CHAPTER-2

RELATIVISTIC KINEMATICS IN EINSTEIN-CARTAN THEORY

“A theory is something nobody believes, except the person who made it. An experiment is something everybody believes, except the person who made it”.

Albert Einstein
1. INTRODUCTION

In general relativity, a time-like congruence in a four dimensional Lorentzian manifold can be interpreted as a family of world lines of certain ideal observers in our space-time. In particular, a time-like geodesic congruence can be interpreted as a family of free-falling particles. Describing the mutual motion of the test particles in a null geodesic in a space-time such as the Schwarzschild vacuum is a very important problem in general relativity. It is solved by defining certain kinematical quantities which completely describe how the integral curves in a congruence may converge (or diverge) or twist about one another. It should be stressed that the kinematical decomposition we are about to describe is a pure mathematics valid for any Lorentzian manifold. In mathematics, the covariant derivative is a way of describing the kinematics along tangent vectors of a manifold. Alternatively, the covariant derivative is a way of introducing and working with a specific connection on a manifold by means of a differential operator.

The local behavior of bundle of time-like (or null-like) curves in general relativity was first considered by Ehlers, Jordan, Kundt and Sachs [8]. They wrote an extensive article on the relativistic mechanics of continuous media where the derivation of the evolution equations of shear and rotation seem to appear for the first time. For null geodesic flows, the kinematical quantities: expansion, rotation and shear (related to the so-called optical scalars) and the corresponding Raychaudhuri equation, were first introduced by Sachs [20]. On the same line the general theory of time-like congruence was first initiated by Greenberg [14], who derived the propagation equations for the kinematical quantities in relativistic hydrodynamics. Further, the generalized form of propagation and constraint equations of kinematical parameters and Raychaudhuri equation in general relativity was presented by Ellis [9]. The strain variation equations of the kinematical parameters and its relativistic analogue were discussed by Carter and Quintana [4].

The standard description of the universe usually contains only the kinematical parameter $\theta$ (i.e. Hubble expansion). The torsion could have specific role in expansion of isotropic and anisotropic universe. A general form of Raychaudhuri equation for expansion in EC cosmology is physically relevant for the study of many issues such as cosmological perturbations. The simpler version of Raychaudhuri
equation in EC theory has already been discussed by several authors such as Stewart and Hajicek [21], Tafel [22], Raychaudhuri [19]. In the presence of torsion, the Raychaudhuri equation plays an important role to make the cosmological models singularity-free (Minkowski [16], de Ritis [6], Fennely and Smalley [13], De Sabbata and Sivaram [7], Ellis and Bruni [10], Ellis, Hwang and Bruni [11], Ellis, Bruni and Hwang [12], Capozziello, Iovane, Lambiase, and Stornaiolo [2], Capozziello and Stornaiolo [3], Kar and SenGupta [15]).

The comprehensive description of propagation and constraint equations of kinematical parameters in Einstein theory is given by Greenberg [14] and Ellis [9]. The similar version of Greenberg and Ellis work has been discussed by Palle [17], [18], Chrobok, Herrmann and Ruckner [5] and Brechet, Hobson and Lasenby [1] in case of Weyssenhoff fluid. In EC theory of gravitation, we developed the general theory of time-like congruence and obtain the propagation equations of kinematical parameters in the presence of torsion tensor.

In Section 2, we have described our developments in the general theory of time-like congruence in EC theory of gravitation with full generality. The expressions for the kinematical parameters (such as expansion scalar $\theta$, shear tensor $\sigma_{ij}$, rotation tensor $\omega_{ij}$ and acceleration vector $\dot{u}^i$) are derived and studied the contribution of torsion tensor to each kinematical quantity. The role of the irreducible parts of the torsion tensor on each of the kinematical parameters is examined in the Section 3. In particular, it is shown that the kinematical quantity acceleration $\dot{u}^i$ depends on $T$-torsion and $V$-torsion, expansion $\theta$ depends on $V$-torsion, $\sigma_{ij}$ depends on $T$-torsion only and $\omega_{ij}$ depends on $T$-torsion and $A$-torsion. The comprehensive description of a Weyssenhoff fluid is described in Section 4. In Section 5, we have derived the propagation equations of kinematical parameters with the contribution of torsion tensor. It is shown that, the additional sources namely material source $\pi_{ij}$ and the tidal force $E_{ij}$ are appeared in the propagation of rotation tensor, in contrast to Einstein theory of gravitation. Since the torsion and spin tensor are linked up with the field equations, the propagation equations of kinematical parameters are extended in EC theory of gravitation. Finally, the general discussion and conclusions are given in section 6.
2. KINEMATICS WITH TORSION

In this section we briefly discuss the general theory of time-like congruence in EC theory of gravitation.

2.1 General Theory of Time-like Congruence in EC Theory

Let us consider a set of neighboring time-like curves, forming a time-like congruence, and it may be described by the parametric relations

\[ x^i = x^i(\xi^\alpha, s) , \]

where, the parameter \( \xi^\alpha (\alpha = 1, 2, 3) \) represents the particular time-like curve and \( s \) is some parameter (or arc length) along this curve. At a point \( P \) along one of the curves \( \xi^\alpha \), the unit vector tangent to world line is denoted by

\[ u^i = \frac{dx^i}{ds} , \quad (2.1) \]

which, satisfies the normalization condition \( u^i u_i = 1 \).

When we consider the two neighboring curves of the congruence specified by \( \xi^i \) and \( \xi^i + \delta \xi^i \), the vector \( \delta x^i \) from point \( P \) on the curve \( \xi \) to the point \( P' \) on the curve \( \xi' \), where \( P \) and \( P' \) have same value of the parameter \( s \), is called a connecting vector and is given by

\[ \delta x^i = \frac{\delta x^i}{\delta \xi^\alpha} \delta \xi^\alpha . \]

Here \( \delta x^i \) need not be orthogonal to the vector \( u^i \) at point \( P \). We now require a unique connecting vector \( \delta x^i \) orthogonal to \( u^i \) for the description of the kinematical parameters is given by

\[ \delta_{\perp} x^i = h^i_j \delta x^j , \quad (2.2) \]

where, \( (\delta_{\perp} x^i) u_i = 0 \) and \( h_{ij} = g_{ij} - u_i u_j \) is the 3-dimensional projection operator.

Then the connecting vector \( \delta x^i \) can be decomposed in the manner

\[ \delta_{\perp} x^i = (\delta_{\perp} x^i - u^i u_j) \delta x^j , \]

\[ \delta x^i = \delta_{\perp} x^i + (\delta x^i u_j) u^j . \quad (2.3) \]
In Einstein’s general theory of relativity, the connecting vector $\delta \mathbf{x}'$ of two particles on neighboring curves satisfies
\[ \hat{\mathbf{x}} \delta \mathbf{x}' = 0 \tag{2.4} \]

In EC theory, using the above result the equation (4.7) of chapter 1 gives
\[ \hat{\mathbf{x}} \delta \mathbf{x}' = \delta \mathbf{x}' - 2Q_{ik} \delta \mathbf{x}' u^k = 0 \]
\[ \Rightarrow \delta \mathbf{x}' = u_{ij} \delta \mathbf{x}' + 2Q_{ik} \delta \mathbf{x}' u^k \tag{2.5} \]

where, $A' = A'_{ij}u^j$.

