Chapter 4

On bilinear hazard quantile functions

4.1 Introduction

Reliability engineering, survival analysis and other disciplines mostly deal with positive random variables, which are often called failure times. As a random variable, a lifetime is completely characterized by its distribution function. Therefore, for example, information on the probability of failure of an operating item in the next (usually sufficiently small) interval of time is really important in reliability analysis. The hazard (failure) rate function defines this probability of interest. If this function is increasing, then our object is usually degrading in some suitable probabilistic

\textsuperscript{2}Part of the contents of this chapter has been published in Metron, by Bijamma Thomas, Midhu N.N. and P.G.Sankaran (2014).
sense, as the conditional probability of failure in the corresponding infinitesimal interval of time increases with time. There are several reliability measures in literature to describe the patterns of failure of systems. One of the popular measures is the hazard rate function. Many of the failure time models based on quantile functions do not have explicit expression for hazard rate function so that those models cannot employed for the analysis of lifetime data. Hazard quantile function which is equated to hazard rate function is used in such situation. The usual procedure in data modelling is to choose one among the candidate distributions, estimate the parameters and then carry out a goodness of fit. If the choice is not adequate, the same step is undergone with another model, sometimes with a different strategy for estimation and checking model adequacy. When we have a family of distributions, that provides approximation to many types of distributions, only one functional form for $Q(u)$ and the related inferential aspects are sufficient for modelling and analysis, as the quantile function will adapt automatically to the suitable model. Motivating by this factor we introduce a family of distributions having a bilinear form for hazard quantile function $H(u)$. The failure rate function is interesting from a reliability point of view since it is strongly connected with ageing properties of life distributions. Many classes of life distributions have been defined and studied based on the shape of the hazard function, such as IFR (increasing failure rate) distributions and DFR (decreasing failure rate) distributions.

The basic concepts of hazard function based on distribution function as well as quantile function is given in Chapter 2. In this chapter, we study a family of
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distributions having a bilinear form for hazard quantile function $H(u)$ given by

$$H(u) = \frac{a + bu}{c + du}$$

(4.1)

where $0 \leq u \leq 1$. The system of equations (4.1) become linear hazard quantile function when $d = 0$, ([Midhu et al., 2014]).

Also the system of equations (4.1) become inverse linear hazard quantile distribution when $b = 0$, ([Midhu et al., 2013]). This family of distributions can be approximated to some well known life time distributions like,

1. The Weibull distribution with probability density function

$$f(x) = \frac{e^{-\left(\frac{x}{\beta}\right)^\alpha}}{\beta^{\alpha-1}} \left(\frac{x}{\beta}\right)^{\alpha-1}, x > 0,$$

2. The beta distribution with probability density function

$$f(x) = \frac{x^{(\alpha-1)}(1-x)^{(\beta-1)}}{B(\alpha, \beta)} \frac{1}{B(\alpha, \beta)}, 0 < x < 1,$$

3. The gamma distribution with probability density function

$$f(x) = \frac{\beta^{-\alpha}x^{\alpha-1}e^{-\frac{x}{\beta}}}{\Gamma(\alpha)} x > 0, etc.$$
When \( k = \frac{a}{c}, A = \frac{b}{a} \) and \( B = \frac{d}{c} \) the expression (4.1) is of the form

\[
H(u) = k \frac{(1 + Au)}{1 + Bu}
\]  

(4.2)

where \( 0 \leq u \leq 1 \). The properties of the class of distributions (4.2) are discussed.

The rest of the article is organized as follows. In Section 4.2 we present the distributional properties of the model. Various reliability characteristics of the model are discussed in Section 4.3. In Section 4.4 we present approximation to some well known distributions. The inference on parameters of the model of the class of distributions are discussed in Section 4.5. We also apply the model to a real life dataset. Section 4.6 provides brief conclusion of the study.

