TOWARDS PLANNING FOR OPTIMISING HIGHWAY-LAYOUT.

The problem of optimisation in road transportation has many facets. For instance 'given the demand conditions, there will be optimum volume of traffic for any given plan of the road. The best possible plan will be that which yields the greatest excess of benefit over cost when carrying the optimum volume of the traffic.'\(^1\) This approach takes the cost of highway transport as function of two variables, volume and density of the traffic on road and volume of the traffic refers to the total amount of traffic in standard vehicle units passing a point in a finite period of time, normally a year.\(^2\) Density of traffic refers to the rate of flow of traffic passing a point at a moment of time and is measured in standard vehicle units per hour. Thus it implies that density is concerned with distribution of total volume from moment to moment over time. Density may be very high at one moment and low at another with given volume of traffic.

While some costs of pavement-wear are functions of aggregate volume of traffic, others notably those costs associated with congestion are entirely function of density.

\(^1\) Winch David M. Economics of Highway Planning. University of Toronto Press - 1965, p. 35.

\(^2\) Ibid, p. 22.
The optimum has been defined, by 'Winch', in terms of maximum net benefits with given level of income and cost situation. This optimising problem relates to road surface specifications, width of the road and if necessary additional road to divert heavy traffic. All these problems are mainly considered as the problems of highways having heavy traffic. Though this approach of volume and density of traffic has received greater attention, this approach is inadequate as it accepts existing road system as given, and examine the possibilities of improving the quality of road surface, and width with some predetermined criteria.

This approach ignores the following problems.

1) The problem is of a highly hypothetical nature.
Suppose that there are many cities and towns that are to be connected by roads where no single road exists. What would be the optimum locational pattern of roads connecting towns and roads to each other so that total cost comprising track-cost and operating cost by all vehicles would be minimum?

11) The second problem relates to the realistic question. Suppose that a town is to be connected by road to existing road leading to main destination or destinations. What would be angle at which it would be connected to the existing road so that total cost comprising track cost and operating cost of all vehicles would be minimum?

111) The third question relates to the problem of minimising the deviation from optimum pattern by improving the existing road system which is not expected to be optimum as we have defined, assuming that no problem of congestion exists.

The first question has a considerable theoretical value but it lacks applicability to real world where a certain road system already exists. Hence, we shall not try to seek a solution to it. We shall concentrate on the second and third questions as they have highly theoretical value as well as practical applicability, especially to the under-developed countries like India.

The problem of optimising road network basically arises because there cannot be only two cities in the region that are to be connected directly by the road. If there would be only two cities in the region to be connected by road,
The total cost would be minimum if direct road connecting two cities would be constructed, as the distance to be constructed would be minimum so that track-cost of turn would be minimum, and as distance to be traversed by vehicles would also be minimum, the operating cost of all the vehicles also would be minimum. The minimum track-cost plus minimum operating cost gives us minimum total cost. But in practice, there exist not only two cities between which traffic operates but there are many cities, towns, and villages which require accessibility by road. We cannot suggest that all cities, towns, and villages are to be connected directly and independently by roads, operating cost of the vehicles would be minimum, because this kind of road plan would involve a very high construction cost of roads leading to very high track-cost; and all roads would tend to be under utilized. Alternatively, if there would be minimum new milage to be constructed, using the excess capacities of existing roads, the actual distance to be traversed by a vehicles would be too long involving a heavy burden of operating cost, as it is a function of distance to be traversed for internodal transportation. This cannot be justified as the main component of operating cost is that of petroleum products of which sources are exhaustible and secondly, particularly for India it has high foreign exchange components.
Thus, there is a need for new approach for optimising the road network so that the total cost for inter-nodal traffic would be minimum. In this approach we have to take into account the existing roads having excess capacity and vehicle linkage intensity where linkage intensity as defined earlier, implies number of vehicles plying between two given nodes.

**INTER-NODAL ROAD TRANSPORT COST FUNCTION**

For optimising layout planning for highway system, there is a necessity of analysis of the nature of the cost function from macro-point of view for inter-nodal vehicle traffic. The inter-relation among the many components of the total cost function is crucial. This kind of analysis is required to develop the optimising model for lay-out planning for new roads to be constructed with a given preexisting road system.

The total inter-nodal cost of vehicle traffic is a function of two variables; one refers to total number of vehicles traversing the distance between any two nodes with given cost conditions which can be termed as vehicle linkage intensity in well-defined units; the other variable refers to the distance to be traversed by concerned vehicles between the two nodes.
Given the inter-nodal vehicle linkage intensity, inter-nodal road transport cost is a function of the distance to be traversed by the given number of vehicles. Thus the distance by road between two nodes is a crucial factor for the determination of road transport cost function.

The actual distance, with given nodes is a function of locational pattern of roads that we desire to optimise.

The total cost of inter-nodal road transportation can be classified as follows:

1) Track-cost: Which implies the interest on construction cost comprising expenditure on land purchase, raw material and labour to be employed and fixed maintenance cost independent of vehicular density.

2) Operating cost of vehicles which is function of distance. It comprises fuel cost, depreciation of tyres, wear and tear of vehicles, variable maintenance cost as function of vehicular density.

3) Investment cost for vehicles: It implies interest charges for capital invested in vehicles.
4) Terminal Cost: It implies the cost for loading and unloading by trucks, commission by transport agent, in the case of passenger bus transportation, cost involved provision of waiting rooms. We shall ignore the terminal cost in the our analysis as it is independent of distance between two nodes.

Furthermore, for analytical simplicity and on the basis of macro-point of view, we combine investment cost for vehicles with operating cost for vehicles as function of distance, as road system on the basis of optimum model would reduce the inter-nodal distance in to entire economy. The reduction in inter-nodal distance would require lesser number of vehicles to satisfy the given level of demand for transportation. This justifies the inclusion of interest cost of capital investment in vehicles.

Thus we analytically divide the total macro-cost of inter-nodal road transport into only two categories as i) Track cost, ii) Operating cost of vehicles including interest cost on investment in vehicles.
Now let us deal with the problem related to optimising the inter-nodal road transport total cost function in the situation where the town is to be linked by the new road to the preexisting road leading to the given destination. The situation can be illustrated by the diagram No. 9.1.

