1.1 Introduction

In 1965, Prof. L.A. Zadeh [49], generalized the usual notion of set by introducing 'fuzzy sets'. Fuzzy subsets are the classes of objects with grades of membership ranging between the nil memberships (0) and the full memberships (1). Fuzzy sets allow us to represent vague concepts expressed in natural language. The representation depends not only on the concept, but also on the context in which it is used. Hence, their application is increasing in the field of probability theory, information theory, computer science etc.

In the year 1968, C. L. Chang [11], introduced the concept of fuzzy topological spaces as an application of fuzzy sets to general topological spaces. Theory of fuzzy topological spaces was subsequently developed by several research workers like J. A. Goguen [20], C. K. Wong [47, 48], M. D. Weiss [46], R. H. Warren [43, 44, 45], R. Lowen [25], K. K. Azad [3], S. R. Malgan and S. S. Benchalli [28, 29], G. Balasubramaniam [5, 6], A. S. Bin Sahana [8, 9], M. Caldas [10], D. Coker [12] and many others. The special case of fuzzy topology is general topology.

This chapter is divided into two parts. In the first part, a brief
introduction to fuzzy subsets and fuzzy topology is provided. The concept of a fuzzy subsets, operations on fuzzy subsets, fuzzy subsets induced by mappings and fuzzy topological spaces which are work due to L.A.Zadeh [49], A.Kaufmann [23], C.L.Chang[11] and R.H.Warren [43, 44, 45] and others are refereed here.

In the second part, various forms of fuzzy open sets, their weaker and stronger forms in a fuzzy topological spaces, their properties, mappings and interrelations are noted. Using the concept of semiopen sets, semi closed sets, regular open sets and regular closed sets from general topology, K.K.Azad [3] in 1981 first introduced fuzzy semi-open, fuzzy semi-closed and fuzzy regular closed sets. It is shown that the intersection of a fuzzy open set and a fuzzy semi-open set need not be a fuzzy semi-open set which is not so in general topology. He observed that every fuzzy regular open (closed) set is a fuzzy semi-open (semi-closed) set, but the reverse implication is not true.

In the year 1991, A.S.Bin Shahana [9], Singal and Rajvansi [41] introduced and studied the concept of fuzzy $\alpha$-open set. They showed that every open set is an $\alpha$-open set. But the converse is not true. The concept of $\beta$-open set in general topology is by Abd El-Monsef et all [1], in 1983. In 1984 FathAlla [14], generalized these sets in fuzzy settings and called them fuzzy $\beta$-open sets. In 1996, S.S.Thakur, S.Singh [42] called these fuzzy sets as fuzzy semi-pre-open sets. They showed that every fuzzy semi-open (resp. fuzzy pre-closed) set and every fuzzy pre-
open (resp. fuzzy pre-closed) set is fuzzy semi-pre-open (resp. fuzzy semi-pre-closed). But the separate converse is not true.

The intersection of two fuzzy semi-open (pre-open, $\alpha$-open, semi-pre-open) sets need not be fuzzy semi-open (pre-open, $\alpha$-open, semi-pre-open) set. Also the intersection of a fuzzy open set with a fuzzy semi-open (pre-open, $\alpha$-open, semi-pre-open) set may fail to be fuzzy semi-open (pre-open, $\alpha$-open, semi-pre-open) set. This is a contrast to the result in general topology.

Weaker forms of continuity in topology have been considered by many topologists using the concept of pre-open, semi-open, $\alpha$-open, semi-pre-open sets. The same is generalized to define semi-continuous, pre-continuous, semi-precontinuous, $\alpha$-continuous, semi-open, pre-open, semi-pre-open, $\alpha$-open mapping.

### 1.2 The concept of fuzzy subsets

Let $X$ be a set and $A$ be a subset of $X$. Let $\mu_A : X \to \{0, 1\}$ be the characteristic function of $A$ defined by $\mu_A(x) = 1$ if $x \in A$ and $0$ if $x \notin A$.

Thus for an element $x \in X$, there are only two possibilities, namely, $x \in A$ or $x \notin A$. Therefore $x \in A$ if $\mu_A(x) = 1$ and $x \notin A$ if $\mu_A(x) = 0$. Hence $A$ is characterized by its characteristic function.

**Example 1.2.1.** $X = \{a, b, c, d, e\}$ and $A = \{b, d\}$. Consider the char-
acteristic function of $A$, $\mu_A : X \to \{0, 1\}$ defined by $\mu_A(a) = 0 = \mu_A(c) = \mu_A(e)$, $\mu_A(b) = \mu_A(d) = 1$. The subset $A$ of $X$ may also be represented by accompanying the element of $X$ with their characteristic function values. Thus $A = \{b, d\} = \{(a, 0), (b, 1), (c, 0), (d, 1), (e, 0)\}$.