### 4.2 The model and distributional properties

Consider a non-negative continuous random variable \( X \) with right continuous distribution function \( F(x) \) and quantile function \( Q(u) \) as described in Section 1. If \( f(x) \) is the probability density function of \( X \), then \( f(Q(u)) \) is called the density quantile function. The derivative of \( Q(u) \), expressed as \( Q'(u) = q(u) \), is known as the quantile density function of \( X \). Differentiating \( F(Q(u)) = u \) we get

\[
f(Q(u))q(u) = 1
\]  

(4.3)

where \( f(x) \) is the density of \( X \).
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Like hazard function, $H(u)$ determines the quantile function using the identity

$$Q(u) = \int_0^u \frac{1}{H(p)(1-p)} dp. \tag{4.4}$$

This means that $H(u)$ determines $Q(u)$ uniquely. For the class of distributions \((4.2)\), $Q(u)$ is obtained as

$$Q(u) = \frac{(A - B) \log(1 + Au) - A(B + 1) \log(1 - u)}{A(A + 1)k}. \tag{4.5}$$

The support of the distribution \((4.5)\) is $Q(0) = 0$ and $Q(1) = \infty$. The quantile density function for \((4.5)\) is obtained as

$$q(u) = \frac{1 + Bu}{k(1 - u)(1 + Au)}. \tag{4.6}$$

The validity of the quantile function is given as $k > 0$, $B \geq -1$ and $A > 0$.

The quantile density function for different values of parameters are given in Figure 4.1. The parameters \((k, A, B)\) are given at the top of the figure.

For the class of distributions in \((4.5)\) the quantile-based measures of the distributional characteristics such as location, dispersion, skewness and kurtosis are given below.
The quantile based measure of location is the median defined by

\[ M = Q\left(\frac{1}{2}\right) = \frac{(A + 1)B \log(2) + (A - B) \log(A + 2)}{A(A + 1)k}. \]

Dispersion is measured by the inter quartile range which is expressed as

\[ IQR = Q\left(\frac{3}{4}\right) - Q\left(\frac{1}{4}\right) \]

\[ = \frac{A(B + 1) \log(3) + (A - B) \log\left(\frac{4 + 3A}{4 + A}\right)}{A(A + 1)k}. \]

Galtons coefficient of Skewness is given by
\[
S = \frac{Q[\frac{3}{4}] + Q[\frac{1}{4}] - 2M}{IQR}.
\]

The coefficient of skewness \( S \) for the class (4.5) is obtained as

\[
S = \frac{A(1 + B)\log[\frac{4}{3}] + (A - B)\log[\frac{(4+3A)(4+A)}{4(2+A)^2}]}{A(B + 1)\log(3) + (A - B)\log[\frac{4+3A}{4+A}]}.
\]

In the case of extreme positive skewness \( Q[\frac{1}{4}] \to M \) and in the extreme negative skewness \( Q[\frac{3}{4}] \to M \) so that \( S \) lies between \(-1\) and \(+1\). When the distribution is symmetric \( M = \frac{Q[\frac{1}{4}] + Q[\frac{3}{4}]}{2} \) and hence \( S = 0 \). The measure of kurtosis can be defined as

\[
T = \frac{Q[\frac{7}{8}] - Q[\frac{5}{8}] + Q[\frac{3}{8}] - Q[\frac{1}{8}]}{IQR}
\]

and for the proposed class of distribution, we have

\[
T = \frac{A(1 + B)\log[\frac{41}{5}] + (A - B)\log[\frac{(8+7A)(8+3A)}{(8+5A)(8+A)}]}{A(B + 1)\log(3) + (A - B)\log[\frac{4+3A}{4+A}]}.
\]

The L-moments are often found to be more desirable than the conventional moments in describing the characteristics of the distributions as well as for inference. The L-moments exists whenever \( E(X) \) is finite, where as for many distributions additional restrictions are required for the conventional moments to be finite, see [Hosking, 1990]. L-moments generally score over the usual moments in providing smaller sampling variance, robustness against outliers and easier
characterization of distributional characteristics, especially for quantile functions that have no tractable distribution functions, for more details one could refer to [Nair and Sankaran, 2009b].

