters II, III and IV of the present thesis. Chapter II of the thesis consists of two sections. This chapter deals with the linear capacitated transportation problem with bounds on rim conditions.

A transportation problem refers to a class of linear programming problems that involves selection of most economical shipping routes for transfer of a uniform commodity from a number of sources to a number of destinations. Moreover, the cost of transportation is directly proportional to the number of units of the commodity transported. Necessitated by situations, when there is a limited capacity of resources (such as vehicles, docks, equipment capacity, location shipping etc.), transportation problems with bounds on total source availabilities and total destination requirements arise. If the total flow in a transportation problem with bounds on rim conditions is also specified, the resulting problem makes the transportation problem more realistic. In addition to this, if the total capacity of each route is also specified then optimal solution of such problems is of greater importance which gives rise to capacitated transportation problems. It can be used extensively in telecommunication networks, production –
distribution system, rail and urban road system. In real world situations, when a commodity is transported, a fixed cost is incurred in the objective function. The fixed cost may represent the cost of renting a vehicle, landing fees at an airport, set up cost for machines etc. Section I of chapter II is devoted to the study of fixed charge capacitated transportation problem with bounds on rim conditions. Initially, a fixed charge capacitated transportation problem with bounds on rim conditions is formulated which is of the form –

\[(P1): \min \{ \sum_{i \in I} \sum_{j \in J} c_{ij}x_{ij} + \sum_{i \in I} F_i \}\]

subject to

\[a_i \leq \sum_{j \in J} x_{ij} \leq A_i \quad \forall \ i \in I \quad (2.1.1)\]

\[b_j \leq \sum_{i \in I} x_{ij} \leq B_j \quad \forall \ j \in J \quad (2.1.2)\]

\[l_{ij} \leq x_{ij} \leq u_{ij} \quad \text{and integers} \quad \forall \ i \in I, j \in J \quad (2.1.3)\]

where \( I = \{1, 2, \ldots, m\} \) is the index set of m origins.

\( J = \{1, 2, \ldots, n\} \) is the index set of n destinations.

\( x_{ij} = \text{number of units transported from } i^{th} \text{ origin to } j^{th} \text{ destination} \).
\( c_{ij} \) = cost of transporting one unit of commodity from \( i^{th} \) origin to \( j^{th} \) destination.

\( l_{ij} \) and \( u_{ij} \) are the bounds on number of units to be transported from \( i^{th} \) origin to \( j^{th} \) destination.

\( a_i \) and \( A_i \) are the bounds on the availability at the \( i^{th} \) origin, \( i \in I \)

\( b_j \) and \( B_j \) are the bounds on the demand at the \( j^{th} \) destination, \( j \in J \)

\( F_i \) is the fixed cost associated with \( i^{th} \) origin.

For the formulation of \( F_i \) (\( i=1,2 \ldots m \)), we assume that \( F_i \) (\( i = 1, 2 \ldots m \)) has \( p \) number of steps so that

\[
F_i = \sum_{l=1}^{p} F_i \delta_{il}, i = 1, 2, 3 \ldots \ldots m
\]  \hspace{1cm} (2.1.4)

where,

\[
\delta_{il} = \begin{cases} 
1 & \text{if } \sum_{j=1}^{n} x_{ij} > a_{il} \\
0 & \text{otherwise}
\end{cases}
\]  \hspace{1cm} \text{for } l=1,2,3\ldots \ldots p, \; i=1,2,\ldots \ldots m

Here, \( 0 = a_{i1} < a_{i2} \ldots < a_{ip} \). Also \( a_{i1}, a_{i2} \ldots , a_{ip} \) (\( i = 1, 2, \ldots m \)) are constants and \( F_{il} \) are the fixed costs \( \forall \; i=1, 2 \ldots m, \; l=1,2 \ldots p \)

In order to solve problem (\( P_1 \)), a related transportation problem (\( P_2 \)) is constructed which is as follows.

\[
(P2): \min \left( \sum_{i \in I} \sum_{j \in J'} c_{ij}' y_{ij} + \sum_{i \in I} F_i' \right) \text{ subject to}
\]
\(\sum_{j \in J} y_{ij} = A'_i \quad \forall i \in I'\)  \hspace{1cm} (2.1.5)