Consider the situation where a linear road AD already exists leading to destination 'A' and let us assume that town B is not linked by any road. Furthermore, we assume that the probable vehicular traffic plies only between A and B. The estimated probable vehicular traffic is known. The AD road has excess capacity implying that additional traffic between B and A will not increase the track cost of road AD. As already pointed out, we have assumed that maintenance cost is partly dependent on vehicular traffic density and certain portion of it is independent of vehicular density. Former would be termed as variable maintenance cost and later as fixed maintenance cost. We assume that new road to be constructed is of the same surface specification as the existing road AD so that the operating cost per vehicle would be the same.

If the new road from B linking the existing road AD is perpendicular, from B to AD road, the construction cost would be minimum because the perpendicular distance to the line from
Figure 9.1
any point is minimum. But by this perpendicular road the total operating cost by all vehicles would be more. This perpendicular road need not be optimum road as the total cost comprising track cost of the new road and operating cost of the vehicles running between AB need not be minimum.

In this context track-cost refers to the track-cost of the new road only and operating cost refers to operating cost by all vehicles running between A and B by this new proposed road and using a certain section of existing road AD within range indicated by section AJ, as shown in the diagram if J is the joining point of the new road which forms a perpendicular from B to AD.

The track-cost relates to the interest cost of investment for construction of road of the length BJ. The relevant operating cost by all vehicles would be that of traversing the distance by the vehicles BJ + JA.

If a new direct road for B to A is available the actual distance to be traversed by all vehicles would be minimum implying minimum operating cost for all vehicles but the track-cost would be more as the new road BA to be constructed would be longer than BJ so the direct road from B to A need not necessarily be an optimum road implying minimum total cost.

If we compare the direct road with the right angle road, we can say that the operating cost of vehicle would be less in
the case of the direct road as compared to perpendicular road system as the distance to be traversed would be less because \( BA > BJ + JA \). (The sum of any two sides of a triangle is more than length of third side).

But the track-cost would be more in the case of all other non-perpendicular roads including direct road AB as the length of new non-perpendicular road would be more as compared to perpendicular road having minimum length.

Then the perpendicular road and the direct road provide two alternatives with minimum track cost and minimum operating cost respectively but both alternatives need not be necessarily the alternative of minimum total cost representing the optimum.

The above discussion reveals the significant inter-dependence of two components of cost vis. operating cost of the vehicles and track cost for given two nodes with preexisting road.

Obviously we can say that the total operating cost by all vehicles plying between A and B is inversely proportional to the track cost of new road to be linked to relevant section i.e. \( BJ \) of existing road. In the diagram No. 1, B and A are given nodes to be linked with additional road (AS) joining the section \( BJ \) of the existing road.
AJ is perpendicular distance from destination A to the existing road, with given length.

AB is a direct distance between the two destinations A and B with given distance.

AS is the new road to be constructed providing the link between A and B joining at point S on the section BJ of existing road with varying distance 'a'. This implies that the distance to be traversed by vehicles between A and B on the new road (AS) and also on the existing road SB varies as d varies.

Suppose AS + SB = Y i.e. equal to the distance to be travelled by vehicles plying between A and B, by the new road.

Q = total track cost

Z = Operating cost by all vehicles plying between A and B.

so Y \propto \frac{1}{d} \quad (d \text{ should not be greater than } AB)

Z \propto Y

but

Y \propto \frac{1}{d}

Z \propto \frac{1}{d}

further the total track cost would be direct function of new road to be constructed of given specification so -

Q \propto d, Z \propto \frac{1}{Q}.
Thus the total operating cost of the given number of vehicles for given internodal road transportation would be inversely proportional to the track cost provided that the length of the new road is not more than the length of direct road. The new road with more length than that of the direct one creates the problem of detouring increasing the distance to be traversed between A and B by the vehicles. Hence the interdependence between the operating cost and the track-cost is obvious.

Conversely, the reduction in track cost tends to increase the burden of operating cost for internodal traffic as a result of longer distance to be traversed.

This kind of interdependence provides the guidelines for the optimising solution for layout planning.

We shall, therefore, try to develop a model which will provide the solution for optimum road which is defined as the road by which the total cost comprising operating cost by given vehicles and track cost would be minimum. Our model would confine to only two nodes A and B. The model would provide a solution to the planning of a new road linking to the existing one in such a manner as would ensure the minimum total cost attributable for the traffic between B and A. The assumptions for the model are as follows :-
1) The operating cost per vehicle per unit distance is given and constant.

ii) The interest cost of construction and part of maintenance cost independent of vehicular traffic per unit distance is given and constant.

This is termed as track cost.

iii) The new road is to be constructed of the same surface specification implying that the average operating cost would be same as on the existing road.

iv) There is no problem of congestion. And therefore no additional cost would be imposed due to the additional traffic of vehicles from B to A owing to the new road.

v) We assume that the maintenance cost is partly dependent on the traffic flow. Therefore certain portion of maintenance cost for the existing road would be included in operating cost of the vehicle, though it appears an unusual.

The model is based on the minimum distance line which is perpendicular from given node B to the existing road, the length of which can be measured, and can be taken as given and constant (L). The distance between the destination A and the
meeting point of the perpendicular from destination B on the existing road can be measured and regarded as given and constant (T). Thus L and T both are constant as shown in the diagram. 9.2

Two sections of distance between A - B by roads are revealed as BM and MA. BM section represents the length of the new road to be constructed and MA section represents the section of existing road that would be used for traffic between B and A. For BM section, operating cost by the vehicles as well as track cost is to be incurred, while for MA, as there is no additional track cost for additional traffic on route B - A only operating cost is to be incurred.

Furthermore as explained earlier, the total distance to be traversed is a function of the length of the new road. The total distance varies inversely as the length of new road to be constructed. It is a decreasing function of length of new road (BM) where length of new road is not more than direct road linking B and A.

The total cost function for given inter nodal traffic will comprise the track cost for new road to be constructed and operating cost for all vehicles running between two nodes A and B.
Figure 9.2

Reference existing Road

Optimum Road

Perpendicular (L)

(A)

(M)

(B)

(J)

(D)
The purpose of building an optimising model is to find out the angle at which the new road joins the existing road so that the total cost would be minimum.