In general, if $X$ is a set and $A$ is a subset in $X$ then $A$ has the following representation $A = \{(x, \mu_A(x) : x \in A}\}$.

Zadeh [49] introduced class of objects with grades of belongingness ranging between 0 and 1. He called such a class as fuzzy subset. Let $X$ be any set and $\mu_A : X \to [0, 1]$ be a function from $X$ into the closed unit, which may take any value between 0 and 1 for an element of $X$. Such a function is a "membership characteristic function". A fuzzy subset $A$ in $X$ is characterized by membership function $\mu_A : X \to [0, 1]$ which associates, to every point $x$ in $X$, a real number $\mu_A(x)$ between 0 and 1 which represents the degree or grade of membership or belongingness of $x$ to $A$. Thus nearer the value of $\mu_A(x)$ to unity, the higher the degree of belongingness of $x$ to $A$. If $A$ is an ordinary subset of $X$ then $\mu_A$ can take on either 0 or 1 only, but not a real number between 0 and 1. That is, in this case $\mu_A$ reduces to the usual characteristic function of $A$.

Thus a fuzzy subset $A$ of $X$ has the following representation. $A = \{(x, \mu_A(x) : x \in A}\}$, where $\mu_A : X \to [0, 1]$ is the membership function. So a fuzzy subset $A$ in $X$ is characterized by its membership function. Hence a fuzzy subset $A$ in $X$ can be defined to be a function
from $X$ into $[0,1]$. 

**Definition 1.2.2.** [49] Let $X$ be a set. Fuzzy subset $A$ in $X$ is a function $A : X \rightarrow [0,1]$.

**Example 1.2.3.** Let $X = \{a,b,c,d,e\}$ and $\mu_A : X \rightarrow [0,1]$ be defined by $\mu_A(a) = 0$, $\mu_A(b) = 0.2$, $\mu_A(c) = 0.3$, $\mu_A(d) = 1$, $\mu_A(e) = 0.2$. Then $A = \{(a,0), (b,0.2), (c,0.3), (d,1), (e, 0.2)\}$ is a fuzzy subset of $X$.

**Remark 1.2.4.** Let $X$ be any set. Then $X$ is also a fuzzy subset of $X$ whose membership function $\mu_X : X \rightarrow [0,1]$ is identically 1 on $X$. That is $\mu_X = 1$ for each $x \in X$. The fuzzy subset $X$ of $X$ is denoted by $1_X$ or 1 or $x$. Similarly the empty set $\Phi$ is also regarded as a fuzzy subset of $X$ whose membership function $\mu : X \rightarrow [0,1]$ is identically 0 on $X$. That is $\mu_{\Phi}(x) = 0$, for each $x \in X$. The empty fuzzy subset is denoted by $\mu_{\Phi}$ or 0.

Every subset of $X$ is a fuzzy subset of $X$ but not conversely. Hence the concept of a "fuzzy subset" is a generalization of the concept of a "subset".

**1.3 Operations on fuzzy subsets**

In this section, the extension of the notions of inclusion, union, intersection and complementation of ordinary subsets to fuzzy subsets and some of the properties are studied. The definitions and properties contained in this section are by Zadeh [49] and Kaufmann [23].
Throughout this section, let $X$ be a set and $A, B, C$ be fuzzy subsets of $X$ with the membership function $\mu_A, \mu_B, \mu_C$ respectively.

**Definition 1.3.1.** The fuzzy set $A$ is said to be included in the fuzzy set $B$ or $A$ is said to be contained in $B$ or $A$ is said to be less than or equal to $B$ if $\mu_A(x) \leq \mu_B(x)$ for all $x \in X$.

**Remark 1.3.2.** (i) Every fuzzy subset is included in itself. (ii) Empty fuzzy subset is included in every fuzzy subset.

**Definition 1.3.3.** The fuzzy subset $A$ is said to be equal to the fuzzy subset $B$, written $A = B$, if $\mu_A(x) = \mu_B(x)$ for all $x \in X$.

**Definition 1.3.4.** The fuzzy subset $A$ is said to be unequal to the fuzzy subset $B$, written $A \neq B$, if $\mu_A(x) \neq \mu_B(x)$ for all $x \in X$.

**Definition 1.3.5.** The complement of the fuzzy subset $A$ in $X$ denoted by $\bar{A}$ or $1 - A$, is the fuzzy subset of $X$ defined by $\mu_{\bar{A}}(x) = 1 - \mu_A(x)$ for all $x \in X$. That is $\mu_{\bar{A}}(x) = 1 - \mu_A(x)$ for all $x \in X$.

Note: $\bar{A} = A$.

**Example 1.3.6.** Let $Y = \{a, b, c\}$ and $A = \{(a, 0.4), (b, 0.3), (c, 0)\}$, then $\bar{A} = \{(a, 0.6), (b, 0.7), (c, 1)\}$.