\(\sum_{i \in I'} y_{ij} = B'_j \quad \forall j \in J'\)  \hspace{1cm} (2.1.6)

\(l_{ij} \leq y_{ij} \leq u_{ij} \quad \forall i \in I, j \in J\)  \hspace{1cm} (2.1.7)

\(0 \leq y_{m+1,j} \leq B_j - b_j \quad \forall j \in J\)  \hspace{1cm} (2.1.8)

\(0 \leq y_{i,n+1} \leq A_i - a_i \quad \forall i \in I\)  \hspace{1cm} (2.1.9)

\(y_{m+1,n+1} \geq 0\) and integers

where \(A'_i = A_i \quad \forall i \in I, \quad A'_{m+1} = \sum_{j \in J} B_j \quad \forall j \in J, \quad B'_j = B_j \quad \forall j \in J, \quad B'_{n+1} = \sum_{i \in I} A_i \)

\(c'_{ij} = c_{ij} \quad \forall i \in I, j \in J, \quad c'_{m+1,j} = c'_{i,n+1} = c'_{m+1,n+1} = 0 \quad \forall i \in I, \quad \forall j \in J\)

\(F'_i = F_i \quad \forall i = 1, 2 \ldots m, \quad F'_{m+1} = 0\)

\(I' = \{1, 2, \ldots m, m+1\}, \quad J' = \{1, 2, \ldots n, n+1\}\)

It is shown that a feasible solution of the given problem (P1) bear one – to – one correspondence with the feasible solution of the related problem. Further, it is established that the value of objective function of above problem at a feasible solution is equal to value of objective function of related problem at its
corresponding feasible solution and conversely. It is also shown that there is a one
to one correspondence between the optimal solution to above problem and the
optimal solution to related transportation problem. An algorithm is presented to
solve a capacitated fixed charge transportation problem with bounds on rim
conditions. The algorithm is then illustrated with the help of a real life problem.

Section II of chapter II is concerned with a special class of transportation
problem called bottleneck capacitated transportation problem with bounds on rim
conditions. The time minimizing transportation problem is one in which a time is
associated with each shipping route. Rather than minimizing cost, the objective is
to minimize the maximum time to transport from all origins to all the destinations.
The technique developed in this section minimizes the maximum time of
transporting goods from suppliers to the consumers in a special class of
capacitated transportation problems with bounds on total availabilities at sources
and total requirements at destinations. The technique is based on the assumption
that the carriers have sufficient capacity to carry goods from an origin to a
destination in a single trip. Moreover, they start simultaneously from their
respective origins. The mathematical formulation of this problem is:

\[
(P3): \min T = \max_{(i,j)} \{ t_{ij} | x_{ij} > 0 \} \quad \text{subject to the constraints (2.1.1) to (2.1.3)}. 
\]
In order to solve the problem (P3), a related transportation problem (P4) is formed that bears the following mathematical formulation.

\[(P4): \quad \text{min } T' = \max_{(i,j)} t'_{ij} \quad \text{subject to the constraints (2.1.5) to (2.1.9)} \]

Also

\[t'_{ij} = t_{ij} \quad \forall i \in I, j \in J, \quad t'_{i,n+1} = t'_{m+1,n+1} = 0 = t'_{m+1,j} \quad \forall i \in I, j \in J\]

The procedure involves finite iterations and is based on moving from one extreme point to another extreme point till the optimal solution is obtained. A solution \(X^2\) is said to be better than \(X^1\) if either (i) \(T^2 < T^1\) i.e. time is improved or (ii) if \(T^2 = T^1, p_2 < p_1\) i.e. amount of commodity in the pipeline at time \(T^2\) is less than the amount of commodity in the pipeline at time \(T^1\) in solution \(X^1\). It is shown that the problem (P3) and related transportation problem (P4) are equivalent. For this, it is derived that every feasible solution of the problem (P3) bears one – to – one correspondence with the feasible solution of the related problem (P4). Further, it is established that the value of the objective function of problem (P3) at a feasible solution is equal to value of objective function of related problem (P4) at its corresponding feasible solution and conversely. Finally, it is shown that there is a one to one correspondence between the optimal solution to problem (P3) and the optimal solution to problem (P4). An algorithm is developed in which a related transportation problem is formed that minimizes the total time necessary for transporting goods from the suppliers to the consumers. The procedure is bound
to converge in a finite number of steps as it involves movement from one basic feasible solution to another better basic feasible solution. Military transportation problem of Indian Army has been discussed to illustrate the algorithm.