The total = The track cost + Operating cost of the vehicles cost of new road for distance to be covered.

= (Length of new road x track cost per unit distance)
(Number of the vehicles x operating cost of vehicle)
Per unit distance x total distance to be traversed

By rearranging, the total cost function can be expressed section-wise of the total distance to be covered.

Total cost = The cost confined to section BM + The cost confined to section MA.

Putting BM = d_2
MA = d_1

The total = d_2 (track cost per unit distance) + (number of vehicles x operating cost per vehicle per unit distance) + d_1 (Number of vehicles x operating cost per vehicle per unit distance).

Suppose c = track-cost per unit of distance,
n = number of vehicles plying between A and B,
a = operating cost per vehicle per unit distance.

So total cost = d_2 (c + n x a) + d_1 (n x a)
For analytical purpose we can express the cost function into average total cost, dividing total track cost and operating cost by the number of vehicles. This gives us separately average track cost per vehicle per unit of distance and average operating cost generally is already known. But the average track-cost is to be calculated by following relationship.

Suppose \( c \) is track-cost per unit distance and if the number of vehicles are \( n \),

\[
\text{Average track-cost} = \frac{c}{n} = c_1,
\]

\( c_1 \) = average track-cost.

Thus we can express the cost function in terms of average total cost function comprising average track-cost per vehicle of unit distance + operating cost per vehicle per unit distance.

The optimising model in terms of average cost provides a solution to the problem of minimising total cost as follows:

\( Y = \) Total cost of the inter-modal road transport by the automobiles between A and B. Comprising total track-cost and total operating cost by the vehicles.

\( Y_1 = \) Average total cost per vehicle \( i.e. \ \frac{Y}{n} \) if

\( n = \) number of vehicles.

\( n = \) Number of vehicles plying between A and B nodes in a given period.
C = Track-cost per unit of distance.

$c_1$ = Average track-cost per vehicle per unit distance = C/n.

a = Operating cost per vehicle per unit distance.

d_1 = Distance between joining point (M) of existing road and new road and destination A on the existing road (d_1 = MA).

d_2 = The length of new road, linking B to existing road AD at M (d_2 = MB).

L = The perpendicular distance from B to existing road AD i.e. BJ = L. This is constant.

T = The distance between A and joining point of perpendicular from B to existing road AD at J i.e. AJ = T which is constant.

θ = The angle between new road to be planned and perpendicular from B to existing road at B.

θ = [JBM which is variable and to be determined.

We can put the total cost function (Y) as -

Y = na (d_1 + d_2) + cd_2

We get average total cost function by dividing n

$Y/n = a (d_1 + d_2) + c/n d_2$
Putting $c_1 = c/n$ we get

$$Y_1 = a (d_1 + d_2) + c_1 d_2$$

By rearranging we get

$$Y_1 = (a + c_1) d_2 + ad_1$$

We can express $d_2$ and $d_1$ in terms of $\theta$ as $(d_1 + d_2)$ is a function of $\theta$.

$Y_1$ is a dependent variable with given values of $a$ and $c_1$; it varies according to $(d_1 + d_2)$; hence it can be expressed terms of $\theta$ as $d_2 = L/\cos \theta + a (T - L \tan \theta)$

$$Y_1 = (a + c_1) L/\cos \theta + a (T - L \tan \theta)$$

$$Y_1 = (a + c_1) L/\sec \theta + a (T - L \tan \theta)$$

We get minima or maxima by differentiating $Y_1$ with $\theta$

$$\frac{dY_1}{d\theta} = \sin \theta = \frac{a}{a + c_1}$$

As $d^2Y_1$ gives a positive value $\sin \theta = a/a + c_1$

$$\frac{d^2Y_1}{d\theta^2}$$

gives minima of the function $Y_1 = (a + c_1) L \sec \theta + a (T - L \tan \theta)$

Thus the angle $\theta$ giving minimum average cost is

$$-\theta = \sin^{-1} \frac{a}{a + c_1}$$

This $\theta = \frac{-1}{\sin a} \frac{a}{a + c_1}$ is the optimum angle between the new
road and perpendicular from B to existing road AD at J i.e. $\theta = MBJ$.

The evidently the new optimum road with angle $\theta$ at B joins the existing road with angle $(90 - \theta)$ as shown in the diagram.

$$\theta = \sin^{-1} \frac{a}{a + c_1}$$

and

$$c_2 = BM = T \sec \theta$$

when

$$\theta = \sin^{-1} \frac{a}{a + c_1}$$

This road will be optimum provided the length of optimum road $BM$ is not more than direct road $BA$.

The determinants of the optimum angle

$$\theta = \sin^{-1} \frac{a}{a + c_1}$$

indicate that the optimum angle is independent of distance $L$ and $T$. It shows that location of $A$ and $B$ are not important for optimum angle.

Further it shows that, if track-cost per unit distance, per vehicle ($c_1$) is constant, $\theta$ is increasing function of operating cost per vehicle per unit distance. The higher value of 'a' with given $c_1$ the wider the angle $\theta$ implying greater length of the new road to be constructed. This is reasonable as increase in the length of the new road can be

For detailed mathematical analysis, please see Mathematical Appendix on page 501.
justified owing to extra saving in operating cost made possible by it.

The track-cost per vehicle per unit distance influences the value of $\theta$ inversely with given value of 'a' i.e. operating cost per unit distance per vehicle. It implies that with given value of 'a', increase in track-cost per unit distance per vehicle tends to reduce the angle $\theta$ thereby reduce the length of potimum road. This is also logical as the saving in operating cost owing to reduced distance to be traversed will allow the lesser length of the new road to be constructed.

The effect of number of vehicles on the value of $\theta$ is implicit. The effect can be revealed by the relationship i.e. $c_1 = c/n$.