**Definition 1.3.7.** The union of two fuzzy subsets $A$ and $B$ in $X$, denoted by $A \lor B$, is a fuzzy subset in $X$ defined by $\mu_{A \lor B}(x) = \max \{\mu_A(x), \mu_B(x)\}$ for all $x \in X$. 
Definition 1.3.8. The union of an indexed family \( \{A_\lambda : \lambda \in \Lambda \} \) of fuzzy subsets of \( X \), denoted by \( \bigvee_{\lambda \in \Lambda} A_\lambda(x) \) is a fuzzy subset of \( X \) defined by \( \bigvee_{\lambda \in \Lambda} A_\lambda(x) = \sup_{\lambda \in \Lambda} \{ A_\lambda(x) \} \) for all \( x \in X \).

Example 1.3.9. Let \( Y = \{a, b, c\} \). Let \( A = \{(a, 0.4), (b, 0), (c, 0.7)\} \), \( B = \{(a, 0.6), (b, 0.4), (c, 0.3)\} \) be two fuzzy subsets of \( Y \). Then \( A \cup B = \{(a, 0.6), (b, 0.4), (c, 0.7)\} \).

Definition 1.3.10. The intersection of two fuzzy subsets \( A \) and \( B \) in \( X \), denoted by \( A \land B \), is a fuzzy subset in \( X \) defined by \( \mu_{A \land B}(x) = \min \{ \mu_A(x), \mu_B(x) \} \) for all \( x \in X \).

Example 1.3.11. Let \( Y = \{a, b, c\} \). Let \( A = \{(a, 0.4), (b, 0), (c, 0.7)\} \), \( B = \{(a, 0.6), (b, 0.4), (c, 0.3)\} \) be two fuzzy subsets of \( Y \). Then \( A \land B = \{(a, 0.4), (b, 0), (c, 0.3)\} \).

Definition 1.3.12. The intersection of an indexed family \( \{A_\lambda : \lambda \in \Lambda\} \) of fuzzy subsets of \( X \), is denoted by \( \bigwedge_{\lambda \in \Lambda} A_\lambda \) is a fuzzy subset of \( X \) defined by \( \bigwedge_{\lambda \in \Lambda} A_\lambda(x) = \inf_{\lambda \in \Lambda} \{ A_\lambda(x) \} \) for all \( x \in X \).

Some basic properties of union, intersection and complementation, which are extension of the basic set theoretic properties of fuzzy subsets, are listed below.

Theorem 1.3.13. Let \( A, B \) and \( C \) be fuzzy set in \( X \). Then following are true.

(i) \( A \land (B \lor C) = (A \land B) \lor (A \land C) \).
(ii) \( A \lor (B \land C) = (A \lor B) \land (A \lor C) \).

(iii) \( A \lor 0 = A \), where 0 is the empty fuzzy subset.

(iv) \( A \land 0 = 0 \), where 0 is the empty fuzzy subset.

(v) \( A \land X = A \).

(vi) \( A \lor X = X \).

(vii) \( 1 - (A \lor B) = (1 - A) \land (1 - B) \).

(viii) \( 1 - (A \land B) = (1 - A) \lor (1 - B) \).

(ix) \( A - B = A \land (1 - B) \).

1.4 Fuzzy subsets induced by mappings

The image and inverse image of a fuzzy subset under a mapping were defined by Zadeh [49]. The definition of mapping by Chang [11] and Warren [43 and 49] and some of their related properties are mentioned below.

**Definition 1.4.1.** [49] Let \( f \) be a mapping from a set \( X \) into a set \( Y \). Let \( A \) be a fuzzy set in \( X \) and \( B \) be a fuzzy set in \( Y \).

(i) The inverse image of \( B \) under \( f \), written \( f^{-1}(B) \) is a fuzzy set in \( X \), defined by \( [f^{-1}(B)](x) = B(f(x)) = (B \bullet f)(x) \) for each \( x \in X \).

(ii) The image of \( A \) under \( f \), written \( f(A) \) is a fuzzy set in \( Y \), defined by \( f(A)(y) = \text{Sup} \{ A(z) : z \in f^{-1}(y) \} \) for each \( y \in Y \), where \( f^{-1}(y) = \{ x \in X : f(x) = y \} \).

The following properties were proved by Chang [11].
Theorem 1.4.2. Let $f$ be a mapping from a set $X$ into a set $Y$. The following are true.

(i) $f^{-1}(1 - B) = 1 - f^{-1}(B)$ for any fuzzy set $B$ in $Y$.

(ii) $f(1 - A) \geq 1 - f(A)$ for any fuzzy set $A$ in $X$.

(iii) $A \leq B$ implies $f(A) \leq f(B)$ for any two fuzzy sets $A$, $B$ in $X$.