The relative position vector between the two near by points in the fluid, as seen by observer may expressed in EC theory as
\[ \delta \mathbf{x}' = h'_{ij}(\delta \mathbf{x'}) \tag{2.6} \]

With the help of (2.3), the equation (2.6) gives
\[ \delta \mathbf{x}' = h'_{ij}[\delta \mathbf{x'} - (\delta \mathbf{x'} u_i)u^j] \]
\[ \delta \mathbf{x}' = h'_{ij}(\delta \mathbf{x'}) - h'_{ij} (\delta \mathbf{x'} u_i) \hat{u}^j \tag{2.7} \]

By substituting the value of $(\delta \mathbf{x'})$ from (2.5) in equation (2.7) we obtain
\[ \delta \mathbf{x}' = h'_{ij}(u_{ij} - 2Q_{ik}u^k)\hat{\mathbf{x}}' - h'_{ij}(\delta \mathbf{x'} u_i) \hat{u}^j \]

Now, the desired form of the above equation can be written in the manner
\[ \delta \mathbf{x}' = h'_{ij}(\delta \mathbf{x'} - u^i u_k)(u_{ij} - 2Q_{im}u^m)\hat{\mathbf{x}}' \]
\[ \delta \mathbf{x}' = h'_{ik}h'_{ij}(u_{ij} - 2Q_{im}u^m)\hat{\mathbf{x}}' \tag{2.8} \]

This equation represents the relative velocity between the two points of the two neighboring flow lines in terms of gradient of the fluid velocity $u^i$. The equation (2.8) may also be expressed as
\[ \delta \mathbf{x}' = h'_{ik}h'_{ij}[u_{ij} + (K_{im} - K_{ml})u^m]\hat{\mathbf{x}}' \]

i.e.
\[ \delta \mathbf{x}' = h'_{ik}h'_{ij}(u_{ij} - K_{ml}u^m)\hat{\mathbf{x}}' \tag{2.9} \]

If we define the operator
\[ \nu_{ij} = h'_{ik}h'_{ij}u_{k/l} \tag{2.10} \]
\[ \Rightarrow \nu_{ij} = u_{i/j} - \hat{u}_j u_j \]
then the relationship between the operators \( \nu_{ij} \) in EC theory and \( \hat{\nu}_{ij} \) in Einstein theory of gravitation can be established as

\[
\nu_{ij} = \hat{\nu}_{ij} + \Omega_{ij}, \tag{2.11}
\]

where,

\[
\hat{\nu}_{ij} = h^k h^l u_{k:l}, \tag{2.12}
\]

\[
\Omega_{ij} = h^j h^l K_{ilm} u^m. \tag{2.13}
\]

Now, the \( \nu_{ij} \) can be decomposed into its irreducible parts as follows:

\[
\nu_{ij} = \sigma_{ij} + \omega_{ij} + \frac{1}{3} \theta h_{ij}, \tag{2.14}
\]

where,

\[
\theta = u^l_{/l} = g^{ll} \nu_{ij}
\]

\[= g^{ll}(\hat{\nu}_{ij} + \Omega_{ij})
\]

\[= g^{ll}(h^k h^l u_{k:l} + h^k h^l K_{ilm} u^m)
\]

\[= h^{kl} u_{k:l} + h^{kl} K_{ilm} u^m
\]

\[= u^{\hat{k}}_{/k} + h^{kl} (-Q_{lkm} + Q_{km} - Q_{mlk}) u^m
\]

\[= u^{\hat{k}}_{/k} - 2Q_{km} u^k,
\]

i.e.

\[
\theta = \hat{\theta} - \hat{Q}_k u^k, \tag{2.15}
\]

where, \( Q_k = 2Q_{km} m \) and the scalar \( \theta \) describes the rate of expansion (or contraction) of volume of the fluid.

The symmetric space-like part of \( \nu_{ij} \) called the shear tensor \( \sigma_{ij} \) is given by

\[
\sigma_{ij} = \nu_{(ij)} - \frac{1}{3} \theta h_{ij}
\]

\[= h^k h^l u_{(k/l)} - \frac{1}{3} \theta h_{ij},
\]

i.e.

\[
\sigma_{ij} = \frac{1}{2} (u_{(i:j)} + u_{j:i}) - \frac{1}{2} (\hat{u}_{i} u_{j} + u_{j} \hat{u}_{i}) - \frac{1}{3} \theta h_{ij}. \tag{2.16}
\]

The above expression for shear tensor may also be expressed as

\[
\sigma_{ij} = \hat{\sigma}_{ij} + \Omega_{(ij)} + \frac{1}{3} Q_k u^k h_{ij}, \tag{2.17}
\]
where, the expression of shear tensor $\sigma_{ij}$ in Einstein theory is mentioned in chapter 1 (vide equation (6.9)) and the additional torsion term $\Omega_{(ij)}$ is given by

$$
\Omega_{(ij)} = h^k_i h^j_l K_{(kl)m} u^m
$$

$$
= \frac{1}{2} (K_{ik} + K_{jk}) u^k - \frac{1}{2} (K_{ik} u_j + K_{jk} u_i) u^k u^{l}.
$$

$$
\Omega_{(ij)} = -(Q_{ij} + Q_{ji}) u^k - (Q_{ik} u_j + Q_{jk} u_i) u^k u^{l} + \frac{1}{3} Q_k u^k h_{ij} .
$$

Consequently, the expression (2.17) becomes

$$
\sigma_{ij} = \hat{\sigma}_{ij} - (Q_{ij} + Q_{ji}) u^k - (Q_{ik} u_j + Q_{jk} u_i) u^k u^{l} + \frac{1}{3} Q_k u^k h_{ij} .
$$

The anti-symmetric space-like part of $\nu_{ij}$ called the rotation tensor $\omega_{ij}$ of the flow of the fluid and is defined as

$$
\omega_{ij} = \nu_{ij} ,
$$

$$
\omega_{ij} = h^k_i h^j_l u_{(kl)} ,
$$

$$
\omega_{ij} = \frac{1}{2} (u_{i,j} - u_{j,i}) - \frac{1}{2} (\hat{u}_{i,j} - u_{i,j} ) .
$$

Also, it can be written as

$$
\omega_{ij} = \hat{\omega}_{ij} + \Omega_{(ij)} .
$$

Hence, the expression (2.17) becomes

$$
\omega_{ij} = \hat{\omega}_{ij} + Q_{ij} u^k - (Q_{ik} u_j + Q_{jk} u_i) u^k u^{m} .
$$

The rotation tensor $\omega_{ij}$ describes the rotation of matter relative to a non-rotating frame.

The kinematical quantity, acceleration $\hat{u}_i$ is defined as

$$
\hat{u}_i = (\nabla \hat{u}_i) u^k = u^{\hat{i}} + K_{\hat{i}l} u^k u^{l} ,
$$

$$
\hat{u}_i = u^{\hat{i}} - 2Q_{\hat{i}l} u^k u^{l} .
$$
From equations (2.10) and (2.14) we can have

\[ u_{ij} = \sigma_{ij} + \omega_{ij} + \frac{1}{3} \theta h_{ij} + \dot{u}_i u_j. \]  

(2.27)

These kinematical quantities satisfy the following properties:

\[ \dot{u}_j u^j = 0, \]  

(2.28a)

\[ \sigma_{ij} u^j = \{ \hat{\sigma}_{ij} - (Q_{kj} + Q_{kj}) u^k - (Q_{ik} u_j + Q_{jk} u_i) u^k u^j + \frac{1}{3} Q_k u^k h_{ij} \} u^j \]

\[ = \hat{\sigma}_{ij} u^j - (Q_{kj} + Q_{kj}) u^j u^k - (Q_{ik} u_j + Q_{jk} u_i) u^j u^k u^j, \]  

(2.28b)

\[ \omega_{ij} u^j = \{ \hat{\omega}_{ij} + Q_{jk} u^k - (Q_{ik} u^j - Q_{jm} u_i) u^j u^m \} u^j \]

\[ = \hat{\omega}_{ij} u^j + Q_{jk} u^j u^k - (Q_{ik} u^j - Q_{jm} u_i) u^j u^k u^j, \]  