Placing the value of $c_1 = c/n$ in the relationship

$$\theta = \sin^{-1} \frac{a}{a + c_1}$$

$$\theta = \sin^{-1} \frac{a}{a + c_1} \quad \text{but } c_1 = c/n$$

$$\theta = \sin^{-1} \frac{a}{a + c/n} = \sin^{-1} \frac{na}{na + c}$$

This indicates that with given value of 'a' and track-cost per unit distance $c$, the value of $\theta$ would progressively increase the number of vehicles increases. This shows that
the length of the optimum road would be more at the point when the number of vehicles is larger as compared to other point where number of vehicles is relatively less with given values of a and c.

This is natural as absolute savings in total operating cost would be relating more as number of vehicles is greater justifying relatively more length of the optimum new road with given a and c.

Let us now consider the condition under which right angled road system would coincide with optimum angle road system.

In the right angle system new link road is generally connected with 90° with existing road. It implies $\theta = 0$ since $\sin 0 = 0$.

$$\frac{a}{a + c_1} = 0$$

This implies that if $c_1$ has positive value and $c_1$

$$a = 0$$

So we can say that right angle system would be optimum only when operating cost of the vehicle is zero. This is impossible.

If we take the form

$$\sin \theta = \frac{na}{na + c} = 0$$

as $\theta = 0$
With positive $a$ and $c$, it is possible only when $n = 0$.

Only on the rather absurd assumption that no vehicle is going to use the road and that there is no operating cost for non-vehicle transportation would the right angled system would coincide with optimum road system. This is highly improbable in the modern world.

Further with given positive $a$ and $n$, $\theta = 0$ only if

$$\sin \theta = \frac{a}{a + c_1} = 0$$

It implies only purely theoretical possibility that is $c_1 = \infty$. If the track-cost per unit distance per vehicle is infinite then optimum angle would be equal to zero. This is practically impossible.

Thus our model shows that for optimising road lay-out planning both costs track-cost as well as operating cost by the vehicles are highly significant. We can develop lay-out planning so that total cost of inter-modal vehicle traffic can be minimised with assumed conditions.
MODIFICATIONS OF OPTIMUM ROAD MODEL.

It is necessary to examine some exceptional cases for which this model can not be applied directly. This implies that there are some major possibilities where the angle determines by our optimum road model may not be optimum as defined.

The particular case may be cited as below. If the equation of optimum angle provides the angle greater than the angle between perpendicular from destination B to the existing road AD and direct road connecting B and A as shown in the figure.

A and B destinations to be connected.
AD existing road.
BJ perpendicular line from B to joining point of existing road AD.
Angle SBJ is angle determined by the optimum model suggesting BS new road = 0.
AD is direct road connecting destinations A and B.
The angle ABJ that is the angle between direct road from B to A and perpendicular BJ = θ₁

If θ₁ is less than 0 the road SB suggested by optimum model would have greater length than direct road and also it creates the necessity of detouring for all vehicles as also with the need of additional construction of road from S to A.

This implies the more operating cost along with greater construction cost for road suggested by optimum model as compared to direct road BA.
Figure 9.3
This suggests us that if the angle between the new direct road for the given destinations and perpendicular from destination to existing road is less than the angle suggested by optimum model. The direct road is minimum cost road and so optimum road and not road suggested by optimum model.

In short the relationship \( \theta = \sin^{-1} \frac{a}{a + e_1} \) gives optimum road only if length of the optimum road is less than direct linear road. Otherwise direct linear road between two nodes concerned is to be accepted as minimum cost road.

II. The optimum model may not give necessarily optimum road in the case of missing link between the perpendicular line for destination B to extended section of the existing road AD as shown in the diagram.

Consider the two destination A and B that are to be connected by new road AD road exists as shown in the diagram. The peculiarity of this case is the road that exists is not up to the joining point of perpendicular from B to extended line from road AD up to J as shown in the diagram.

DJ indicates the section having no existing road.

BJ is perpendicular from B to extended line BJ.

So the optimum angle (\( \theta \)) has to be measured with reference to perpendicular BJ. Suppose that the direct road
line from B to D as shown in the diagram makes the angle \( \theta_1 \) at B with perpendicular BJ that is angle JBD and suppose optimum angle at B is angle JM.

This model cannot be applied directly if \( \theta \) that is optimum angle is less than \( \theta_1 \) that is the angle between perpendicular and direct road from B to starting point of existing road AD that is point D because the road with optimum angle given by optimum model necessitates the additional construction of road from D to M and further increases the distance as compared to the route by direct road BD. As HM + MD is greater than BD (as the sum of two sides of triangle is always more than any one side of the triangle).

This comparatively direct road from B to D involved less total cost including track cost and operating cost of vehicles if the optimum angle tis less than the angle between direct road and perpendicular line with reference to extended line from existing road.

If the optimum angle is greater than the angle between direct road from destination B to starting point of the road and perpendicular line, the road suggested by optimum model can be taken as optimum road.
BREAK-EVEN ROAD

The wide divergence can be observed between actual distance by road and minimum straight line distance between any two nodes even in the case of towns having significant inter-nodal vehicular traffic.

This divergence suggests the scope for the reducing the inter-nodal distance by roads leading to significant saving in operating cost by the vehicles concerned.

As all important towns are served by roads, the optimum road model can not be directly applied to rationalize the existing road system.

The relevance of optimum road model lies in the fact that the optimum road results in saving in inter-nodal distance leading in turn to significant savings in the operating cost of the vehicles but the net savings in operating cost of the vehicles as a result of anticipated reduced distance must be sufficient to justify the additional capital expenditure required for construction of optimum roads. In short, as a minimum condition, every optimum road must be, at least, break-even road as the road reducing the inter-nodal distance so that saving in operation cost by the concerned vehicles is just equal to track cost of the new optimum road. As the savings in the operating cost is not the exclusive function of distance reduced but also the function of vehicles running between two nodes or multiple nodes, we can have also concept of critical break-even number of vehicles. We define
critical break even number of the vehicles as number of vehicles to generate an aggregate saving in operating cost just equal to meet track cost of new proposed optimum road.

If there are vehicles equal to minimum critical number of the vehicles, the optimum road would become break-even road.

If the actual number of vehicles exceeds that represented by critical number the optimum road would become surplus generating road the savings in total cost would be maximum.

If the vehicles are less than minimum critical number, the road may be optimum, but it cannot be considered as break-even road but the deficit would be minimum.

For break-even road minimum condition is that new savings in operating cost by must be equal to track cost of the new road.