(iv) $C \leq f^{-1}(C) \leq f^{-1}(D)$ for any two fuzzy sets $C$, $D$ in $Y$.

(v) $A \leq f^{-1}[f(A)]$ for any fuzzy set $A$ in $X$.

(vi) $B \geq f[f^{-1}(B)]$ for any fuzzy set $B$ in $Y$.

(vii) Let $g$ be function from $Y$ into $Z$. Then $(g \circ f)^{-1}(C) = f^{-1}(g^{-1}(C))$ for any fuzzy set $C$ in $Z$.

Warren [43] proved the following properties.

Theorem 1.4.3. Let $f$ be a function from a set $X$ into a set $Y$. If $A$, $A_k, k \in K$ are fuzzy sets in $X$ and if $B$, $B_j, j \in J$ are fuzzy sets in $Y$, then the following are true. (i) $f[f^{-1}(B)] = B$, when $f$ is onto.

(ii) $f(\land A_k) \leq \land f(A_k)$.

(iii) $f(\lor A_k) \leq \lor f(A_k)$.

(iv) $f^{-1}(B_j) \leq \land f^{-1}(B_j)$.

(v) $f^{-1}(B_j) \leq \lor f^{-1}(B_j)$.

(vii) $f[f^{-1}(B) \land A] = B \land f(A)$.
1.5 Fuzzy topological spaces

Chang [11] defined fuzzy topological spaces using fuzzy sets introduced by Zadeh [49]. In this section, some basic concepts and results on fuzzy topological spaces which may be used in the sequel, are included.

Definition 1.5.1. [11] Let $X$ be a set and $\tau$ be a family of a fuzzy subsets of $X$. $\tau$ is called a fuzzy topology on $X$ if it satisfy the following conditions.

(i) $0, 1 \in \tau$

(ii) If $G_j \in \tau$ for each $j \in J$ then $\bigvee G_j \in \tau$

(iii) If $G, H \in \tau$ then $G \wedge H \in \tau$.

The pair $(X, \tau)$ is called a fuzzy topological space (abbreviated as fts). The members of $\tau$ are called fuzzy open sets and a fuzzy set $A$ in $X$ is said to be closed if $1 - A$ is a fuzzy open set in $X$.

Remark 1.5.2. Every topological space is a fuzzy topological space but not conversely.

Example 1.5.3. Let $X = \{a, b, c\}$ be a set and let $A = \{(a, 0), (b, 0.4), (c, 1)\}$, be a fuzzy set in $X$. Let $\tau = \{0, A, 1\}$. Then $(X, \tau)$ is a fts which is not a topological space.

The closure of a fuzzy set was defined by Nazaroff [34] as follows.
Definition 1.5.4. [11] Let $X$ be a fts and $A$ be a fuzzy set in $X$. The closure of $A$, denoted by $\text{Cl}(A)$ is defined to be

$$\text{Cl}(A) = \bigwedge \{ B : B \text{ is a closed fuzzy set in } X \text{ and } B \geq A \}.$$ 

Some of the basic properties of closure in fts which are extensions of the corresponding results in general topological spaces are contained in the following.

Theorem 1.5.5. [44, 45] Let $A$ and $B$ be fuzzy sets in a fts $(X, \tau)$. The following results hold good.

(i) $\text{Cl}(A)$ is a closed fuzzy set in $X$.

(ii) $\text{Cl}(A)$ is the least closed fuzzy set in $X$ which is greater than or equal to $A$.

(iii) $A$ is closed iff $A = \text{Cl}(A)$.

(iv) $\text{Cl}(O) = O$.

(v) $\text{Cl}(\text{Cl}(A)) = \text{Cl}(A)$.

(vi) $\text{Cl}(A) \vee \text{Cl}(B) = \text{Cl}(A \vee B)$.

(vii) $\text{Cl}(A) \wedge \text{Cl}(B) \geq \text{Cl}(A \wedge B)$.

(viii) $A \leq B \Rightarrow \text{Cl}(A) \leq \text{Cl}(B)$.

The interior of a fuzzy set was defined by Chang [11] as follows

Definition 1.5.6. [11] Let $A$ and $B$ be two fuzzy sets in a fts $(X, \tau)$ and let $A \geq B$. Then $B$ is called an interior of fuzzy set $A$ if there exists $G \in \tau$ such that $A \geq G \geq B$. The least upper bound of all interior fuzzy sets of $A$ is called the interior of $A$ and is denoted by $A^0$. 


Some of the basic properties proved by Chang and Warren are contained in the following theorem.

**Theorem 1.5.7.** [11, 44, 45] Let \( A \) and \( B \) be fuzzy sets in a fts \( (X, \tau) \).

The following results hold good.