Section III of chapter II is an extension of the first two sections. The objective of section I and II are unified in section III with the same set of constraints. Section III of chapter II is concerned with finding the optimum time cost trade off pairs in a capacitated fixed charge transportation problem giving the same priority to both time and cost. The two objectives of cost minimization and time minimization in a capacitated fixed charge transportation problem is unified. The Mathematical Formulation of this Problem is given below.

(P5): \[ \min \{ \sum_{i \in I} \sum_{j \in J} c_{ij}x_{ij} + \sum_{i \in I} F_i, \max (t_{ij}/x_{ij} > 0) \} \] subject to (2.1.1) to (2.1.3).

In order to solve problem (P5), we first separate it into two problems (P6) and (P7) which are as follows.

(P6): \[ \min \left( \sum_{i \in I} \sum_{j \in J} c_{ij}x_{ij} + \sum_{i \in I} F_i \right) \] subject to (2.1.1), (2.1.2) and (2.1.3) and

(P7): \[ \max \left( t_{ij}/x_{ij} > 0 \right) \] subject to (2.1.1), (2.1.2) and (2.1.3)

The problem is solved by re-optimization procedure which is explained below.
1. First, minimize cost without considering time and then minimize time with respect to the minimum cost obtained.
2. Secondly, modify cost with respect to minimum time so obtained as follows.

\[ c'_{ij} = \begin{cases} 
M & \text{if } t_{ij} \geq T^1 \\
c_{ij} & \text{if } t_{ij} < T^1 
\end{cases} \]

where M is a sufficiently large positive number.

Now minimize time with respect to the minimum cost of the last result. This step is repeated until the problem becomes infeasible.

To obtain the set of efficient time cost trade off pairs, first solve (P6) and read the time with respect to the minimum cost Z where time T is given by problem (P7). At the first iteration, let \( Z_1^* \) be the minimum total cost of the problem (P6) and \( T_1^* \) be the optimal time of the problem (P7) with respect to \( Z_1^* \) then any schedule which is completed earlier than \( T_1^* \) would cost more than \( Z_1^* \). Then \( (Z_1^*, T_1^*) \) is called the first time cost trade off pair.

After modifying the cost with respect to the time obtained, a new optimal solution is obtained and time is read with respect to the new optimal solution so obtained. Let after \( q^{th} \) iteration, the solution becomes infeasible. Thus, we get the complete set of time cost trade off pairs as \( (Z_1^*, T_1^*), (Z_2^*, T_2^*), (Z_3^*, T_3^*), \ldots \)

\( (Z_q^*, T_q^*) \) where \( Z_1^* \leq Z_2^* \leq Z_3^* \leq \ldots \leq Z_q^* \) and \( T_1^* \geq T_2^* \geq T_3^* \ldots \geq T_q^* \) with strict inequality holding in at least one of the two conditions. The pairs so obtained are the pareto-optimal solutions of the given problem. Then we identify the minimum cost \( Z_1^* \) and minimum time \( T_q^* \) among the above trade off pairs.
The pair \((Z_1^*, T_{q}^*)\) with minimum cost and minimum time is termed as the ideal solution which can not be achieved in practical situations. An algorithm is developed to obtain the cost – time trade off pairs in a capacitated fixed charge transportation problem with bounds on rim conditions. The algorithm will converge after a finite number of steps because we are moving from one extreme point to another and the choice of \(c_{ij}\)'s will ensure that an infeasible solution will be obtained after a finite number of iterations. The algorithm is then illustrated with the help of a real world problem.