(2.28c)

\[ \sigma_{ij} g^{ij} = g^{ij} \{ \hat{\sigma}_{ij} - (Q_{kj} + Q_{kj}) u^k - (Q_{ik} u_j + Q_{jk} u_i) u^k u^j + \frac{1}{3} Q_k u^k h_{ij} \} \]

\[ = g^{ij} \hat{\sigma}_{ij} - (Q_{ki} + Q_{ki}) u^k - (Q_{ik} u^j + Q_{jk} u^j) u^k u^j + \]

\[ + \frac{1}{3} Q_k u^k h_{ij} g^{ij} \]

\[ = -Q_k u^k + \frac{1}{3} (3Q_k u^k), \]  

(2.28d)

\[ \sigma_{ij} g^{ij} = 0, \]  

(2.28e)

\[ \omega_{ij} g^{ij} = 0. \]  

Finally, from equations (2.6), (2.8) and (2.14), we arrive at useful identity:

\[ h^j (\delta_{x^j} x^j) = h^j h^i (u^j / l - 2Q_{lm} u^m) \delta_{x^j} x^k \]

i.e.

\[ h^j (\delta_{x^j} x^j) = A_y \delta_{x^j} x^j, \]  

(2.29)

where,

\[ A_y = h^k h^j u_{k/l} - 2h^k h^j Q_{mk} u^m \]

\[ = h^k h^j u_{k/l} - 2h^k Q_{jk} u^l, \]  

\[ A_y = \nu_{ij} - 2h_{kl} Q_{jk} u^l. \]

Hence
This identity will be utilized to show the vortex, magnetic and electric field lines to be “frozen-into” the fluid in the later chapter.

2.2 Some Important Relations and Definitions

By using the completely skew-symmetric permutation tensor $\eta^{ijkl}$ we define certain terms and prove some relations in EC theory.

A) Vorticity and Torsion Vector:

The skew-symmetric property of rotation tensor $\omega_{ij}$ and the torsion tensor $\Omega_{ijkl}$ implies that we can define the vorticity vector $\omega^i$ and torsion vector $\Omega^i$ as follows:

The vorticity vector $\omega^i$ is defined as

$$\omega^i = \frac{1}{2} \eta^{ijkl} u_j \omega_{kl} , \tag{2.31}$$

also,

$$\omega^i = \frac{1}{2} \eta^{ijkl} u_j \left\{ \frac{1}{2} (u_{k,l} - u_{l,k}) - \frac{1}{2} (\dot{u}_k u_i - \dot{u}_i u_k) \right\} , \tag{2.32}$$

i.e. 

$$\omega^i = \frac{1}{2} \eta^{ijkl} u_j u_{k/l} . \tag{2.33}$$

In the same way, the torsion vector $\Omega^i$ is defined in the form

$$\Omega^i = \frac{1}{2} \eta^{ijkl} u_j \Omega_{kl} . \tag{2.33}$$

This is equivalent to

$$\Omega^i = \frac{1}{2} \eta^{ijkl} u_j h^m_k h^n_l K_{mn} u^p \tag{2.34}$$

$$= \frac{1}{2} \eta^{ijkl} \left[ Q_{kln} u^m - (Q_{kln} u_i - Q_{kln} u_k) u^m u^n \right] ,$$

i.e. 

$$\Omega^i = \frac{1}{2} \eta^{ijkl} u_j Q_{kln} u^m . \tag{2.34}$$

Using the properties of an alternating tensor we may prove the following relations:

$$\omega_{ij} = \eta_{ijkl} \omega^k u^l , \quad \omega_j \omega^j = 0 , \quad \omega_j u^j = 0 ,$$
\[ \eta^{ijkl} \omega_{kl} = -2(u^i \omega^j - \omega^i u^j) \quad . \]  

(2.35)

For the torsion tensor \( \Omega_{[ij]} \):

\[ \Omega_{[ij]} = \eta^{ijkl} \Omega^{kl} u^i \quad , \]

\[ \Omega_{[ij]} \Omega^j = 0 \quad , \quad \Omega_{[ij]}^{kl} u^j = 0 \quad , \]

\[ \eta^{ijkl} \Omega_{[kl]} = -2(u^i \Omega^j - \Omega^i u^j) \quad . \]  

(2.36)

Similarly, using the relation (2.22) we have

\[ \eta^{ijkl} \hat{\omega}_{kl} = \eta^{ijkl} [\omega_{kl} - \Omega_{[kl]}] \]

\[ = -2[(u^i \omega^j - \omega^i u^j) - 2(u^j \Omega^i - \Omega^i u^j)] \quad , \]

i.e.

\[ \eta^{ijkl} \hat{\omega}_{kl} = -2[u^i (\omega^j - \Omega^j) - u^i (\omega^j - \Omega^j)] \quad . \]  

(2.37)

B) Scalars from Kinematical Quantities

The magnitude of the rotation tensor \( \omega_{ij} \) is given by

\[ \omega^2 = -\frac{1}{2} \omega_{ij} \omega^{ij} = -\omega_i \omega^i \quad . \]  

(2.38)

Further, the magnitude of the torsion tensor \( \Omega_{[ij]} \) is in the form

\[ \Omega^2 = -\frac{1}{2} \Omega_{[ij]} \Omega^{[ij]} = -\Omega_i \Omega^i \quad . \]  

(2.39)

Also we defined as

\[ \omega^2 = -\frac{1}{2} \omega_{ij} \omega^{ij} \]

\[ = -\frac{1}{2} (\hat{\omega}_{ij} + \Omega_{[ij]})(\hat{\omega}^{ij} + \Omega^{[ij]}) \quad , \]

i.e.

\[ \omega^2 = \hat{\omega}^2 - \hat{\omega}_{ij} \Omega^{[ij]} + \Omega^2 \quad . \]  

(2.40)

Similarly,

\[ \hat{\omega}^2 = -\frac{1}{2} \hat{\omega}_{ij} \hat{\omega}^{ij} \]

\[ = -\frac{1}{2} (\omega_{ij} - \Omega_{[ij]})(\omega^{ij} - \Omega^{[ij]}) \quad , \]

i.e.

\[ \hat{\omega}^2 = \omega^2 + \omega_{ij} \Omega^{[ij]} + \Omega^2 \quad . \]  

(2.41)

The magnitude of the shear tensor \( \sigma_{ij} \) is given by
\[ \sigma^2 = \frac{1}{2} \sigma_{ij} \beta_{ij} \geq 0 \ , \quad (2.42a) \]

Also from equations (2.17) and (2.42) that
\[
\sigma^2 = \frac{1}{2} (\hat{\sigma}_{ij} + \Omega_{(ij)} + \frac{1}{3} Q^k u^k h_{ij} (\hat{\sigma}^{ij} + \Omega^{ij} + \frac{1}{3} Q^k u^k h_{ij}) \\
= \frac{1}{2} [\hat{\sigma}_{ij} + \frac{1}{2} \hat{\sigma}_{ij} \Omega^{ij} + \Omega_{(ij)}^{ij} + \frac{2}{3} Q^k u^k \Omega_{(ij)}^{ij} h_{ij} + \\
+ \frac{1}{9} (Q^k u^k)^2 h_{ij} h_{ij} ] \\
= \frac{1}{2} [\hat{\sigma}_{ij} + \frac{1}{2} \hat{\sigma}_{ij} \Omega^{ij} + \Omega_{(ij)}^{ij} - \frac{2}{3} (Q^k u^k)^2 + \frac{1}{3} (Q^k u^k)^2 ] , \quad (2.42b) \\
\sigma^2 = \hat{\sigma}^2 + \hat{\sigma}_{ij} \Omega^{ij} - \Omega^2 - \frac{1}{6} (Q^k u^k)^2 . \]

Its value is found to be dependent on the contribution of three irreducible parts of torsion tensor.