(i) \( A^0 \) is a open fuzzy set in \( X \)

(ii) \( A^0 \) is the largest fuzzy set in \( X \) which is less than or equal to \( A \)

(iii) \( A \) is open iff \( A = A^0 \)

(iv) \( A \leq B \) implies \( A^0 \leq B^0 \)

(v) \( (A^0)^0 = A^0 \)

(vi) \( A^0 \wedge B^0 = (A \wedge B)^0 \)

(vii) \( A^0 \vee A^0 \leq (A \vee B)^0 \)

(viii) \( (1 - A^0) = 1 - Cl(A) \)

(ix) \( Cl(1 - A) = 1 - A^0 \)

The concept of a base and a subbase for a fuzzy topology were introduced by Goguen [20].

**Definition 1.5.8.** [20] Let \( (X, \tau) \) be a fts. A subfamily \( \beta \) of \( \tau \) is called a base for \( \tau \) iff for each \( A \) in \( \tau \), there exists \( \beta_A \subset \beta \) such that \( A = \vee \beta_A \).

A subfamily of \( \tau \) is called a subbase for \( \tau \) iff the family \( \beta = \{ \wedge : \text{is a finite subfamily of } \tau \text{ is a bases for } \tau \} \).

The following characterization of bases is due to Warren [44].

**Theorem 1.5.9.** [44] Let \( \tau \) be a fuzzy topology on \( X \) and \( B \) be a subfamily of \( \tau \). Then the following two properties of \( B \) are equivalent.
(i) \( B \) is a base for \( \tau \)

(ii) For each \( G \in \tau \) and for each \( x \in X \) such that \( G(x) > 0 \) and for each real number \( \epsilon > 0 \), there is \( \beta \in B \) such that \( \beta = G \) and \( G(x) - \beta(x) < \epsilon \).

**Definition 1.5.10.** [44] A fuzzy set \( N \) in a fts \( (X, \tau) \) is a neighbourhood of a point \( x \) in \( X \) iff there is \( G \in \tau \) such that \( G \leq N \) and \( N(x) = G(x) > 0 \).

A neighbourhood of a point \( x \) is denoted by \( N_x \). A neighbourhood \( N_x \) is called an open neighbourhood of \( x \) iff \( N_x \in \tau \).

**Definition 1.5.11.** [11] Let \( (X, \tau) \) be a fts. A fuzzy set \( H \) in \( X \) is a neighbourhood of a fuzzy set \( G \) in \( X \), iff there is \( G \in \tau \) such that \( A \leq G \leq H \).

The concept of relative fuzzy topology is due to Warren [45].

**Definition 1.5.12.** [45] Let \( (X, \tau) \) be a fts and let \( A \) be a crisp subset of \( X \). Then the family \( \tau_A = \{ G/A : G \in \tau \} \) is a fuzzy topology on \( A \), where \( G/A \) is the restriction of \( G \) to \( A \).

The fuzzy topology \( A \) is called the relative fuzzy topology on \( A \) or the fuzzy topology on \( A \) induced by the fuzzy topology on \( X \). Also \( (A, \tau_A) \) is called the fuzzy subspace of \( (X, \tau) \).

Continuity of maps for fuzzy topological spaces was defined by Chang [11].
Definition 1.5.13. [11, 20] Let \((X, \tau)\) and \((Y, \rho)\) be two fts and let 
f \: (X, \tau) \rightarrow (Y, \rho) be a mapping. Then \(f\) is said to be fuzzy-continuous
\((f\text{-continuous})\) if \(f^{-1}(B) \in \tau\), for each \(B \in \rho\).

In the following theorem, some basic characterizations of \(f\)-continuous
maps established by Warren are noted.

Theorem 1.5.14. [45] Let \(f\) be a mapping from fts \((X, \tau)\) into a fts
\((Y, \rho)\). Then the following statements are equivalent.

(i) \(f\) is \(f\)-continuous.

(ii) The inverse image of every fuzzy closed set is closed.

(iii) The inverse image of every element of a sub base for \(S\) is in \(T\).

(iv) For every \(x \in X\) and every neighbourhood \(N\) of \(f(x)\), \(f^{-1}(N)\) is a
neighbourhood of \(x\).

(v) For every \(x \in X\) and every neighbourhood \(N\) of \(f^{-1}(x)\), there is a
neighbourhood \(M\) of \(X\) such that and \(M(x) = [f^{-1}(N)](x)\).

(vi) For every fuzzy set \(A\) in \(X\), \(f(A) \leq f(A)\).

(vii) For every fuzzy set \(B\) in \(Y\), \(f^{-1}(B) \leq f^{-1}(B)\).

(viii) If the set \(G = \{x, f(x) : x \in X\}\) has the fuzzy topology inherited
as a subspace of \((X \times Y, T \times S)\), then the function \(g : X \rightarrow G\) given by
\(g(x) = (x, f(x))\) is \(f\)-continuous.