The L-moments have generally lower sampling variances and robust against outliers, see [Hosking, 1996] and [Sankarasubramanian and Srinivasan, 1999] for more details. The first L-moment is the mean which is given by

$$\mu = L_1 = \frac{AB + (A - B) \log(A + 1)}{A^2k}. \quad (4.7)$$

The second L-moment can be expressed as

$$L_2 = \int_0^1 (2u - 1)Q(u)du = \frac{A(A(B + 2) - 2B) + 2(B - A)\log(A + 1)}{2A^3k}. \quad (4.8)$$

The third and fourth L-moments are obtained as

$$L_3 = \int_0^1 (6u^2 - 6u + 1)Q(u)du$$

and

$$L_4 = \int_0^1 (20u^3 - 30u^2 + 12u - 1)Q(u)du.$$  

For the class of distributions (4.5)

$$L_3 = \frac{A \left( (A^2 + 12)B - 12A \right) + 6(A + 2)(A - B) \log(A + 1)}{6A^4k} \quad (4.9)$$
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and

\[
L_4 = \frac{A^3B + 2A^3 - 2A^2B + 30A^2 - 30AB + 60A - 60B}{12A^4k} - \frac{(A^2 + 5A + 5)(A - B) \log(A + 1)}{A^5k}.
\]  

(4.10)

L-coefficient of variation, analogous to the coefficient of variation based on ordinary moments is obtained as

\[
\tau_2 = \frac{\lambda_2}{\lambda_1}.
\]

For the class of distributions (4.5),

\[
\tau_2 = \frac{A(B + 2)}{2AB + 2(A - B) \log(A + 1)} - \frac{1}{A}.
\]  

(4.11)

Here \(\tau_2 \to 1\) as \(A \to -1\) and \(\tau_2 \to 1/2\) when \(A = B\).

L-coefficient of skewness is measured as

\[
\tau_3 = \frac{\lambda_3}{\lambda_2}.
\]

From (4.5), we have

\[
\tau_3 = \frac{2A^2(2B + 3)}{3A((A - 2)B + 2A) + 6(B - A) \log(A + 1)} - \frac{2}{A} - 1.
\]  

(4.12)
Thus $\tau_3$ tends to $1/3$ when $A = B$.

L-coefficient of kurtosis is given by

$$\tau_4 = \frac{\lambda_4}{\lambda_2}$$

For our class of distributions in (4.5)

$$\tau_4 = \frac{5(A + 1)}{A^2} - \frac{5A(A(B + 2) + 4B + 6)}{6A(A(B + 2) - 2B) + 12(B - A)\log(A + 1)} + 1. \tag{4.13}$$

$\tau_4$ tends to $1/6$ when $A = B$.

### 4.3 Reliability characteristics

One of the important measures in reliability analysis to study ageing pattern of the system is the hazard quantile function. The shape of the hazard quantile function is determined by the derivative of $H(u)$. For the hazard quantile function of (4.2) the derivative $H'(u)$ is given by

$$H'(u) = \frac{k(A - B)}{(Bu + 1)^2}.$$ 

$H'(u)$ changes shape according as $k(A - B)$ positive, negative or equal to zero. When $A = B$, the hazard quantile function $H(u)$ has constant value and when $k > 0$ and $A > B$, $H(u)$ has increasing failure rate and if $k > 0$ and $A < B$ then $H(u)$ has
decreasing failure rate. Also when $k < 0$ and $A < B$, $H(u)$ has increasing failure rate and if $k < 0$ and $A > B$ then $H(u)$ has decreasing failure rate. Plots of the hazard quantile functions for different values of parameters are presented in Figure 4.2. The parametric values of $(k, A, B)$ are given at the top of the Figure 4.2.