**C) Contribution of Spin to Kinematical Quantities**

We have already derived the expressions for the kinematical quantities with torsion. We recall here the equations (2.15), (2.19), (2.24), (2.26) for ready reference
\[
\theta = \hat{\theta} - Q_k u^k , \quad (2.43a) \\
\sigma_{ij} = \hat{\sigma}_{ij} - (Q_{ij} + Q_{ji}) u^k - (Q_{ik} u_j + Q_{jk} u_i) u^k u^l + \\
+ \frac{1}{3} Q^k u^k h_{ij} , \quad (2.43b) \\
\omega_{ij} = \hat{\omega}_{ij} + Q_{ik} u^k - (Q_{lm} u_j - Q_{jm} u_i) u^m u^l , \quad (2.43c)
\]
and
\[ u_i = u^i + 2 Q_{ik} u^k u^l . \quad (2.43d) \]

However, from EC field equations we have
\[
Q^k_{ij} = k S_{ij} u^k , \\
S_{ij} u^j = 0 , \\
\Rightarrow Q_i = 0 . \quad (2.43e) \]
Consequently, the above equations reduce to
\[
\theta = \hat{\theta} , \quad (2.43f) \]
\[ \sigma_{ij} = \hat{\sigma}_{ij} , \]  
(2.43g)

\[ \omega_{ij} = \hat{\omega}_{ij} + kS_{ij} , \]  
(2.43h)

\[ \dot{u}_i = u'_i . \]  
(2.43i)

where \( \Omega_{ij} = kS_{ij} \). We can see that the expressions for the expansion, acceleration vector and the shear tensor are the same in both Einstein and EC theory of gravitation and only the rotation tensor is affected by the spin tensor.

Similarly, the torsion vector \( \Omega' \) associated with \( \Omega_{ij} \) is given by

\[ \Omega' = \frac{1}{2} \eta_{ijkl} k u_j S_{kl} , \]  
(2.44)

which is the spin vector. The magnitude of rotation tensor \( \omega_{ij} \) and torsion tensor \( \Omega_{ij} \) with contribution of spin is defined as

\[ \omega^2 = -\frac{1}{2} \omega_{ij} \omega^{ij} \]

\[ = -\frac{1}{2} (\hat{\omega}_{ij} + kS_{ij})(\hat{\omega}^{ij} + kS^{ij}) , \]

\[ \omega^2 = \hat{\omega}^2 - kS_{ij} \hat{\omega}^{ij} + k^2 S^2 . \]  
(2.45)

Similarly,

\[ \dot{\omega}^2 = \omega^2 + kS_{ij} \omega^{ij} + k^2 S^2 , \]  
(2.46)

\[ \sigma^2 = \hat{\sigma}^2 . \]  
(2.47)
2.3. Kinematical Quantities at a Glance

The work discussed above can be summarized below which establishes the contribution of torsion and spin to the kinematical quantities.

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<th>Kinematics with torsion</th>
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<td><strong>magnitude of rotation tensor</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\hat{\omega}^2 = \frac{1}{2} \hat{\omega}_{ij} \hat{\omega}^{ij}$</td>
<td>$\omega^2 = \frac{1}{2} \omega_{ij} \omega^{ij}$</td>
<td>$\omega^2 = \frac{1}{2} \omega_{ij} \omega^{ij}$</td>
</tr>
</tbody>
</table>

$\eta^{ijkl}$, $\omega_{ij}$, $\omega^i$, $\omega^2$, $\omega_{ij}$, $\omega^{ij}$, $\omega^i$, $\omega^2$ are the tensors and vectors representing the kinematical quantities.
3. CONTRIBUTION OF T-TORSION, A-TORSION AND V-TORSION TO THE KINEMATICAL QUANTITIES

In the following we will see how the irreducible parts of the torsion tensor influence the various kinematical quantities.

A) Contribution to Acceleration $\dot{u}_i$

The equation (2.25) gives

$$\dot{u}_i = u_i^\alpha + K_{ki}u^k u^i,$$

$$= u_i^\alpha + (-Q_{ki} + Q_{ik} - Q_{ik})u^k u^i,$$

$$= u_i^\alpha + 2Q_{ik}u^k u^i,$$

$$= u_i^\alpha + \frac{2}{3}(3Q_{ik})u^k u^i,$$

$$= u_i^\alpha + \frac{2}{3}[2Q_{ik} - (Q_{ki} - Q_{ik})]u^k u^i,$$

$$= u_i^\alpha + \frac{2}{3}[2Q_{ik} - Q_{k[i]} - Q_{l[i]}g_{jl}k]u^k u^i + \frac{2}{3}Q_{ik}g_{ij}k u^k u^i,$$

i.e.

$$\dot{u}_i = u_i^\alpha + 2(\dot{g}_{ik}Q_{ik} + \dot{g}_{ik}Q_{ik})u^k u^i.$$ (3.1)

This shows that the expression for acceleration depends on $T$-torsion and $V$-torsion only.

B) Contribution to Expansion $\theta$

The equation (2.15) gives

$$\theta = \dot{\theta} - Q_{ik}u^k,$$

$$= \dot{\theta} - \frac{2}{3}g^{ik}Q_{ij}g_{jk}u^j,$$

i.e.

$$\theta = \dot{\theta} - 2g^{ik}(\dot{g}_{ik}Q_{ik}u^i).$$ (3.2)

This explains that the expression of expansion depends on $V$-torsion only.

C) Contribution to Shear Tensor $\sigma_{ij}$

The contribution of irreducible parts of the torsion tensor to the shear tensor is discussed below. The equation (2.17) gives
\[ \Omega_{(ij)} + \frac{1}{3} Q_k u^k h_{ij} = h^k h^l K_{[ik]m} u^m + \frac{1}{3} Q_k u^k h_{ij} \]
\[ = h^k h^l \frac{1}{2} (K_{lk} + K_{kl}) u^m + \frac{1}{3} Q_k u^k h_{ij} \]
\[ = -h^k h^l (Q_{mlk} + Q_{mlk}) u^m + \frac{1}{3} Q_k u^k h_{ij} , \]
\[ \Omega_{(ij)} + \frac{1}{3} Q_k u^k h_{ij} = -2 h^k h^l (Q_{mlk}) u^m + \frac{1}{3} Q_k u^k h_{ij} , \] (3.3)

\[ \Rightarrow \]
\[ \Omega_{(ij)} + \frac{1}{3} Q_k u^k h_{ij} = - \frac{2}{3} h^k h^l (3 Q_{mlk}) u^m + \frac{1}{3} Q_k u^k h_{ij} g_{ik} \]
\[ = - \frac{2}{3} [h^k h^l (3 Q_{mlk}) u^m - \frac{1}{4} (h^k h^l + \]
\[ + h^k h^l (Q_{m} u^m g_{ik} - Q_{ik})] \]
\[ = - \frac{2}{3} h^k h^l [(2 Q_{mlk} - Q_{kml} + Q_{kml} - \]
\[ - \frac{1}{2} (Q_{m} u^m g_{ik} - Q_{ik})] \]
\[ = - \frac{2}{3} h^k h^l [(2 Q_{mlk} - Q_{kml} + Q_{kml} - Q_{m} g_{ik}) u^m] , \]
\[ \Omega_{(ij)} + \frac{1}{3} Q_k u^k h_{ij} = -2 h^k h^l (Q_{mlk}) u^m . \] (3.4)

Therefore, the expression of shear tensor (2.17) becomes
\[ \sigma_{ij} = \bar{\sigma}_{ij} - 2 h^k h^l (Q_{mlk}) u^m . \] (3.5)

This shows that only \( T \)-torsion influences the shear tensor.