Definition 1.5.15. [48] A mapping \(f : (X, \tau) \rightarrow (Y, \rho)\) is said to be
\(f\)-open (resp. \(f\)-closed) iff for each open (resp. closed) fuzzy set \(A\) in
\(X\), \(f(A)\) is an \(f\)-open (resp. \(f\)-closed) in \(Y\).
Definition 1.5.16. [28, 29] Let \((X, \tau)\) and \((Y, \rho)\) be two fts and \(f : (X, \tau) \rightarrow (Y, \rho)\) be a mapping. Then the following statements are equivalent.

(i) \(f\) is \(f\)-open map.

(ii) \(f(A^0) \subseteq [f(A)]^0\) for each fuzzy set \(A\) in \(X\).

(iii) If \(\beta\) is a base for \(\tau\), then \(f(A)\) is an open fuzzy set in \(Y\) for each \(A\) in \(\beta\).

(iv) \(f^{-1}([B]) \subseteq f^{-1}([B])\) for each fuzzy set \(B\) in \(Y\).

(v) \((f^{-1}(B))^0 \subseteq f^{-1}(B^0)\) for each fuzzy set \(B\) in \(Y\).

Theorem 1.5.17. [28] Let \(f : (X, \tau) \rightarrow (Y, \rho)\) be a mapping, then \(f\) is \(f\)-closed iff \(f(A) \subseteq f(A)\) for each fuzzy set \(A\) in \(X\).

Theorem 1.5.18. [29] Let \(f : (X, \tau) \rightarrow (Y, \rho)\) be a mapping. Then \(f\) is \(f\)-closed (\(f\)-open), iff for each fuzzy set \(A\) in \(Y\), and for any open (closed) fuzzy set \(B\) in \(X\) such that \(f^{-1}(A) \leq B\), there is an open (closed) fuzzy set \(C\) in \(Y\) such that \(A \leq C\) and \(f^{-1}(C) \leq B\).

Definition 1.5.19. [35] A fuzzy set \(A\) is a quasi-coincident with a fuzzy set \(B\) denoted by \(A \sqcap B\) iff there exist \(x \in X\) such that \(A(x) + B(x) > 1\). If \(A\) and \(B\) are not quasi-coincident then we write \(A \sqcup B\). \(A \leq B \iff A \sqcup (1 - B)\).

Definition 1.5.20. [35] A fuzzy point \(x_p\) is a quasi-coincident with a fuzzy set \(B\) denoted by \(x_p \sqcap B\) iff there exist \(x \in X\) such that \(p + B(x) > 1\).
1.6 Various forms of fuzzy subsets and their properties

Definition 1.6.1. A fuzzy set $A$ in a fts $X$ is called
(i) fuzzy regularly open (briefly fr-open) set of $X$, if $A = \text{IntCl}(A)$ and fuzzy regularly closed (briefly fr-closed) $(A = \text{ClInt}(A))$ [3].
(ii) fuzzy semi-open (briefly fs-open) if $A \leq \text{ClInt}(A)$ and fuzzy semi-closed (briefly fs-closed) if $\text{IntCl}(A) \leq A$ [3].
(iii) fuzzy pre-open (briefly fp-open) if $A \leq \text{IntCl}(A)$ and fuzzy pre-closed (briefly fp-closed) if $\text{ClInt}(A) \leq A$ [8].
(iv) fuzzy $\alpha$-open (briefly fa-open) if $A \leq \text{IntCl}(\text{IntCl}(A))$ and fuzzy $\alpha$-closed (briefly fa-closed) if $\text{ClInt}(\text{ClInt}(A)) \leq A$ [8].
(v) fuzzy semi-pre-open (briefly fsp-open) if $A \leq \text{ClInt}(\text{ClInt}(A))$ and fuzzy semi-pre-closed (briefly fsp-closed) if $\text{IntCl}(\text{IntCl}(A)) \leq A$ [42].

Lemma 1.6.2. [3, 8, 42] In a fuzzy topological space $(X, \tau)$
(i) every fuzzy regular open (closed) set is fuzzy open (closed).
(ii) every fuzzy open (closed) set is fuzzy $\alpha$-open (\alpha-closed).
(iii) every fuzzy $\alpha$-open (\alpha-closed) set is both fuzzy semi-open (semi-closed) and fuzzy pre-open (pre-closed).
(iv) every fuzzy semi-open (semi-closed) set is fuzzy semi-pre-open (semi-preclosed).
(v) every fuzzy pre-open (pre-closed) set is fuzzy semi-pre open (semi-preclosed).

The closure and interior of fuzzy sets in fts $X$ as in [3], [8] and [42]
are given below.