![Figure 4.2: Plots of $H(u)$ for different values of parameters](image)

Mean residual function is a well known measure, which has been widely used in various fields of reliability and survival analysis. For a non negative random variable $X$, the mean residual life function is defined as

$$m(x) = E[X - x | X > x] = \frac{1}{F(x)} \int_x^\infty \frac{1}{F(t)} dt$$
which is explained as the expected remaining life time of a unit given survival up to time \( x \). In certain families of distributions, mean residual life functions do not have a closed form. For example, Govindrajalu distribution discussed in [Nair and Vineshkumar, 2011] with quantile function \( Q(u) = \theta + \sigma((\beta + 1)u^\beta - \beta u^{\beta+1}), \theta, \sigma, \beta > 0, 0 \leq u \leq 1 \) does not have a closed form expression for \( m(x) \). In quantile set up the mean residual quantile function is defined as

\[
M(u) = (1 - u)^{-1} \int_u^1 [Q(p) - Q(u)] dp
\]

which can also be expressed as

\[
M(u) = \frac{1}{1 - u} \int_u^1 (1 - p) q(p) dp.
\]

For the class of distributions (4.5), the mean residual quantile function is expressed as

\[
M(u) = \frac{AB(1 - u) + (A - B) \log \left( \frac{A+1}{Au+1} \right)}{A^2k(1 - u)}.
\]

The hazard quantile function and mean residual quantile function defined in reverse time (Nair and Sankaran, 2009) for the class of distributions are respectively expressed as

\[
A(u) = (uq(u))^{-1} = \frac{k(1 - u)(Au + 1)}{u(Bu + 1)} \quad (4.14)
\]
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and

\[ R(u) = (u)^{-1} \int_0^u pq(p)dp = \frac{(B - A) \log(Au + 1) - A^2(B + 1) \log(1 - u)}{A^2(A + 1)k}\frac{B}{u} - A^2k. \]  

(4.15)

The total time on test transform (TTT) is a very popular statistical tool, which has wide applications in reliability analysis. For more details one could refer to [Lai and Xie, 2006]. The quantile based total time on test transform (TTT) introduced by [Nair et al., 2008], has the form

\[ T(u) = \int_0^u (1 - p)q(p)dp = \frac{ABu + (A - B) \log(Au + 1)}{A^2k}. \]  

(4.16)

Instead of usual moments, we may consider the L-moments of \( X \) given \( X > t \). First L-moment of \( X|X > t \) is vitality function studied by [Kupka and Loo, 1989] which is given by

\[ \alpha_1(u) = (1 - u)^{-1} \int_u^1 Q(p)dp = -A(B + 1)(u - 1)(\log(1 - u) - 1) + \frac{(A-B)(A(u-1)-(Au+1)\log(Au+1)+(A+1)\log(A+1))}{A(A + 1)k(u - 1)}. \]  

(4.17)

The second L-moment of residual life [Nair and Vineshkumar, 2010]is given by
\[ \alpha_2(u) = \frac{1}{(1-u)^2} \int_u^1 (2p-u-1)Q(p)dp. \] (4.18)

For the class of distributions given in (4.5), we have

\[ \alpha_2(u) = \frac{A(u-1)(A(B(u-1) - 2) + 2B) + 2(A-B)(Au+1) \log \left( \frac{Au+1}{A+1} \right)}{2A^3k(u-1)^2}. \] (4.19)

In reversed time, L-moments can be obtained as

\[ \theta_1(u) = (u)^{-1} \int_0^u Q(p)dp \]
\[ = \frac{A^2(-(B+1))(u-1) \log(1-u) + (A+1)ABu + (A-B)(Au+1) \log(Au+1)}{A^2(A+1)ku} \] (4.20)

and

\[ \theta_2(u) = (u)^{-2} \int_0^u pR(p)dp \]
\[ = \frac{(A-B)(Au+1) \log(Au+1)}{A^3(A+1)ku^2} - \frac{ABu - 2AB - 2A + 2B}{2A^2ku} - \frac{(B+1)(u-1) \log(1-u)}{(A+1)ku^2}. \] (4.21)