It would be instructive to analyse the cost-saving function. We can say that saving in total cost is generated owing to two important variables, with given cost conditions and specifications.

1. Saving owing to reduction in distance between two nodes.
2. With given road the total saving in function of number of vehicles.
Diagram 9.5

(Critical break even Road with fixed node)

Diagram 9.6

(Critical Break even Road with given Road length and without fixed node)
Let us analyse first the case where two exist joining to nodes A and B. The two roads are linked with each other at right angle at point O as shown in the diagram.  

Suppose that we have decided to construct new road from B for traffic between A and B joining existing road AO with a view to saving in operating cost.

If new road makes angle $\theta$ with OB that is perpendicular existing road joining OA at O. And number of the vehicles is n.

\[ \text{Saving in operating cost} = a \times n(BO + OK - BK) \]
\[ = a \times n(S + S \tan \theta - S \sec \theta) \]
where $S = d$, distance from B to O on the existing road.
\[ = an \times s(1 + \tan \theta - \sec \theta). \]

The track cost of new road BK = $C \times BK$
\[ = C \times S \sec \theta \]
as $BK = S \sec \theta$.  

The new road BK would be break even road only if

\[ \text{Track-cost} = \text{saving in operating cost by vehicles}. \]

So the condition is -

\[ C \times S \sec \theta = na \times S(1 + \tan \theta - \sec \theta) \]
Cancelling 'S' from both sides we get

\[ C \sec \theta = na(1 + \tan \theta - \sec \theta) \]
This equation provides us the relationship between savings in operating cost and break-even number of vehicles.

\[ n = \frac{c/a}{\frac{\text{Sec } \theta}{1 + \text{Tan } \theta - \text{Sec } \theta}} \quad \ldots \ldots \quad I \]

The equation 'I' indicates that with given values of \( c \) and \( a \), break-even number of vehicles is a function of only \( \theta \). Contrary to expectations, it is independent of the absolute distance by existing road between two nodes \( A \) and \( B \), and also not function of \( S \) i.e., distance from \( B \) to joining point of perpendicular to existing road \( AO \).

The equation 'I' indicates with the given values of \( c \) and \( a \) a break-even number of vehicles changes as \( \theta \) changes. In the light of this equation we can say that there is a possibility of a particular value of \( \theta \) than would require minimum break-even number of vehicles. This \( \theta \) may be termed as critical angle and this particular minimum number of break-even number corresponding to critical angle may be termed as critical break-even number of vehicles.

We shall try to verify whether there is break-even angle and further to find critical number of vehicles, for the right angled road system. So that without resorting to cost benefit analysis of multiple alternatives of additional road with given vehicle linkage intensity we can decide whether there can be critical break-even road.
We can determine, if possible, the critical angle of new proposed road by following functional relationship:

\[ n = \frac{c}{a} \frac{\sec \theta}{1 + \tan \theta - \sec \theta} \]

We can get minima or maxima by differentiating \( n \) with respect to \( \theta \) as

\[ \frac{dn}{d\theta} = \sec \theta (\tan \theta - 1) = 0 \]

As \( \sec \theta = 0 \)
\( \tan \theta - 1 = 0 \)
\( \tan \theta = 1 \)

hence \( \theta = 45^\circ \)

When \( \theta = 45^\circ \), \( \frac{d^2n}{d\theta^2} \) is found positive

Hence \( n \) for \( 45^\circ \) is minimum value of \( n \)

Thus we have obtained the critical angle providing critical road that requires minimum break-even number of vehicles. This critical angle corresponding to existing right angle road system is determined and found equal to \( 45^\circ \). It indicates that when new road joining right angled roads from given point is constructed with \( 45^\circ \), the number of vehicles required to make the road break-even is minimum. It implies that road with only \( 45^\circ \) requires minimum break-even number of vehicles. The new roads with angles other than \( 45^\circ \) would require more number of vehicles as compared to vehicles.
required for road with $45^\circ$ to make the road break-even.

The road which requires minimum number of vehicles to become break-even road will be termed as critical break-even road. The distinction between break-even road and critical break-even road is to be borne in mind. Any road with any angle can be termed as break-even when sufficient number of vehicles are running between two concerned points so that saving in operating cost of the vehicles is just equal to meet the track-cost of new road. This number is termed as break-even number of vehicles. The number of break-even vehicles would be different for different angles ($\theta$) with $a$ and $c$ given.

$$n = \frac{c/a}{\frac{\sec \theta}{1 + \tan \theta - \sec \theta}}$$

This can be confirmed by calculating the value of i.e. break-even number of vehicles with respect to different values of $\theta$. The results are shown in the Table.

The table shows that with given value of $a$ and $c$ the break-even number of vehicles in a function of

$$\frac{\sec \theta}{1 + \tan \theta - \sec \theta}$$

From the table we can get the different features of

$$\frac{\sec \theta}{1 + \tan \theta - \sec \theta}$$

for different values of $\theta$ as follows:

1) The value of it with reference to $45^\circ$ is minimum and it confirms our earlier findings.
2) The value of \[ \frac{\sec \theta}{1 + \tan \theta - \sec \theta} \] tends to increase as the degree of deviation of \( \theta \) from 45° increases. It is not only value of \( \theta \) that determines the value of it, but it also depends upon the degrees of deviation from angle 45°.

3) The values of \( \theta \) it with reference to \( \theta \) and \( (90 - \theta) \) are same.

4) The nature of function can be described as symmetrical parabolic with minima at 45°. From 0° to 45° the value of \[ \frac{\sec \theta}{1 + \tan \theta - \sec \theta} \] tends to decrease as value of \( \theta \) increases. The value at 45° is minimum and from 45° to 90° it tends to increase as \( \theta \) increases.

**Relationship between the critical break-even road and optimum road:**

Let us examine the relationship between critical break-even road and optimum road with reference to right angle road system. As shown above the critical number of vehicles refers to the break-even road having angle 45°. So, the critical number of vehicles is given by following equation.