**D) Contribution to Rotation Tensor** \( \omega_{ij} \)

Finally, the expression of rotation tensor (2.22) yields
\[ \Omega_{(ij)} = h^k h^l K_{[ik]m} u^m \]
\[ = h^k h^l Q_{ikm} u^m , \]
\[ \Omega_{(ij)} = h^k h^l (\bar{Q}_{ikm} + h^k h^l (Q_{ikm} + \bar{Q}_{ikm}) u^m) . \] (3.6)

Since
\[ h^k h^l (\bar{Q}_{ikm}) u^m = \frac{1}{3} h^k h^l Q_{ikm} u^m\]
\[ h^k h^l Q_{kln} u^m = \frac{1}{6} h^k h^l (Q_k g_{lm} - Q_l g_{km}) u^m = 0, \quad (3.7) \]

then the equation (3.6) becomes
\[ \Omega_{(ij)} = h^k h^l (Q_{kl} + A Q_{kl}) u^m. \quad (3.8) \]

Hence the expression of rotation tensor (2.24) becomes
\[ \omega_i = \hat{\omega}_i + h^k h_l (Q_{kl} + A Q_{kl}) u^m. \quad (3.9) \]

This shows that the torsion term \( \Omega_{(ij)} \) does not depend on \( V \)-torsion. It depends only on \( T \)-torsion and \( A \)-torsion. Alternatively, it can be stated that \( \Omega_{(ij)} \) depends on the torsion deviator defined by
\[ ^{+} Q_{ij}^k = T Q_{ij}^k + A Q_{ij}^k \]
\[ = T Q_{ij}^k + A Q_{ij}^k + V Q_{ij}^k - \frac{1}{3} Q_{ij} \delta^k_j \]
\[ \text{i.e.} \quad ^{+} Q_{ij}^k = Q_{ij}^k - \frac{1}{3} Q_{ij} \delta^k_j. \quad (3.10) \]

It is evident from this equation that
\[ ^{+} Q_{ij}^{i'} = Q_{ij}^{i'} - \frac{1}{6} (Q_{ij} \delta_{i'}^j - Q_{ij} \delta_{i}^j) \]
\[ = \frac{1}{2} Q_{ij} - \frac{1}{6} (4Q_{ij} - Q_{ij}). \]

Hence, \( ^{+} Q_{ij}^{i'} = 0. \)

4. SPINNING FLUID (WEYSSENHOFF FLUID)

4.1. Spinning Fluid in Einstein-Cartan Theory

We recall here the field equations in EC theory of gravitation as
\[ R^{ij} - \frac{1}{2} R g^{ij} = -k t^{ij}, \quad (4.1) \]
\[ Q_{ij}^{k} + \delta_{i}^{k} Q_{ij}^{i'} - \delta_{j}^{k} Q_{ij}^{i'} = k S_{ij}^{k}, \quad (4.2) \]

where we consider the energy-momentum tensor \( t_{ij} \) in the form
\[ t_{ij} = (\rho_{\text{eff}} + p_{\text{eff}}) u_i u_j - p_{\text{eff}} g_{ij} + u_i q_j + l_i u_j + \pi_{ij}, \quad (4.3) \]
Here, $\rho_{\text{eff}}$ and $p_{\text{eff}}$ are respectively, the effective energy density and the effective isotropic pressure of the medium and are given as

$$\rho_{\text{eff}} = t_{ij} u^i u^j, \quad p_{\text{eff}} = -\frac{1}{3} t_{ij} h^{ij},$$

(4.4)

and the energy flux vector $q_i$, the momentum flux vector $l_i$ and trace-free tensor material $\pi_{ij}$ are defined as

$$l_i = h^i_j t_{jk} u^k, \quad q_i = h^i_j t_{jk} u^j,$$

$$\pi_{ij} = h^k_i h^j_k t_{kl} - \frac{1}{3} t_{kl} h^{kl} h_{ij}.$$  

(4.5)

Again we have the following conditions

$$q_i u^i = 0, \quad l_i u^i = 0, \quad \pi_{ij} u^i = 0 \quad \text{and} \quad \pi^i_i = 0 .$$  

(4.6)

In EC theory the energy-momentum tensor of Weyssenhoff fluid is

$$t_{ij} = (\rho + p) u_i u_j - p g_{ij} - S_{kij} u^k u^j.$$  

(4.7)

The decomposition of the energy-momentum of Weyssenhoff fluid in effective quantities gives (Chrobok el at. [5])

$$\rho_{\text{eff}} = t_{ij} u^i u^j = [ (\rho + p) u_i u_j - p g_{ij} - S_{kij} u^k u^j ] u^i u^j = (\rho + p) - p ,$$

$$\rho_{\text{eff}} = \rho ,$$

(4.8a)

$$p_{\text{eff}} = -\frac{1}{3} t_{ij} h^{ij} = -\frac{1}{3} [ (\rho + p) u_i u_j - p g_{ij} - S_{kij} u^k u^j ] h^{ij} = -\frac{1}{3} (-3 p) ,$$

$$p_{\text{eff}} = p ,$$

(4.8b)

$$l_i = h^i_j t_{jk} u^k = h^i_j [ (\rho + p) u_j u_k - p g_{jk} - S_{mjk} u^m u^k ] u^k = -S_{mjk} u^i u^m u_k$$

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\[ l^i = \dot{u}_j S^j, \quad (4.8c) \]
\[ q_i = h^i_j t_{jk} u^j \]
\[ = h^i_j [(\rho + p)u_j u_k - pg_{jk} - S_{mj\ell} u^\ell u_k] u^j, \]
\[ q_i = 0, \quad (4.8d) \]
\[ \pi_{ij} = h^k_i h^l_j t_{kl} - \frac{1}{3} t_{kl} h^{kl} h_{ij}, \quad (4.8e) \]
\[ = h^k_i h^l_j [(\rho + p)u_k u_l - p g_{kl} - S_{mk\ell} u^\ell u_l] - \]
\[ - \frac{1}{3} [(\rho + p)u_k u_l - p g_{kl} - S_{mk\ell} u^\ell u_l] h^{kl} h_{ij} \]
\[ \Rightarrow \pi_{ij} = 0. \quad (4.8f) \]

### 4.2. Conformal Curvature Tensor (Weyl Tensor)

The expression for the Weyl tensor in terms of the Riemann curvature tensor \( R_{ijkl} \) as

\[ C_{ijkl} = R_{ijkl} - \frac{1}{2} [R_{jk} g_{il} - R_{il} g_{jk} - R_{ik} g_{jl} + R_{jl} g_{ik}] - \]
\[ - \frac{1}{6} R (g_{ik} g_{jl} - g_{il} g_{jk}) . \quad (4.9) \]

The Weyl tensor \( C_{ijkl} \) is the trace-free part of curvature tensor. The electric type tensor \( E_{ij} \) is defined as

\[ E_{jk} = C_{ijk} u^i u^l, \quad (4.10) \]

From equations (4.9) and (4.10) we obtain

\[ E_{jk} = C_{ijk} u^i u^l \]
\[ = R_{ijk} u^i u^l - \frac{1}{2} [R_{jk} - R_{il} g_{jk} + R_{ik} u^i u^l] + \frac{1}{6} R h_{jk}, \]
\[ E_{jk} = R_{ijk} u^i u^l - \frac{1}{2} h^i_j h^l_k R_{jk} - \frac{1}{2} R_{ik} u^i u^l h_{jk} + \frac{1}{6} R h_{jk}, \]
\[ E_{jk} = R_{ijk} u^i u^l - \frac{1}{2} h^i_j h^l_k R_{jk} - \frac{1}{2} R_{ij} u^i u^l h_{jk} + \frac{1}{6} R h_{jk}. \quad (4.11) \]