Linear functions are widely used in modeling a mathematical optimization problem. Also quadratic functions and quadratic problems are the least difficult to handle out of all non linear programming problems. A fair number of functional relationships occurring in the real world are truly quadratic. For example-Kinetic energy carried by a rocket or an atomic particle is proportional to the square of its velocity. There are many non linear relationships occurring in nature that are capable of being approximated by quadratic functions. Chapter III of the present work deals with non linear capacitated transportation problems. Section I and II of chapter III deals with indefinite quadratic transportation problem. Section I is devoted to capacitated fixed charge bi–criterion indefinite quadratic transportation problem where the objective function is the product of two linear functions that gives more insight in to the situation than the optimization of each criterion.
Mathematical model of a capacitated fixed charge bi-criterion indefinite quadratic transportation problem is

\[(P8): \min \left\{ \left( \sum_{i \in I} \sum_{j \in J} c_{ij} x_{ij} \right) \left( \sum_{i \in I} \sum_{j \in J} d_{ij} x_{ij} \right) + \sum_{i \in I} F_i, \max \left( t_{ij} / x_{ij} > 0 \right) \right\} \text{ subject to} \]

\[
\sum_{j \in J} x_{ij} \leq a_i; \forall i \in I \quad (3.1.1)
\]

\[
\sum_{i \in I} x_{ij} = b_j; \forall j \in J \quad (3.1.2)
\]

\[
l_{ij} \leq x_{ij} \leq u_{ij}; \forall (i, j) \in I \times J \quad (3.1.3)
\]

\(c_{ij}\) = variable cost of transporting one unit of commodity from the \(i^{th}\) origin to the \(j^{th}\) destination.

\(d_{ij}\) = the per unit damage cost or depreciation cost of commodity transported from \(i^{th}\) origin to the \(j^{th}\) destination.

\(l_{ij}\) and \(u_{ij}\) are the bounds on number of units to be transported from \(i^{th}\) origin to \(j^{th}\) destination.

\(t_{ij}\) is the time of transporting goods from \(i^{th}\) origin to the \(j^{th}\) destination.

\(x_{ij}\) = number of units transported from origin \(i\) to the destination \(j\).
The fixed cost $F_i$ depends upon the amount supplied from the $i^{\text{th}}$ origin to different destinations and is defined in the same way as we define (2.1.9) in chapter I. In the problem (P8), we minimize the total transportation cost and depreciation cost simultaneously. Further we find the different cost – time trade off pairs. A dummy destination is introduced with demand $b_{n+1} = \sum_{i=1}^{m} a_i - \sum_{j=1}^{n} b_j$ to balance the problem (P8). The cost and time allocated in this dummy column is zero. The resulting balanced problem is then separated into two problems (P9) and (P10) as follows.

(P9): minimize the cost function
$$\left\{ \sum_{i \in I} \sum_{j \in J} c_{ij} x_{ij} \right\} \left\{ \sum_{i \in I} \sum_{j \in J} d_{ij} x_{ij} \right\} + \sum_{i \in I} F_i$$
subject to (3.1.1) to (3.1.3) and

(P10): minimize the time function
$$\left\{ \max_{i \in I, j \in J} \left( \frac{t_{ij}}{x_{ij}} > 0 \right) \right\}$$
subject to (3.1.1) to (3.1.3)

The problem is then solved by Re – optimization procedure as discussed in section III of chapter II. Moreover, optimality condition for the solution of problem (P9) is derived in this section. Further an algorithm is presented to find the efficient cost time trade off pairs in a capacitated fixed charge bi-criterion indefinite quadratic transportation problem. The algorithm is then illustrated with the help of a numerical example. Section II of chapter III is an extension of section I in which an additional flow constraint is introduced. Moreover, bounds
on rim conditions is also considered. The mathematical model of this problem is

(P11): \( \min \left\{ \sum_{i \in I} \sum_{j \in J} c_{ij} x_{ij} \left( \sum_{i \in I} \sum_{j \in J} d_{ij} x_{ij} \right) + \sum_{i \in I} F_i, \max \left(t_{ij} / x_{ij} > 0\right) \right\} \) subject to

\[ a_i \leq \sum_{j \in J} x_{ij} \leq A_i \quad \forall i \in I \quad (3.2.1) \]

\[ b_j \leq \sum_{i \in I} x_{ij} \leq B_j \quad \forall j \in J \quad (3.2.2) \]

\[ l_{ij} \leq x_{ij} \leq u_{ij} \quad \forall (i, j) \in I \times J \quad (3.2.3) \]

\[ \sum_{i \in I} \sum_{j \in J} x_{ij} = P \left( \min \left( \sum_{i \in I} A_i, \sum_{j \in J} B_j \right) \right) \quad (3.2.4) \]

Sometimes, situations arise when one wishes to keep reserve stocks at the origins for emergencies, thereby restricting the total transportation flow to a known specified level, say \( P \left( \min \left( \sum_{i \in I} A_i, \sum_{j \in J} B_j \right) \right) \). This flow constraint changes the structure of the transportation problem.