When \( \theta = 45° \), \( m_1 = 0 / a \), \[ \frac{\sec 45°}{1 + \tan 45° - \sec 45°} \]

From the Table No., the value of \[ \frac{\sec 45°}{1 + \tan 45° - \sec 45°} \] is 2.4141. So \( n_1 = 2.4141 \text{ c/a.} \)
This indicates that the value of critical break-even number of vehicles and the number of vehicles that makes road with $45^\circ$ optimum is same as

\[
\frac{\sec 45^\circ}{1 + \tan 45^\circ - \sec 45^\circ} = \frac{\sin 45^\circ}{1 - \sin 45^\circ} = 2.4141.
\]

This shows the critical break-even road is also optimum road with $\theta = 45^\circ$.

In this context, we can say that all break-even roads are not necessarily optimum and all optimum roads are not necessarily break-even roads but only critical break-even road is optimum road. In other words we can say that the critical break-even road and optimum road both coincide only when the critical break-even number of vehicles are plying.

So far we have discussed as to how the road is to be constructed from given fixed nodes to the existing right-angled road system. Now we shall discuss as to how the road is to be constructed when its length is given owing to financial stringency etc. assuming, further that no fixed node is given from which the new road is to be started. We shall therefore try to find out the angle the new road makes with the existing right-angled road system by which the distance saved would be maximum.
In the diagram No. 9.6 AO and BO are two roads joining at right angle with each other at point O. Suppose the length of the new road to be constructed is given to be equal to 'b'. The distance from O to the joining point on OA is equal to 'x' and that on OB is equal to 'z'. The value of 'x' and 'z' varies as \( \theta \) changes, \( \theta \) being the angle that the new road makes with the existing road OA. So 'y', i.e. the distance saved is:

\[
y = z + x - b.
\]

As \( z \) and \( x \) are the functions of \( \theta \), we can write
\[
z = b \sin \theta; \quad \text{and} \quad x = b \cos \theta.
\]
\[
y = b \sin \theta + b \cos \theta - b.
\]
\[
y = b(\sin \theta + \cos \theta - 1)
\]

We can identify, with the above function, the angle of the new forked road which could give the maximum saving in distance, with reference to joining points.

We get maxima or minima by differentiating \( y \) with \( \theta \).

For maxima or minima,
\[
y = b(\sin \theta + \cos \theta - 1)
\]
\[
\frac{dy}{d\theta} = (-\cos \theta + \sin \theta) = 0.
\]
\[
\sin \theta = \cos \theta.
\]
\[
\frac{\sin \theta}{\cos \theta} = 1
\]

i.e. \( \tan \theta = 1 \),
and so \( \theta = 45^\circ \)

Now the value of \( \frac{dy}{d\theta} \) is negative.

So \( \frac{dy}{d\theta} \) gives maxima.

Thus it is clear that the distance saved would be maximum only when the new linear road with given length joins the existing right angle road system at 45° (\( \theta \)). There would a loss of the distance saving opportunity; imposing an unnecessary burden with the same track-cost on the vehicles and consequently on national economy from macro-economic point of view, if the roads are constructed at the angles other than 45° corresponding to the existing right-angled road system. Since, the road with given length with 45° saves the maximum distance, it can be termed as "Economically most efficient Road."

As the road length is given the total track-cost i.e. \((c \times b)\) would be fixed as \(c\) and \(b\) are constant. It implies that with the given total track-cost the road with 45° would require critical break-even number of vehicles as the saving in distance is maximum.

We can estimate the break-even number of vehicles for the new road of which the length is given as follows:
Track cost = Saving in distance x number of vehicles x operating cost per vehicle per unit distance.

\[ c \times b = b \left( \sin \theta + \cos \theta - 1 \right) \times n \times a \]

\[ n = \frac{c}{a} \left( \frac{1}{\sin \theta + \cos \theta - 1} \right) \]

We realise that the angle which requires the minimum number of vehicles, is same for both types of functions. If the road is to be constructed from given node, joining right angled road system, the angle that requires critical break-even number of vehicles is 45° and in the case of the road with given length with no fixed node, the angle that requires critical break-even number of vehicles is also 45°.

Let us have the explanation for it. The break-even number of the vehicles in the case of fixed node, as explained earlier is given as -

\[ n = \frac{c}{a} \frac{\sec \theta}{1 + \tan \theta - \sec \theta} \]

In the case of new road to be constructed with fixed length without fixed node n is given by

\[ n = \frac{c}{a} \frac{1}{\sin \theta + \cos \theta - 1} \]

We can have the explanation for it, if we can show

\[ \frac{c}{a} \frac{\sec \theta}{1 + \tan \theta - \sec \theta} = \frac{c}{a} \frac{1}{\sin \theta + \cos \theta - 1} \]
We can prove it cancelling c/a from both sides.

\[
\frac{\sec \theta}{1 + \tan \theta - \sec \theta} = \frac{1}{\cos \theta} = \frac{1 + \frac{\sin \theta}{\cos \theta} - \frac{1}{\cos \theta}}{\cos \theta + \sin \theta - 1}
\]

So we can say that in the case of both types of situations the break-even number of vehicles are same.

This implies also the critical number of vehicles are also same for both types of situations. Further it provides the additional evidence for the equality of critical angle i.e. 45° in the case of new road with fixed node as well as new road with given length without fixed node.

Further it is clear that both functions related to new road with fixed node and road with given length without fixed node do not contain element of fixed distance, namely, s and be respectively.

Thus, break-even number of vehicles is independent of s and b which distinguish the two functions. The implications of this point are very important as both equations are interchangeable. So far extensive analysis if we deal with one type
of function to determine the critical angle same findings can be applicable to other types of functions. To be more specific, if we deal with the problem of determining the critical angle in the case of a new road with given length without predetermined node, the findings can be applied to problem of road with predetermined node, assuming other things constant.
CRITICAL ANGLES FOR THE ROADS OTHER THAN THAT OF
RIGHT ANGLE SYSTEM

We have so far discussed the determination of critical angles of the new road in the context of existing two roads making an angle of 90° with each other. The equation of critical angle of new road with reference to the roads meeting each other at an angle other than 90° remains unanswered. Let us try to find out the programming model to determine the critical angle of new road with reference to any angle as shown in the diagram. 9.7

We shall try to find out the functional relation to determine the critical angle in the case of new road with given length without any predetermined node joining two roads having any angle.