**Definition 1.6.3.** Let $A$ be a fuzzy set in a fuzzy topological space $(X, \tau)$. Then fuzzy $B$-Closure of $A$ is denoted and defined by $B\text{Cl}A = \bigwedge \{L : L \geq A, L \in B\}$ where $B \in \{r^c, \alpha^c, s^c, p^c, sp^c\}$.

**Definition 1.6.4.** Let $A$ be a fuzzy set in a fts $X$. Then fuzzy $B$-interior of $A$ is denoted and defined by $B\text{Int}A = \{L : L \leq A, L \in B\}$ where $B \in \{r, \alpha, s, p, sp\}$.

**Lemma 1.6.5.** [3, 8, 42] Let $A$ be a fuzzy set in a fts $X$ then,

(i) $\alpha\text{Cl}A = A \cup \text{ClIntCl}A$.
(ii) $s\text{Cl}A = A \cup \text{IntCl}A$.
(iii) $p\text{Cl}A \geq A \cup \text{ClInt}A$.
(iv) $sp\text{Cl}A \geq A \cup \text{IntClInt}A$.

**Lemma 1.6.6.** [3, 8, 42] Let $A$ be a fuzzy set in a fts $X$ then,

(i) $sp\text{Cl}A \leq s\text{Cl}A \leq \text{Cl}A \leq \text{Cl}A \leq r\text{Cl}A$
(ii) $sp\text{Cl}A \leq p\text{Cl}A \leq \text{Cl}A \leq \text{Cl}A \leq r\text{Cl}A$

**Lemma 1.6.7.** [3, 8, 42] Let $A$ be a fuzzy set in a fts $X$ then,

(i) $\alpha\text{Int}A = A \cap \text{IntClInt}A$.
(ii) $s\text{Int}A = A \cap \text{ClInt}A$.
(iii) $p\text{Int}A \leq A \cap \text{IntCl}A$.
(iv) $sp\text{Int}A \leq A \cap \text{ClIntCl}A$. 

17
Lemma 1.6.8. [3, 8, 42] Let $A$ be a fuzzy set in a fts $X$ then,
(i) $\text{spInt}A \geq \text{sInt}A \geq \text{Int}A \geq \text{Int}A \geq \text{rlnt}A$.
(ii) $\text{spInt}A \geq \text{pInt}A \geq \text{Int}A \geq \text{Int}A \geq \text{rlnt}A$.

Lemma 1.6.9. [3] Let $f : (X, \tau) \rightarrow (Y, \rho)$ be a mapping and $\{A_i\}$ be a family of fuzzy sets of $Y$, then
(i) $f^{-1}(\bigvee A_i) = \bigvee f^{-1}(A_i)$
(ii) $f^{-1}(\bigwedge A_i) = \bigwedge f^{-1}(A_i)$

Lemma 1.6.10. [3] For mappings and fuzzy sets $A_i$ of $Y_i$, $i = 1, 2$, we have $(f_1 \times f_2)^{-1}(A_1 \times A_2) = f_1^{-1}(A_1) \times f_2^{-1}(A_2)$.

Some of the generalized forms of fuzzy closed sets in fuzzy topological spaces are noted below.

Definition 1.6.11. A fuzzy set $A$ in a fts $X$ is called
(i) regular generalized fuzzy closed set if $\text{Cl}A \leq B$, whenever $A \leq B$ and $B$ is fuzzy regular open (rfg-closed) set in $X$ [42].
(ii) fuzzy generalized closed (briefly fg-closed) set if $\text{Cl}(A) \leq B$, whenever $A \leq B$ and $B$ is fuzzy open set in $X$ [6].
(iii) fuzzy generalized pre-closed [15] (briefly fgp-closed) set if $\text{pCl}(A) \leq B$, whenever $A \leq B$ and $B$ is fuzzy open set in $X$.
(iv) fuzzy pre-generalized closed (briefly fpg-closed) set if $\text{pCl}(A) \leq B$, whenever $A \leq B$ and $B$ is fuzzy semi-open set in $X$ [10].
(v) fuzzy semi generalized-closed (briefly fsg-closed) set if $\text{sCl}(A) \leq B$, whenever $A \leq B$ and $B$ is fuzzy open set in $X$ [27].
(vi) fuzzy generalized semi-closed (briefly fgs-closed) set if $sCl(A) \leq B$, whenever $A \leq B$ and $B$ is fuzzy semi-open set in $X$ [37].

(vii) fuzzy $\alpha$-generalized closed (briefly fag-closed) set if $Cl(A) \leq B$, whenever $A \leq B$ and $B$ is fuzzy open set in $X$ [40]. In [7], this set is known as fuzzy generalized strongly closed set.

(viii) fuzzy generalized $\alpha$-closed (briefly fga-closed) set if $Cl(A) \leq B$, whenever $A \leq B$ and $B$ is fuzzy -open set in $X$ [40]. In [7] this set known as fuzzy generalized almost strongly semi-closed set.