Residual life can also be obtained in terms of percentiles as percentile residual life function given by

\[ P_\alpha(u) = Q[1 - (1-\alpha)(1-u)] - Q(u). \] (4.22)
From the equation (4.5)

\[ P_\alpha(u) = \frac{-A(B + 1) \log(1 - \alpha) + (A - B) \log(A(\alpha - \alpha u + u) + 1) + (B - A) \log(Au + 1)}{A(A + 1)k}. \]

and the reversed percentile residual life function is defined by [Nair and Vineshkumar, 2011] as

\[ [R_\alpha(u) = Q(u) - Q[u(1 - \alpha)]. \]

For our class of distributions

\[
R_\alpha(u) = \frac{(A - B)(\log(Au + 1)) - A(B + 1)(\log(1 - u))}{A(A + 1)k} - \frac{(A - B)(\log((1 - \alpha)Au + 1)) - A(B + 1)(\log(1 - (1 - \alpha)u))}{A(A + 1)k}.
\tag{4.23}
\]

### 4.4 Approximations

Since the class of distributions (4.5) cannot be converted to a tractable form for its distribution function, its relationship with other known standard distributions can be assessed only through approximations. The advantage of seeking such cases is justified from the analytical and practical point of view. The usual procedure in data modelling is to choose one among the candidate distributions, estimate the parameters and then carry out a goodness of fit. If the choice is not adequate, the same procedure is applied with another model, sometimes with a different strategy.
foe estimation and checking model adequacy. When we have a quantile function that provides approximation to many type of distributions, only one functional form for $Q(u)$ and the related inferential aspects are sufficient for modelling and analysis, as the quantile function will adapt automatically to the suitable form. In this section we attempt to fit our distribution to some well-known distributions like gamma distribution and Weibull distribution. The same approach can also be used to approximate other distributions.

### 4.4.1 Gamma Distribution

Probability density function of Gamma is given by

$$f(x) = \frac{\beta^{-\alpha}x^{\alpha-1}e^{-\frac{x}{\beta}}}{\Gamma(\alpha)}, \alpha, \beta, x > 0.$$ 

The proposed model can approximate the Gamma distribution by equating the first three L-moments of (4.5). As an illustration, we assume the parameters of Gamma distribution as $\alpha = 1.5$ and $\beta = 2$ and the corresponding values of $k$, $A$ and $B$ are $k = 0.129298$, $A = 14.3184$ and $B = 3.49017$. Figure 4.3 gives the probability density function of Gamma distribution and the dotted line represents approximated quantile function.

For measuring the closeness between the two models, we find $\sup_x |F_1(x) - F_2(x)|$, 


where \( F_1(x) \) is the distribution function of the proposed model and \( F_2(x) \) is the distribution function of the Gamma distribution. Now we obtain \( \sup_x |F_1(x) - F_2(x)| = 0.0067 \) which is also very small.

![Figure 4.3: Probability density functions of Gamma and it's approximation](image)

### 4.4.2 Weibull Distribution

The probability density function of Weibull distribution

\[
f(x) = \frac{e^{-(\frac{x}{\beta})^\alpha}}{\beta}^{\alpha(\frac{x}{\beta})^{-1+\alpha}}
\]

The Weibull distribution with parameters \( \alpha \) and \( \beta \) has moments

\[
L_1 = \beta \Gamma(1 + \frac{1}{\alpha}), \quad L_2 = \beta (1 - 2^{-\frac{1}{\alpha}}) \Gamma(1 + \frac{1}{\alpha}) \quad \text{and} \quad L_3 = \frac{3 - 2(1 - 3^{-\frac{1}{\alpha}}) L_2}{(1 - 2^{-\frac{1}{\alpha}})}
\]

Since it is difficult to determine analytically the values of \( \alpha \) and \( \beta \), as an illustration we assume \( \alpha = 0.5 \) and \( \beta = 2 \) and the corresponding values of (4.5) is \( k = 4.77234 \),
$A = -0.828578$ and $B = 12.4532$. Figure 4.4 gives the probability density function of approximation of the Weibull distribution and the dotted line represents the approximated quantile function.