Suppose that the length of the new road to be constructed is given and equal to 'b'. The new road of length 'b' is to be constructed to connect the two existing roads OR and OT meeting at O with the given angle without fixed node.

Suppose the angle the new road makes with OT is θ. Hence the new road will make the angle with other existing road OR will be 180° - (θ + θ).

Now we shall try to determine the value of the critical angle θ so that the new road with given length would save the
maximum distance corresponding to the two joining points of the new road to the existing road.

Suppose two existing roads OR and OT meet at O. The distance from O to the joining points of the new road on OT and OR is x and z respectively. We must be aware that the distance x and z vary as $\theta$ changes.

Let us, further see whether there can be any particular angle between the new road and the existing one OT by which the distance saved would be maximum, with the help of following function.

$$Y = x + z - b$$  \hspace{1cm} Y = \text{distance saved.}  \\
\hspace{1cm} b = \text{given length of the new road.}$$

But we have,

$$\frac{b}{\sin \beta} = \frac{x}{\sin(\beta + \theta)} = \frac{z}{\sin \theta} \quad \ast \text{as } \sin 180^\circ - (\beta + \theta) = \sin (\beta + \theta) =$$

$$x = \frac{b \sin (\beta + \theta)}{\sin \beta}, \quad z = \frac{b \sin \theta}{\sin \beta}$$

Putting these values in the distance saving function:

$$Y = \frac{b \sin (\beta + \theta)}{\sin \beta} + \frac{b \sin \theta}{\sin \beta} - b$$

to get maxima or minima

$$\frac{dY}{d\theta} = \frac{b}{\sin \beta} \left( \cos (\beta + \theta) + \cos \theta \right)$$
\[ \frac{b}{\sin \beta} = 0 \]

for minima or maxima

\[ \frac{d^2Y}{d\theta^2} = 0, \text{ so } \cos (\beta + \theta) + \cos \theta = 0. \]

\[ \theta = \frac{180^\circ - \beta}{2} \]

Now

\[ \frac{d^2Y}{d\theta^2} = \frac{b}{\sin \beta} \left( -\sin (\beta + \theta) - \sin \theta \right) \]

\[ = -\frac{b}{\sin \beta} \times 2 \sin \frac{180^\circ}{2} \cos \frac{\beta}{2} \]

We finally get

\[ \frac{-b}{\sin \beta/2} \]

Hence \( \frac{d^2Y}{d\theta^2} \) is negative.

So we get maxima when

\[ \theta = \frac{180 - \beta}{2} \]

In this way we can get the critical angle \( \theta \) that saves the maximum distance, with the given value of \( \beta \). For instance, if \( \beta = 120^\circ \)

\[ \theta = \frac{180^\circ - 120^\circ}{2} = 30^\circ \]

This shows that when two existing roads make an angle of 120° with each other, the distance saved would be maximum when the new road with given length is constructed so as to make an angle of 30°. So angle 30° would be the critical angle when \( \beta \) is 120°.
As distance saved is maximum, this angle is critical angle that requires the critical break-even number of vehicles. This road may be called as critical break-even road. The critical angle is given by the following equation.

\[ \theta_1 = \frac{180^\circ - \beta}{2} \]

This equation corroborates with the critical angle with reference to right angle road system i.e. when \( \beta = 90^\circ \). As we have proved that the critical angle for right angle road system is \( 45^\circ \), we get the same value of critical angle by applying above equation as

\[ \theta_1 = \frac{180^\circ - 90^\circ}{2} = 45^\circ \]

Further we can show that the distance between joining point of two existing road (0) and joining points of new road on two existing roads OR and OT i.e. distance s and x are same. As obviously the angles that new road makes with existing road are also equal, i.e. \( \theta \) and \( 180^\circ - (\beta + \theta) \) can be shown equal as follows :-

Suppose \( 180^\circ - (\beta + \theta) \delta = \theta_2 \)

\[ \theta_2 = 180^\circ - \theta - \beta \]

as \( \theta_2 = \frac{180^\circ - \beta}{2} \)

\[ \theta_2 = 180^\circ - \frac{(180^\circ - \beta)}{2} - \beta \]
by rearranging we get

\[ \theta_2 = (180^\circ - \beta) - \left(\frac{180^\circ}{2} - \beta\right) \]

\[ \theta_2 = \left(\frac{180^\circ}{2} - \beta\right) \]

but we know that

\[ \theta = \frac{180^\circ}{2} - \beta \]

\[ \theta = \theta_2 \]

So \( z = x \).

This implies that when road is constructed with critical angle both angles between the new road and two existing roads are equal and distances \((x : z)\) from joining point of existing two roads to connecting points of new road on both roads are the same.

We can draw the special features of critical angle from the equation that gives the value of it. The equation for critical angle \( \theta \) is

\[ \theta = \frac{180^\circ}{2} - \beta \quad \beta \neq 0 \]

There are some special features.

1. The critical angle is independent of \( b, z, x \) i.e. length of the new road distances between joining points of existing roads and connecting points between new roads and existing roads. The implication of this features is very important as the resultant equation
can be applied to the new road with given predetermined node, as \( \theta \) is function of only \( \beta \).

2. The equation shows that \( \theta \) is inversely related to \( \beta \). Higher the value of \( \beta \), lower the value of critical angle. For instance, if \( \beta = 120^\circ \), \( \theta = 30^\circ \). If \( \beta = 40^\circ \), \( \theta = 70^\circ \).