The results required to develop the propagation equations for kinematical parameters in the next section are obtained here.
\[ R_{ijkl} u^i u^j - \frac{1}{3} R_{ij} u^i u^j h_{jk} = E_{jk} + \frac{1}{2} h_j^i h^i_k R_{ij} + \frac{1}{2} R_{ij} u^i u^j h_{jk} - \frac{1}{6} R h_{jk} - \frac{1}{3} R_{ij} u^i u^j h_{jk} , \]

\[ \Rightarrow R_{ijkl} u^i u^j - \frac{1}{3} R_{ij} u^i u^j h_{jk} = E_{jk} + \frac{1}{2} h_j^i h^i_k R_{ij} + \frac{1}{6} R_{ij} u^i u^j h_{jk} - \frac{1}{6} R h_{jk} . \] (4.12)

Using the field equation (4.1) in (4.12) we get

\[ R_{ijkl} u^i u^j - \frac{1}{3} R_{ij} u^i u^j h_{jk} = E_{jk} - \frac{k}{2} h_j^i h^i_k (t_{il} - \frac{1}{2} t g_{il}) - \frac{k}{6} (t_{il} - \frac{1}{2} t g_{il}) u^i u^j h_{jk} - \frac{k}{6} t h_{jk} \]

\[ = E_{jk} - \frac{k}{2} h_j^i h^i_k t_{il} + \frac{k}{4} t h_{jk} - \frac{k}{6} t u^i u^j h_{jk} + \frac{k}{12} t h_{jk} - \frac{k}{6} t h_{jk} , \]

i.e.

\[ R_{ijkl} u^i u^j - \frac{1}{3} R_{ij} u^i u^j h_{jk} = E_{jk} - \frac{k}{2} \left[ h_j^i h^i_k t_{il} - \frac{1}{3} t_{il} h^j h_{jk} \right] . \] (4.13)

With the help of the equation (4.8e) the equation (4.13) can be written as

\[ R_{ijkl} u^i u^j - \frac{1}{3} R_{ij} u^i u^j h_{jk} = E_{jk} - \frac{k}{2} \pi_{jk} , \] (4.14)

From equations (4.11) and (4.14) we have

\[ \frac{k}{2} \pi_{jk} = - \frac{1}{6} R_{ij} u^i u^j h_{jk} - \frac{1}{2} h_j^i h^i_k R_{ij} + \frac{1}{6} R h_{jk} . \] (4.14a)

5. PROPAGATION EQUATIONS OF KINEMATICAL PARAMETERS

In Einstein theory of gravitation the propagation and constraint equations of kinematical parameters are presented by Ellis [9]. The well known propagation equations of kinematical parameters in EC theory of gravitation are derived here.

5.1. Propagation Equation for Expansion \( \theta \)

We recall the Ricci identity for 4-velocity vector \( u_i \)

\[ u_{i/j/k} - u_{i/k/j} = R_{ijkl} u^i - 2 Q_{ij} u_{j/k} . \] (5.1)
Multiplying the equation (5.1) by \( u^k \) we get
\[
\left( u_{i/j} \right) u^k - u_{i/k} u^k = R_{ijl} u^l u^k - 2Q_{ij} u_{i,j} u^k. \tag{5.2}
\]

However, we know that
\[
\dot{u}_{i/j} = u_{i/k} u^k + u_{i} u^k_{/j},
\]
\[
\Rightarrow \quad u_{i/k} u^k = \dot{u}_{i,j} - u_{i} u^k_{/j}. \tag{5.3}
\]

Substituting this in the equation (5.2) we get
\[
\left( u_{i/j} \right) - u_{i} u^k_{/j} = R_{ijl} u^l u^k - 2Q_{ij} u_{i,j} u^k. \tag{5.4}
\]

Using equation (2.27) we get
\[
\dot{u}_{i,j} = \left( \sigma_{ik} + \omega_{ik} + \frac{1}{3} \theta \ h_{ik} + \dot{u}_{i} u_{k} \right)(\sigma^k_j + \omega^k_j + \\
+ \frac{1}{3} \theta \ h^k_j + \dot{u}^k u_j) + R_{ikl} u^l u^k - \\
- 2Q_{ij} (\sigma_{il} + \omega_{il} + \frac{1}{3} \theta \ h_{il} + \dot{u}_{i} u_{j}) u^k. \tag{5.5}
\]

Contracting the indices \( i \) and \( j \) into the above equation we obtain
\[
\dot{\theta} = \ddot{u}^k - \left( \sigma^k_j + \omega^k_j + \frac{1}{3} \theta \ h^k_j + \ddot{u}^k u_j \right)(\sigma^k_j + \omega^k_j + \\
+ \frac{1}{3} \theta \ h^k_j + \dot{u}^k u_j) + g_{ij} R_{ikl} u^l u^k - 2Q_{ij} (\sigma^k_j + \omega^k_j + \frac{1}{3} \theta \ h^k_j + \dot{u}^k u_j) u^k.
\]

This on simplifying gives
\[
\dot{\theta} = \ddot{u}^k - 2(\sigma^2 - \omega^2) - \frac{1}{3} \theta^2 + R_{ikl} u^l u^k - \\
- 2Q_{ij} (\sigma^k_j + \omega^k_j + \frac{1}{3} \theta \ h^k_j + \dot{u}^k u_j) u^k. \tag{5.6}
\]

This is the propagation equation for expansion parameter \( \theta \) in the presence of torsion tensor. The equation (5.6) is purely kinematical equation. Moreover, by making the use of the field equations \( Q^k_j = k \ S^k_j u^k \) (where the torsion vector is vanishes) and the Frenkel condition \( S^k_j u^k = 0 \), the equation (5.6) gives
\[
\dot{\theta} = \ddot{u}^k - 2(\sigma^2 - \omega^2) - \frac{1}{3} \theta^2 + R_{ikl} u^l u^k - \\
- 2 S^k_j u^k (\sigma^k_j + \omega^k_j + \frac{1}{3} \theta \ h^k_j + \dot{u}^k u_j) u^k,
\]
\[
\dot{\theta} = \dot{u}_{ij}^I - 2(\sigma^2 - \omega^2) - \frac{1}{3} \theta^2 + R_{ik}^I u^I u^k.
\]  
\text{(5.7)}

This equation expressed as the propagation equation of expansion \( \theta \) in EC theory of gravitation.

5.2. Propagation Equation for Shear Tensor \( \sigma_{ij} \)

Propagation equation for shear tensor is obtained by differentiating equation (2.16) covariantly in the direction of the flow vector \( u^k \)

\[
\sigma_{ij,k} u^k = \frac{1}{2} (u_{i,j} + u_{j,i} - \dot{u}_i u_j - \dot{u}_j u_i)_{,k} u^k - \frac{1}{3}(\theta h_{ij})_{,k} u^k,
\]

\[
\Rightarrow \quad \dot{\sigma}_{ij} = \frac{1}{2} [(u_{i,j} + u_{j,i} - 2\dot{u}_i u_j - 2\dot{u}_j u_i)_{,k} u^k - u_{i,j,k} u^k - u_{j,i,k} u^k] -
\]

\[
- \frac{1}{3} \dot{\theta} h_{ij} + \frac{2}{3} \theta u_i \dot{u}_j + \frac{1}{2}(R_{ijkl} + R_{klij}) u^I u^k - Q_{ij}^I u_{i,j} u^k.
\]  
\text{(5.8)}