For measuring the closeness between the two models, we find \( \sup_x |F_1(x) - F_3(x)| \), where \( F_1(x) \) is the distribution function of the proposed model and \( F_3(x) \) is the distribution function of the Weibull distribution. Now we obtain \( \sup_x |F_1(x) - F_3(x)| = 0.0314 \) which is also very small.

![Figure 4.4: Probability density functions of Weibull and its approximation](image)

### 4.5 Estimation of parameters and data analysis

For estimating the parameter of the distributions in the quantile set up there are different methods available in [Gilchrist, 2000]. Among these different methods, the commonly used techniques are method of minimum absolute deviation, method of least squares, method of maximum likelihood and method of L-moments. Recently
method of L-moments is widely used as an alternative to the conventional methods in inference problems in view of the robustness in the estimates produce (see [Hosking, 1996]). To estimate the parameters of the model given in (4.5) we use the method of L-moments. The simple algebraic expressions of the L-moments explained in Section 4.2 admit the applicability of the L-moments method for estimating the parameters of the model (4.5). Since there are three parameters in the model, we take three sample L-moments \( l_r, \ r=1, 2 \) and 3 given by

\[
l_1 = \left( \begin{array}{c} n \\ 1 \end{array} \right)^{-1} \sum_{k=1}^{n} x_k (4.24)
\]

\[
l_2 = \frac{1}{2} \left( \begin{array}{c} n \\ 2 \end{array} \right)^{-1} \sum_{k=1}^{n} \left\{ \left( \begin{array}{c} k - 1 \\ 1 \end{array} \right) - \left( \begin{array}{c} n - k \\ 1 \end{array} \right) \right\} x_k (4.25)
\]

and

\[
l_3 = \frac{1}{3} \left( \begin{array}{c} n \\ 3 \end{array} \right)^{-1} \sum_{k=1}^{n} \left\{ \left( \begin{array}{c} k - 1 \\ 2 \end{array} \right) - 2 \left( \begin{array}{c} n - k \\ 2 \end{array} \right) \left( \begin{array}{c} k - 1 \\ 1 \end{array} \right) + \left( \begin{array}{c} n - k \\ 2 \end{array} \right) \right\} x_k. (4.26)
\]

We equate sample L-moments to population L-moments given by

\[
l_r = L_r: \ r = 1, 2 \text{ and } 3. \quad (4.27)
\]

Solutions of set of equations (4.27) give the estimates of \( k, A \) and \( B \).
Hosking(1990) has studied asymptotic properties of L-moment estimates. The following theorem provides asymptotic normality of sample L-moments.

**Theorem 4.1** (Hosking(1990)). Let $X$ be a real-valued random variable with quantile function $Q(u, \theta)$, where $\theta$ is a vector of $m$ parameters. Assume that variance of $X$ is finite. Let $\ell_r, r = 1, 2, \ldots, m$ be sample L-moments calculated from a random sample of size $n$ drawn from the distribution of $X$. Then $\sqrt{n}(\ell_r - L_r), r = 1, 2, \ldots, m,$ converge in distribution to the multivariate normal $N(0, \Lambda)$, where the elements $\Lambda_{rs}(r, s = 1, 2, \ldots, m)$ of $\Lambda$ are given by

$$
\Lambda_{rs} = \int \int_{0 < u < v < 1} \{P^*_r(u)P^*_s(v) + P^*_s(u)P^*_r(v)\}u(1-v)q(u)q(v)dudv,
$$

where $P^*_r(x)$ being the $r$th shifted Legendre polynomial defined by

$$
P^*_r(x) = \sum_{k=0}^{r} (-1)^{r-k} \binom{r}{k} \binom{r+k}{k} x^k.
$$

Since the set of equations (4.27) are non-linear in $k$, $A$ and $B$ asymptotic variance of the L-moment estimates are difficult to compute. One can use bootstrap method to obtain the asymptotic variance of the estimates.