(ix) fuzzy generalized semi-pre-closed (briefly fgsp-closed) set if $spCl(A) \leq B$, whenever $A \leq B$ and $B$ is fuzzy open in $X$ [36].

(x) fuzzy semi-pre-generalized closed (briefly fspg-closed) set if $spCl(A) \leq B$, whenever $A \leq B$ and $B$ is fuzzy semi-open set in $X$ [36].

Fuzzy continuous mappings for various fuzzy open sets in fuzzy topological spaces are noted below.

**Definition 1.6.12.** A mapping $f : (X, \tau) \rightarrow (Y, \rho)$ is called

(i) fuzzy continuous (in short f-continuous) if the inverse image of every fuzzy open (closed) set in $Y$ is fuzzy open (closed) in $X$ [11].

(ii) fuzzy pre-continuous (in short fp-continuous) if the inverse image of every fuzzy open(closed) set in $Y$ is fuzzy p-open (p-closed) in $X$ [8].

(iii) fuzzy semi-continuous (in short fs-continuous) if the inverse image of every fuzzy open (closed) set in $Y$ is fuzzy s-open (s-closed) in $X$ [3].

(iv) fuzzy semi-pre-continuous (in short fsp-continuous) if the inverse image of every fuzzy open (closed) set in $Y$ is fuzzy sp-open (sp-closed)
in $X$ [42].

(v) **fuzzy strongly semi-continuous (in short $\alpha$-continuous)** if the inverse image of every fuzzy open (closed) set in $Y$ is fuzzy $\alpha$-open ($\alpha$-closed) in $X$ [8].

Fuzzy open mappings for different opens sets of fuzzy topological spaces are noted below.

**Definition 1.6.13.** A mapping $f : (X, \tau) \rightarrow (Y, \rho)$ is called

(i) **fuzzy open** (in short $f$-open) if the image of every fuzzy open set in $X$ is fuzzy open in $Y$ [47].

(ii) **fuzzy pre-open** (in short $fp$-open) if the image of every fuzzy open set in $X$ is fuzzy $p$-open in $Y$ [8].

(iii) **fuzzy semi-open** (in short $fs$-open) if the image of every fuzzy open set in $X$ is fuzzy $s$-open in $Y$ [3].

(iv) **fuzzy semi-pre-open** (in short $fsp$-open) if the image of every fuzzy open set in $X$ is fuzzy $sp$-open in $X$ [42].

(v) **fuzzy $\alpha$-open** (in short $fa$-open) if the image of every fuzzy open set in $X$ is fuzzy $\alpha$-open in $Y$ [8].

**Definition 1.6.14.** [6] A mapping $f : (X, \tau) \rightarrow (Y, \rho)$ is called generalized fuzzy continuous (in short $gf$-continuous) if the inverse image of every fuzzy closed set in $Y$ is $gf$-closed in $X$.

**Definition 1.6.15.** [17] A fts $(X, \tau)$ is said to be:

(a) $fT_0$ iff for every pair of fuzzy singletons $p_1$ and $p_2$ with different
supports, there exists an open fuzzy set $O$ such that $p_1 \subseteq O \subseteq \text{co } p_2$.

(b) $fT_1$ iff for every pair of fuzzy singletons $p_1$ and $p_2$ with different supports, there exist open fuzzy sets $O_1$ and $O_2$ such that $p_1 \subseteq O_1 \subseteq \text{co } p_2$ and $p_2 \subseteq O_2 \subseteq \text{co } p_1$.

(c) $fT_2$ ($f$-Hausdorff) iff for every pair of fuzzy singletons $p_1$ and $p_2$ with different supports, there exist open fuzzy sets $O_1$ and $O_2$ such that $p_1 \subseteq O_1 \subseteq \text{co } p_2$; $p_2 \subseteq O_2 \subseteq \text{co } p_1$ and $O_1 \subseteq O_2$, or equivalently, iff for every pair of fuzzy singletons $p_1$ and $p_2$ with different supports, there exists an open fuzzy set $O$ such that $p_1 \subseteq O \subseteq \text{Cl}_O \subseteq \text{co } p_2$.

(d) $f$-regular iff for every fuzzy singleton $p$ in $X$ and every fuzzy closed set $F$ in $X$ such that $p \subseteq \text{co } F$ there exist open fuzzy sets $U$ and $V$ such that $p \subseteq U, F \subseteq V$ and $U \subseteq \text{co } V$.

**Definition 1.6.16.** A bijective map $f : (X, \tau) \rightarrow (Y, \rho)$ is called fuzzy-homeomorphism (briefly $f$-homeomorphism) if $f$ and $f^{-1}$ are fuzzy continuous.

**Definition 1.6.17.** A fts $(X, \tau)$ is $fT_{1/2}$ space if every fg-closed set in $X$ is $f$-closed.