In order to solve the problem (P11) we separate it into two problems (P12) and (P13) given below.

(P12): minimize the cost function \( \left\{ \sum_{i \in I} \sum_{j \in J} c_{ij} x_{ij} \left( \sum_{i \in I} \sum_{j \in J} d_{ij} x_{ij} \right) + \sum_{i \in I} F_i \right\} \) subject to

(3.2.1) to (3.2.4)
(P13): minimize the time function \( \max_{i,j} \left( \frac{t_{ij}}{x_{ij}} \right) \) subject to (3.2.1) to (3.2.4).

The restricted flow constraint (3.2.4) in problem (P11) implies that a total
\[
\left( \sum_{i \in I} A_i - P \right)
\]
of the source reserves has to be kept at the various sources and a total
\[
\left( \sum_{j \in J} B_j - P \right)
\]
of destination slacks is to be retained at the various destinations.

Therefore an extra destination to receive the source reserves and an extra source
to fill up the destination slacks are introduced. In order to solve the problem (P12)
, a related problem (P12′) is formed which is given below.

(P12′): \( \min Z = \left( \sum_{i \in I'} \sum_{j \in J'} c'_{ij} y_{ij} \right) \left( \sum_{i \in I'} \sum_{j \in J'} d'_{ij} y_{ij} \right) + \sum_{i \in I'} F'_i \) subject to

\[
\sum_{j \in J'} y_{ij} = A'_i \quad \forall i \in I'
\]
\[
\sum_{i \in I'} y_{ij} = B'_j \quad \forall j \in J'
\]
\[
l_{ij} \leq y_{ij} \leq u_{ij} \quad \forall (i, j) \in I \times J
\]
\[
0 \leq y_{m+1,j} \leq B_j - b_j \quad \forall j \in J
\]
\[
0 \leq y_{i,n+1} \leq A_i - a_i \quad \forall i \in I
\]
\[
y_{m+1,n+1} = 0
\]
\[
A'_i = A_i \quad \forall i \in I, \quad A'_{m+1} = \sum_{j \in J} B_j - P, \quad B'_j = B_j \quad \forall j \in J \quad B'_{n+1} = \sum_{i \in I} A_i - P
\]
\[
c'_{ij} = c_{ij} \quad \forall i \in I, \; j \in J, \quad c'_{m+1,j} = c'_{i,n+1} = 0 \quad \forall i \in I, \; \forall j \in J, \; c'_{m+1,n+1} = M
\]
\[ d'_{ij} = d_{ij} \quad \forall i \in I, j \in J, \quad d'_{m+1,j} = d'_{i,n+1} = 0 \quad \forall i \in I, \quad \forall j \in J; \quad d'_{m+1,n+1} = M \]

\[ F'_i = F_i \quad \forall i = 1, 2 \ldots m, \quad F'_{m+1} = 0 \]

where \( I' = \{1, 2, \ldots m, m+1\} \), \( J' = \{1, 2, \ldots n, n+1\} \)

An algorithm is developed by forming a related fixed charge indefinite quadratic transportation problem and it is shown that to each basic feasible solution called corner feasible solution to related transportation problem, there is a corresponding feasible solution to this restricted flow problem. Further it is established that the efficient cost - time trade off pairs to the given problem are derivable from this related problem. A one to one correspondence between the feasible solution to (P12) and the corner feasible solution to related problem (P12') is established. After that it is derived that the value of the objective function of problem (P12) is equal to the value of the objective function of problem (P12') at its corresponding corner feasible solution and conversely. In addition to this, it is shown that optimal solution to (P12) bears one to one correspondence with the optimal corner feasible solution to (P12'). Finally it is shown that optimizing (P12') is equivalent to optimizing (P12) provided (P12) has a feasible solution.