To illustrate the procedure of estimation and application of the class of distributions in a practical situation, we apply the model (4.5) to a real data set given in [Musa, 1980]. The dataset represents successive inter-failure times for a commercial subsystem, which consists of 73 failure times.
In order to examine the appropriateness of our model for this data, we first identify the nature of the hazard rate \( h(x) = \frac{f(x)}{1-F(x)} \) through empirical methods. We estimate hazard rate non parametrically by \( \hat{h}(x) = \frac{\hat{f}(x)}{1-\hat{F}(x)} \), where \( \hat{f}(x) \) is kernal based estimator of density function \( f(x) \) and \( \hat{F}(x) \) is the empirical distribution function obtained by the relation \( \hat{F}(x) = \int_0^x \hat{f}(u)du \). When \( T_1, T_2, ..., T_n \) are the sample observations, the non parametric estimator of \( \hat{f}(x) \) is given by \( \hat{f}(x) = \frac{1}{nh} \sum_{i=1}^{n} K(\frac{x-T_i}{h(n)}) \) where \( h^*(n) \) is the bandwidth parameter and \( K(.) \) is the kernel density satisfying the conditions, (1) \( K \) is bounded (2) \( K \) is symmetric and (3) \( \int K(x)dx = 1 \). With the help of the Gaussian kernel given by \( K(x) = \frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}} \), the estimate of \( h(x) \) is computed from the sample. Figure 4.5 provides the plot of \( \hat{h}(x) \) which shows that the hazard rate is approximately decreasing.

![Figure 4.5: Non parametric estimate of hazard rate for the data set](image)

The sample L-moments are \( l_1 = 69.726 \), \( l_2 = 51.0571 \) and \( l_3 = 32.634 \) and we equate them to corresponding population L-moments using equations (4.7), (4.8) and (4.9) as given. Using Newton Raphson method we find the solutions of the equations.
Since the failure pattern exhibited by the data observed to be decreasing failure rate the search for the candidate distribution was confirmed to the decreasing failure rate members which leads to the restriction $A < B$.

The estimates are obtained as $\hat{k} = 0.0645002$, $\hat{A} = -0.916688$ and $\hat{B} = 0.957025$.

The hazard quantile function $H(u)$ for the dataset is shown in Figure 4.6. From Figure 4.6, it follows that $H(u)$ is decreasing in $u$.

![Figure 4.6: Hazard quantile function for the data set](image)

We employed chi-square goodness of fit to check the adequacy of the model. The chi-square statistic value is 7.712 with p-value 0.462. This indicate that the proposed model is a reasonable one for the given data set.

To check the goodness of fit, we also use Q-Q plot which is given in Figure 4.7. Since most of the data points are close to the 45° straight line, which indicates that the quantile function given in (4.5) is a reasonable fit to the data.
4.6 Conclusion

There are several reliability measures in literature to describe the patterns of failure of systems. One of the popular measures is hazard rate function. Many of the lifetime models based on quantile functions do not have explicit expression for hazard rate function so that those models can not employed for the analysis of lifetime data. Hazard quantile function which is equated to hazard rate function is used in such situation.

In the present work we have introduced a class of distributions (4.5) based on the hazard quantile function and studied its various properties. Several existing well known lifetime distributions are members of the class of distributions as special cases or through approximations. Various reliability characteristics were discussed.
The parameters of the model were estimated using the method of L-moments and the model was applied to a real data set. Non parametric estimator of $H(u)$ given in [Nair and Sankaran, 2009a] can be employed in practice to identify the approximate lifetime model for a given dataset.