Using (5.4) in (5.8) we can get

\[
\dot{\sigma}_{ij} = \frac{1}{2} [u_{i,j} + u_{j,i} - 2\dot{u}_i u_j - 2\dot{u}_j u_i - u_{i,j,k} u^k - u_{j,i,k} u^k] +
\]

\[
- \frac{1}{3} \dot{\theta} h_{ij} + \frac{2}{3} \theta u_i \dot{u}_j + \frac{1}{2}(R_{ijkl} + R_{klij}) u^I u^k -
\]

\[
Q_{ij}^I u_{i,j} u^k - Q_{ij}^I u_{j,i} u^k.
\]  
\text{(5.9)}

By using the covariant decomposition of the 4-velocity gradient (2.27) in (5.9), we obtained

\[
\dot{\sigma}_{ij} = \dot{u}_{i,j} - \dot{u}_i u_j - \dot{u}_j u_i - \frac{1}{2} (\sigma^i_{jk} + \omega_{jk} + \frac{1}{3} \theta h_{jk} + \dot{u}_j u_k)(\sigma_j^k + \omega_j^k +
\]

\[
+ \frac{1}{3} \theta h^i_j + \dot{u}^i u_j) - \frac{1}{2} (\sigma^i_{jk} + \omega_{jk} + \frac{1}{3} \theta h_{jk} + \dot{u}_j u_k)(\sigma_j^k + \omega_j^k +
\]

\[
+ \frac{1}{3} \theta h^i_j + \dot{u}^i u_j) - \frac{1}{3} \dot{\theta} h_{ij} + \frac{2}{3} \theta u_i \dot{u}_j + \frac{1}{2}(R_{ijkl} + R_{klij}) u^I u^k -
\]

\[
- Q_{ij}^I (\sigma_{ij} + \omega_{ij} + \frac{1}{3} \theta h_{ij} + \dot{u}_i u_j) u^k -
\]

\[
- Q_{ij}^I (\sigma_{ji} + \omega_{ji} + \frac{1}{3} \theta h_{ji} + \dot{u}_j u_i) u^k.
\]  
\text{(5.10)}

Simplification of (5.10) gives
\[ \dot{\sigma}_{ij} = \dot{u}_{(i,j)} - u_{j} \dot{u}_{i} - \dot{u}_{(i)} u_{j} - \{ \sigma_{ik} \sigma'_{k} + \frac{2}{3} \theta \sigma_{ij} + \omega_{ik} \omega_{j} + \frac{1}{3} \theta h_{ij} + \]
\[ + \frac{1}{9} \theta^{2} h_{ij} + u_{(i} \sigma_{j)k} \dot{u}^{k} + u_{(i} \omega_{j)k} \dot{u}^{k} + \frac{1}{3} \dot{\theta} u_{(i \dot{u} j)} + \]
\[ + \frac{1}{2} (R_{ijkl} + R_{klji}) \dot{u}^{l} \dot{u}^{k} - \]
\[ - 2 [Q_{klj} \sigma'_{ij} + Q_{ijkl} \omega_{ij} + \frac{1}{9} \theta Q_{k(i} h_{j)l} + Q_{ijkl} \dot{u}_{j} u_{l} ] \dot{u}^{k}. \] (5.11)

It follows from the Raychaudhuri equation (5.6) that
\[ \dot{\sigma}_{ij} = \dot{u}_{(i,j)} - u_{j} \dot{u}_{i} - \dot{u}_{(i)} u_{j} - \{ \sigma_{ik} \sigma'_{k} + \frac{2}{3} \theta \sigma_{ij} + \omega_{ik} \omega_{j} + u_{(i} \sigma_{j)k} \dot{u}^{k} + \]
\[ + u_{(i} \omega_{j)k} \dot{u}^{k} + \frac{1}{3} \dot{\theta} u_{(i \dot{u} j)} + \frac{1}{3} h_{ij} [2(\omega^{2} - \sigma^{2}) + \dot{u}^{k}] - \frac{1}{3} h_{lj} R_{kij} \dot{u}^{l} \dot{u}^{j} + \]
\[ + \frac{1}{2} (R_{ijkl} + R_{klji}) \dot{u}^{l} \dot{u}^{k} + \frac{2}{3} \dot{h}_{ij} Q_{km} + \sigma_{m}^{k} + \omega_{m}^{k} + \frac{1}{3} \theta h_{ij} + \dot{u}^{k} u_{l} ] \dot{u}^{k} - \]
\[ - 2 [Q_{klj} \sigma'_{ij} + Q_{ijkl} \omega_{ij} + \frac{1}{3} \theta Q_{k(i} h_{j)l} + Q_{ijkl} \dot{u}_{j} u_{l} ] \dot{u}^{k}. \] (5.12)

With the help of the result (4.14) and the equation (5.12), finally we arrived at
\[ \dot{\sigma}_{ij} = \dot{u}_{(i,j)} - u_{j} \dot{u}_{i} - \dot{u}_{(i)} u_{j} - \{ \sigma_{ik} \sigma'_{k} + \frac{2}{3} \theta \sigma_{ij} + \omega_{ik} \omega_{j} + u_{(i} \sigma_{j)k} \dot{u}^{k} + \]
\[ + u_{(i} \omega_{j)k} \dot{u}^{k} + \frac{1}{3} \dot{\theta} u_{(i \dot{u} j)} + \frac{1}{3} h_{ij} [2(\omega^{2} - \sigma^{2}) + \dot{u}^{k}] + E_{(ij)} - \]
\[ - \frac{k}{2} \pi_{(jk)} + \frac{2}{3} h_{ij} Q_{km} + \sigma_{m}^{k} + \omega_{m}^{k} + \frac{1}{3} \theta h_{ij} + \dot{u}^{k} u_{l} ] \dot{u}^{k} - \]
\[ - 2 [Q_{klj} \sigma'_{ij} + Q_{ijkl} \omega_{ij} + \frac{1}{3} \theta Q_{k(i} h_{j)l} + Q_{ijkl} \dot{u}_{j} u_{l} ] \dot{u}^{k}. \] (5.13)

This is the propagation equation of shear tensor with the additional couplings of torsion tensor in EC theory. The terms in the equation (5.13) are divided into the three groups. The first group of the terms is composed of the kinematical quantities such as rotation tensor, expansion and shear tensor of the congruence with full connection. In the second group of terms, we like to stress that the material source \( \pi_{ij} \) and the electric type tensor (tidal force) \( E_{ij} \) are symmetric. The third group of terms depends on the contribution of torsion tensor. The additional sources from torsion occurred in the third group are simply algebraic in character.
Similarly, the equation (5.13) can be written with the contribution of spin tensor as
\[
\dot{\sigma}_{ij} = \dot{u}_{(i,j)} - \dot{u}_i \dot{u}_j - \ddot{u}_{(i,j)} - \frac{1}{2} \sigma_{ik} \sigma^k_{\ j} + \frac{2}{3} \theta \sigma_{ij} + \omega_k \omega^k_{\ j} + \\
+ u_i \sigma_{j;k} \dot{u}^k + u_{(i} \omega_{j)} \dot{u}^k - \frac{1}{3} \dot{\theta} u_i \dot{u}_j + \frac{1}{3} h_{ij} [2(\omega^2 - \sigma^2) + \dot{u}^k_{\ j}] + \\
+ E_{(j;k)} - \frac{k}{2} \pi_{(j;k)} .
\] (5.14)

**Theorem 1:** For essentially expanding flow of the Weyssenhoff fluid, the symmetric part of tidal force \( E_{ij} \) vanishes.

**Proof:** The essentially expanding flow is characterized by
\[
\sigma_{ij} = \omega_{ij} = 0 , \quad \dot{u}_i = 0 , \quad \theta \neq 0 ,
\] (5.15)

It follows from the propagation equation (5.14) and (5.15) that
\[
E_{(j;k)} - \frac{k}{2} \pi_{(j;k)} = 0 ,
\] (5.16)

For the Weyssenhoff fluid, the equations (4.8e) and (5.16) yields
\[
E_{(j;k)} = 0 .
\] (5.17)

When the space-time is filled up of the Weyssenhoff fluid of the essential expanding flow then \( E_{(j;k)} = 0 \).