We can determine the critical break-even number of vehicles when \( \theta = \frac{180^\circ - \beta}{2} \) with the following function, for break-even road.

\[
Track\ cost = Saving\ in\ operating\ cost\ by\ all\ vehicles\ concerned.
\]

\[
c \times b = n \times a \times distance\ saved.
\]

\[
c \times b = n \times a \times (x + z - b)
\]

Putting the value of \( z \) and \( x \) in terms of \( \theta \) and \( \beta \)

\[
c \times b = n \times a \times \left( \frac{b \sin(\beta + \theta)}{\sin \beta} \right) + \frac{b \sin \theta}{\sin \beta} - b
\]

\[
c \times b = na \times b \left( \frac{\sin(\beta + \theta)}{\sin \beta} \right) + \frac{\sin \theta}{\sin \beta} - 1
\]

Cancelling 'b' from both sides.

\[
c = na \frac{\sin(\beta + \theta)}{\sin \beta} + \frac{\sin \theta}{\sin \beta} - 1
\]

\[
n = c/a \left( \frac{1}{\frac{\sin(\beta + \theta)}{\sin \beta} + \frac{\sin \theta}{\sin \beta} - 1} \right)
\]

The equation II gives equation for number of break-even vehicles which makes the road break-even.
With the above equation we get critical break-even number of vehicles putting \( \theta \)

\[
\theta = \frac{180^\circ - \beta}{2}
\]

With the known values of \( \beta \) and \( \theta \) we can determine the critical break-even number of vehicles, as \( \theta \) is equal to critical angle with given values of \( c \) and \( a \).

The equation II refers to the new road with given length and with no predetermined node. As we have pointed out that the critical angle would be independent of the fact whether new road is having fixed node or without fixed node, the equation II is applicable even in the case of the road with fixed node. This can be confirmed by showing equality between the equations for determining critical number of vehicles with respect to two types of roads.

\[
n = \text{Break-even numbers of vehicles for the roads having fixed length without fixed node is given by}
\]

\[
n = \frac{c}{a} \cdot \frac{1}{\frac{\sin (\beta + \theta)}{\sin \beta} + \frac{\sin \theta}{\sin \beta} - 1}
\]

Break-even number of vehicles for new road with given fixed node is given by the following function.

Now as shown in the diagram No., let us take the case of new road to be constructed from fixed node B. In this case,
the length of the new road is not given but node b is predetermined, so length of the new road x is a function of θ. The angle between existing roads AO and OB meeting at O is β. The distance of OB is fixed and equal to S, so the equation for break-even number of vehicles can be derived as

\[ c\times x = na\,(s + z - x) \]

where z is varying distance from O to M at which new road meets existing road OA.

In the triangle OMB with angles β, θ, 180° - (β+θ) at O, B, M respectively

We have the relationship

\[ \frac{s}{\sin(\beta+\theta)} = \frac{z}{\sin \theta} = \frac{x}{\sin \beta} \]

\[ \sin 180° - (\beta+\theta) = \sin (\beta+\theta) \]

So putting the values in terms of s, β and θ.

\[ x = \frac{s \sin \beta}{\sin(\beta+\theta)}, \quad z = \frac{s \sin \theta}{\sin(\beta+\theta)} \]

Putting the values of x and z in equation III

\[ \frac{c\times s \sin \beta}{\sin(\beta+\theta)} = na \frac{s + s \sin \theta}{\sin(\beta+\theta)} - \frac{s \sin \beta}{\sin(\beta+\theta)} \]
Cancelling $s$ and rearranging we get equation for break-even number of vehicles.

\[
\frac{\sin \beta}{\sin (\beta + \theta)} = \frac{c/a}{1 + \frac{\sin \theta}{\sin (\beta + \theta)} - \frac{\sin \beta}{\sin (\beta + \theta)}}
\]

This equation B can be proved equal to the equation for break-even number of vehicles in the case of new road with the given length without fixed node i.e. equation A as follows :-

\[
\frac{\sin \beta}{\sin (\beta + \theta)} = \frac{c/a}{1 + \frac{\sin \theta}{\sin (\beta + \theta)} - \frac{\sin \beta}{\sin (\beta + \theta)}}
\]

By multiplying numerator and denominator by $\sin (\beta + \theta)$ we get :-

\[
n = \frac{c/a \sin \beta}{\sin (\beta + \theta) \cdot \sin \theta - \sin \beta}
\]

Further dividing numerator and denominator by $\sin \beta$ we get :-

\[
n = \frac{c/a \sin (\beta + \theta)}{\sin \beta} + \frac{\sin \theta}{\sin \beta} - 1
\]

This shows that the break-even number of vehicles with the given value of $c$ and $a$ would be same for all values of $\beta$ and $\theta$ in both the equations. This implies the value of critical angle would be same in the case of the new road with fixed node as well as in the case of the new road with given length without fixed node.
The operational utility of the concept of critical number of vehicles lies in its ability to indicate readily whether the new road would be break-even or otherwise, because if the actual number of vehicles would be less than critical break-even number of vehicles new road can not be break-even, although it is optimum indicating minimum total cost for the given inter-nodal vehicular traffic; without resorting to numerous alternative cost benefit analysis.

On the basis of equation of critical break even vehicles, the values of critical break even coefficient with reference to different values of can be calculated. The critical break even numbers of vehicles for given would be as follows

\[
\text{Critical break even} = \frac{\text{Capital cost of track per Km}}{\text{Operating cost of vehicle per Km.}} \times \text{Critical break even coefficient.}
\]

(Please see Table No. 9.1)
The approach optimising highway system cannot ignore the role of complementary nodes in planning for optimising highway system. Complementary nodes are such nodes that optimum highway for one of them provides distance saving advantage to other remaining nodes. This distance saving advantage can be obtained by using the existing road link or by constructing new optimum link road with reference to new trunk optimum road. This can be clarified by the map - A. There are A, B, C, E, F nodes for which new optimum roads with reference to common destination 'D' are to be proposed. We need not construct separate optimum road from each node. One optimum road from 'C' for instance can be suggested and this road may be termed as optimum trunk road and new optimum road from 'E' with reference to this new optimum trunk road can be proposed. The traffic from A, B can use this new optimum trunk road.

Choice of one node.

The problem arises how to choose the node from which new basic trunk highway would be constructed. The choice would have to be based on comparative ratio of vehicle distance saved and additional new road length to be constructed. The particular alternative having highest ratio of distance saved to additional road length is to be preferred. In some cases new trunk road will be used by complementary nodes with existing road link and in some cases the new additional optimum road with reference to new trunk road is to be constructed. In diagrams, this new...

1. The same track and operating cost is assumed.
optimum link road is shown with reference node 'E'. The traffic between F & D would use the existing road link as shown in the diagram. If the vehicular traffic between concerned nodes would not be less than critical break-even number of vehicles, new optimum link road is to be constructed.