Section III of chapter III is an extension of section II. This section discusses restricted flow in a non linear capacitated transportation problem with bounds on rim conditions. Here the objective function is the sum of two fractional
functions consisting of variable costs and fixed charges respectively. The objective function is of the form

\[
\text{(P14)} : \min z = \left[ \sum_{i \in I} \sum_{j \in J} c_{ij} x_{ij} - \sum_{i \in I} \sum_{j \in J} d_{ij} x_{ij} \right] + F_i \sum_{i \in I} G_i
\]  

(3.3.1)

The above objective function is subjected to the constraints (3.2.1) to (3.2.4). This objective function maximizes the return \(G_i\) on the capital investment \(F_i\) of fixed nature. Moreover, the same objective function simultaneously minimizes the variable cost and maximizes the total variable profit that is earned while shipping goods from various supply points to different destinations. A related transportation problem is formed and it is shown that a one to one correspondence between the basic feasible solution to (P14) and the related transportation problem exists. The value of the objective function of problem (P14) at a basic feasible solution and its related transportation problem at its corresponding corner feasible solution are shown to be equal. Further, it is established that there is one to one correspondence between the optimal solution to (P14) and optimal among the corner feasible solution to related transportation problem. As before it is shown that optimizing related transportation problem is same as optimizing the given restricted flow problem provided the latter has a feasible solution. In section III of chapter III of present thesis, optimality condition for the existence of local optimal basic feasible solution of problem (P14) is also developed.
algorithm is presented to solve non linear capacitated transportation problem with restricted flow and a numerical illustration is also included in support of the same.

Chapter IV of the present work is devoted to a special class of transportation problems i.e. fractional capacitated transportation problem. In the fractional function, variables in the numerator represent the change in total cost and the variable in the denominator represent the profit earned while shipping goods. Optimization of a ratio often describes some kind of an efficiency measure for a system. Section I of chapter IV presents an algorithm to find the optimum cost-time trade off pairs in a fractional capacitated transportation problem with bounds on rim conditions. The objective function is a ratio of two linear functions consisting of variable costs and variable profits respectively. In this section also, the total transportation flow is restricted to a known specified level. The mathematical model of the problem is

\[
(P15) : \min \left[ \sum_{i \in I} \sum_{j \in J} c_{ij}x_{ij} \right] \quad \text{subject to (3.2.1) to (3.2.4)}
\]

Here again a related transportation problems are formulated and the efficient time cost trade off pairs to the given problem are shown to be derivable from the related problem.
Section II of chapter IV studies a paradoxical situation in a fractional capacitated transportation problem in the sense that more is being shipped at a lesser cost. Optimality condition for a feasible solution of problem (P15) with objective function value as

\[
\sum_{i \in I} \sum_{j \in J} c_{ij} x_{ij} \quad \text{subject to} \quad \sum_{i \in I} d_{ij} x_{ij} = \text{constant}
\]

is established. The problem considered for the existence of paradox is

(P16): \[
\begin{align*}
\min \quad & \sum_{i \in I} \sum_{j \in J} c_{ij} x_{ij} \\
\text{subject to} \quad & (3.1.1) \text{ to } (3.1.3)
\end{align*}
\]

A sufficient condition for the existence of a paradoxical solution is established which is as follows.

\[
\left[ D^0(u_p^1 + v_q^1) - N^0(u_p^2 + v_q^2) \right] < 0 \quad \text{where} \quad N^+ = \sum_{i \in I} \sum_{j \in J} c_{ij} x_{ij} \quad , \quad D^+ = \sum_{i \in I} \sum_{j \in J} d_{ij} x_{ij} ,
\]

\( u_p^1, u_p^2, v_q^1, v_q^2 \) are the dual variables corresponding to the cell \((p,q)\) such that on changing \( a_p \) by \( a_p + \lambda \) and \( b_q \) by \( b_q + \lambda \) where \( \lambda > 0 \) and basis remaining the same, if the above condition is satisfied then there exists a paradox. An algorithm is presented to find a paradoxical solution. If a paradox exist, one would obviously be interested in the best paradoxical pair. Method to obtain the best paradoxical pair is also derived. Paradoxical range of flow is obtained and
then the paradoxical solution for a specified flow in that paradoxical range is also studied.

The algorithms offers a more universal apparatus for a wider class of real life decision priority problems. Illustrative examples are given in every section in support of the algorithm developed in the thesis. Real life problems are also studied in chapter 2 of the present work.