### 5.3. Propagation Equation for Rotation Tensor \( \omega_{ij} \)

The propagation equation for rotation tensor is derived by differentiating the expression of rotation tensor (2.21) covariantly in the direction of the flow vector \( u^k \)
\[
\omega_{ij;k} u^k = \frac{1}{2} (u_{ij,j} - u_{j,i,j} - \dot{u}_j u_i + u_j \dot{u}_i)_{,k} u^k ,
\]
\[
\dot{\omega}_{ij} = \frac{1}{2} [(u_{i,j})_{,j} - (u_{j,i})_{,j} - 2 \dot{u}_i u_j] .
\] (5.18)

With the help of (5.4), the equation (5.18) becomes
\[
\dot{\omega}_{ij} = \frac{1}{2} [\dot{u}_{i,j} - \dot{u}_{j,i} - 2 \ddot{u}_i u_j - u_{i,k} u^k_{\ j} - u_{j,k} u^k_{\ i} + (R_{kij} - R_{kji}) u^l u^k - \\
- 2 Q_{ij} \dot{u}^l u^k] ,
\]
i.e.
\[
\dot{\omega}_{ij} = \dot{u}_{(i,j)} - \ddot{u}_{(i,j)} - \frac{1}{2} u_{i,k} u^k_{\ j} + \frac{1}{2} u_{j,k} u^k_{\ i} + \frac{1}{2} (R_{kij} - R_{kji}) u^l u^k -
\]

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Using the decomposition (2.27) of \(u_i\) the equation (5.19) yields

\[
\dot{\omega}_{ij} = \dot{u}_{i[j]} - \ddot{u}_{i} u_{j} - \left\{ \frac{2}{3} \theta \omega_{ij} + \sigma_{ik} \omega_{kj} + \sigma_{ik} \sigma_{kj} - u_{i} \sigma_{jk} \dot{u}^{k} - \right. \\
- u_{i} \omega_{jk} \dot{u}^{k} + \frac{1}{3} \theta \dot{u}_{i} u_{j} \right\} + \frac{1}{2} \left( R_{ijk} - R_{ikj} \right) u^{i} u^{k} - \\
- 2 \left[ Q_{ij}^{i} \sigma_{jk} + Q_{klj}^{i} \omega_{ij} + \frac{1}{3} \theta Q_{klj}^{i} h_{ij} + Q_{klj}^{i} \dot{u}_{i} u_{j} \right] u^{k}.
\]

(5.20)

Simplification of (5.20) gives

\[
\dot{\omega}_{ij} = \dot{u}_{i[j]} - \ddot{u}_{i} u_{j} - \left\{ \frac{2}{3} \theta \omega_{ij} + \sigma_{ik} \omega_{kj} + \sigma_{ik} \omega_{kj} - u_{i} \sigma_{jk} \dot{u}^{k} - \right. \\
- u_{i} \omega_{jk} \dot{u}^{k} + \frac{1}{3} \theta \dot{u}_{i} u_{j} \right\} + \frac{1}{2} \left( R_{ijk} - R_{ikj} \right) u^{i} u^{k} - \\
- 2 \left[ Q_{ij}^{i} \sigma_{jk} + Q_{klj}^{i} \omega_{ij} + \frac{1}{3} \theta Q_{klj}^{i} h_{ij} + Q_{klj}^{i} \dot{u}_{i} u_{j} \right] u^{k}.
\]

(5.21)

With the help of the result (4.14) and the equation (5.21), we obtain finally

\[
\dot{\omega}_{ij} = \dot{u}_{i[j]} - \ddot{u}_{i} u_{j} - \left\{ \frac{2}{3} \theta \omega_{ij} + \sigma_{ik} \omega_{kj} + \sigma_{ik} \omega_{kj} - u_{i} \sigma_{jk} \dot{u}^{k} - \right. \\
- u_{i} \omega_{jk} \dot{u}^{k} + \frac{1}{3} \theta \dot{u}_{i} u_{j} \right\} + E_{[jk]} - \frac{k}{2} \pi_{[jk]} - \\
- 2 \left[ Q_{ij}^{i} \sigma_{jk} + Q_{klj}^{i} \omega_{ij} + \frac{1}{3} \theta Q_{klj}^{i} h_{ij} + Q_{klj}^{i} \dot{u}_{i} u_{j} \right] u^{k}.
\]

(5.22)

This is the propagation equation of rotation tensor in EC theory of gravitation. The terms in equation (5.22) are also divided into the three groups. The first group of the terms is composed of rotation, expansion and shear of the congruence. In the second group, it is necessary to note that, the material source \(\pi_{[jk]}\) and the electric part \(E_{[jk]}\) are appeared in equation (5.22) in contrast to Einstein theory. Where as the third group of terms composed of the additional couplings of torsion.

Finally, the equation (5.22) may be expressed with the contribution of spin tensor as

\[
\dot{\omega}_{ij} = \dot{u}_{i[j]} - \ddot{u}_{i} u_{j} - \left\{ \frac{2}{3} \theta \omega_{ij} + \sigma_{ik} \omega_{kj} + \sigma_{ik} \omega_{kj} - u_{i} \sigma_{jk} \dot{u}^{k} - \right. \\
- u_{i} \omega_{jk} \dot{u}^{k} + \frac{1}{3} \theta \dot{u}_{i} u_{j} \right\} + E_{[jk]} - \frac{k}{2} \pi_{[jk]} - \\
- 2 \left[ Q_{ij}^{i} \sigma_{jk} + Q_{klj}^{i} \omega_{ij} + \frac{1}{3} \theta Q_{klj}^{i} h_{ij} + Q_{klj}^{i} \dot{u}_{i} u_{j} \right] u^{k}.
\]

(5.22)
\[-u_{[i} \omega_{j]k} \dot{u}^k + \frac{1}{3} \theta \dot{u}_{[i} u_{j]} + E_{[i|k|} - \frac{k}{2} \pi_{j|k|}. \]  

(5.23)

**Theorem 2:** For an irrotational Weyssenhoff fluid in absence of acceleration, the skew-symmetric part of the tidal force $E_{ij}$ vanishes.

**Proof:** An irrotational Weyssenhoff fluid with zero acceleration is characterized by

\[ \omega_{ij} = 0, \quad \dot{u}_j = 0, \]

(5.24)

With the help of the propagation equation (5.23) and (5.24) we have

\[ E_{[ij]} - \frac{k}{2} \pi_{[ij]} = 0, \]

(5.25)

For the Weyssenhoff fluid admitting unaccelerated irrotational flow, the equation (4.8e) and (5.25) gives

\[ E_{[ij]} = 0. \]

(5.26)

Hence the proof is completed.

**6. DISCUSSION AND CONCLUSIONS**

The general expressions for the kinematical parameters of the time-like congruence in the presence of torsion tensor are derived. In particular, it is shown that the kinematical quantity acceleration $\dot{u}^i$ depends on $T$-torsion and $V$-torsion, expansion $\theta$ depends on $V$-torsion, shear tensor $\sigma_{ij}$ depends on $T$-torsion only and rotation tensor $\omega_{ij}$ depends on $T$-torsion and $A$-torsion.

The derivation of the propagation equations of kinematical parameters, in comparison to Einstein theory, shows they include some different couplings of torsion tensor. In Einstein theory of gravitation the propagation equation of rotation tensor is influenced by only the kinematical quantities, where as in EC theory of gravitation, the propagation equation of rotation tensor is influenced by kinematical quantities with torsion tensor as well as the additional sources, namely the antisymmetric part of material tensor $\pi_{ij}$ and the antisymmetric part of electric part $E_{ij}$ of the Weyl tensor.

For an irrotational Weyssenhoff fluid in the absence of acceleration, the skew-symmetric part of tidal force vanishes. It is shown that, in the case of the Weyssenhoff fluid describing essentially expanding flow, the symmetric part of tidal force vanishes.
REFERENCES