6.1 Introduction

In 1999, Russian researcher Molodtsoy [40] initiated the concept of soft set theory and started to develop the basics of the corresponding theory as a new approach for modeling uncertainties. Soft set theory has received much attention since its appearance. Maji et al. [38] first defined several operations on soft sets. The algebraic structures of soft set theory has been explored in recent years. In [3] Aktas and Cagman gave a definition of soft groups and studied their basic properties. Many researchers have contributed towards the fuzzification of the notion of soft sets.

In 2001, Maji et al. [34] combined the fuzzy set and soft set models and introduced the concept of fuzzy soft set and studied its algebraic properties. These results were further revised by Ahmed and Kharal [2]. Aygiuoglu et al. [9] introduced the notion of fuzzy soft group and studied its properties.

In this chapter, we study the algebraic properties of fuzzy soft sets in module theory and introduce the notion of fuzzy soft modules. Also we
investigate some characteristic properties of fuzzy soft modules. Furthermore
\( \alpha \)-level soft set of a fuzzy soft set is defined and we prove some results related
to them. The definition of fuzzy soft function is recalled and proved some of
its properties. Then we define fuzzy soft homomorphism between fuzzy soft
modules and proved that the homomorphic image and pre-image of a fuzzy
soft module are also fuzzy soft modules.

Throughout this chapter, let \( I \) be the closed unit interval ie. \( I = [0, 1] \)
and \( E \) be a set of parameters for the universe \( X \).
Here we give some basic definitions and results from the theory of soft sets
and fuzzy soft sets which will be used in the subsequent sections.

**Definition 6.1.1 [40]**  Let \( X \) be an initial universe set and \( E \) be a set of
parameters for \( X \). A pair \((F, E)\) is called a soft set over \( X \) if and only if \( F \) is a
mapping from \( E \) into the set of all subsets of the set \( X \). ie. \( F : E \rightarrow P(X) \),
where \( P(X) \) is the power set of \( X \).

In otherwords, a soft set is a parameterized family of subsets of the
set \( X \). For \( e \in E \), \( F(e) \) is regarded as the set of \( e \) – approximate elements of the
soft set \((F, E)\).
Definition 6.1.2 [57] For two soft sets \((F, A)\) and \((G, B)\) over a common universe \(X\), \((F, A)\) is said to be a soft subset of \((G, B)\) and write \((F, A) \subseteq (G, B)\) if

(i) \(A \subset B\), and

(ii) For each \(a \in A\), \(F(a) \subseteq G(a)\).

Definition 6.1.3 [27] Let \((F, A)\) and \((G, B)\) be two soft sets over \(X\) and \(Y\) respectively. And let \(\varphi : X \to Y\) and \(\psi : A \to B\) be two functions. Then the pair \((\varphi, \psi)\) is called a soft function from \(X\) to \(Y\).

Definition 6.1.4 [34] Let \(I^X\) denote the set of all fuzzy sets on \(X\) and \(A \subset E\). A pair \((f, A)\) is called a fuzzy soft set over \(X\), where \(f\) is a mapping from \(A\) into \(I^X\).

ie. \(f : A \to I^X\). That is for each \(a \in A\), \(f(a) = f_a : X \to I\) is a fuzzy set on \(X\).

Remark 6.1.5 [34] Obviously, a classical soft set \((F, E)\) over \(X\) can be seen as a fuzzy soft set \((f, E)\) over \(X\). For each \(e \in E\), the image of \(e\) under \(f\) is defined as the characteristic function of the set \(F(e)\).

ie. \(f_e(x) = \chi_{F(e)}(x) =\begin{cases} 1 & \text{if } x \in F(e) \\ 0 & \text{otherwise.} \end{cases}\)

Definition 6.1.6 [34] For two fuzzy soft sets \((f, A)\) and \((g, B)\) over a common universe \(X\), \((f, A)\) is said to be a fuzzy soft subset of \((g, B)\) and write \((f, A) \subseteq (g, B)\) if
(i) $A \subseteq B$, and
(ii) For each $a \in A$, $f_a \subseteq g_a$ i.e. $f_a$ is a fuzzy subset of $g_a$.

**Definition 6.1.7** [34] Two fuzzy soft sets $(f, A)$ and $(g, B)$ over a common universe $X$ are said to be **equal** if $(f, A) \sqsubseteq (g, B)$ and $(g, B) \sqsubseteq (f, A)$.

**Definition 6.1.8** [34] Let $(f, A)$ and $(g, B)$ be two fuzzy soft sets over a common universe $X$ with $A \cap B \neq \emptyset$. The **intersection** of $(f, A)$ and $(g, B)$ is the fuzzy soft set $(h, C)$ over $X$ where $C = A \cap B$ and $h(c) = h_c = f_c \cap g_c$ \forall $c \in C$ and we write $(f, A) \cap (g, B) = (h, C)$.

**Definition 6.1.9** [34] Let $(f, A)$ and $(g, B)$ be two fuzzy soft sets over a common universe $X$. The **union** of $(f, A)$ and $(g, B)$ is the fuzzy soft set $(h, C)$ over $X$ where $C = A \cup B$ and

$$h(c) = h_c = \begin{cases} f_c & \text{if } c \in A - B \\ g_c & \text{if } c \in B - A \\ f_c \cup g_c & \text{if } c \in A \cap B \end{cases} \forall c \in C,$$ and we write $(f, A) \cup (g, B) = (h, C)$.

**Definition 6.1.10** [34] Let $(f, A)$ and $(g, B)$ be two fuzzy soft sets over a common universe $X$. Then $(f, A)$ **AND** $(g, B)$ denoted $(f, A) \wedge (g, B)$ is defined as $(h, A \times B)$ where $h(a, b) = h_{a,b} = f_a \cap g_b \forall (a, b) \in A \times B$.
Definition 6.1.11 [34] Let \((f, A)\) and \((g, B)\) be two fuzzy soft sets over a common universe \(X\). Then \((f, A)\) OR \((g, B)\) denoted \((f, A) \lor (g, B)\) is defined as \((h, A \times B)\) where \(h(a, b) = h_{a,b} = f_a \cup g_b\) \(\forall (a, b) \in A \times B\).

Definition 6.1.12 [9] Let \((f, A)\) be a fuzzy soft set over \(X\). For each \(\alpha \in (0,1]\), the soft set \((f, A)^\alpha = (f^\alpha, A)\) is called an \(\alpha\)-level soft set of \((f, A)\) where \(f^\alpha(a) = (f_a)^\alpha = \{x \in X : f_a(x) \geq \alpha\}\) for each \(a \in A\). Obviously \((f, A)^\alpha = \{(f_a)^\alpha : a \in A\}\) is a soft set over \(X\) for each \(\alpha \in (0,1]\).

### 6.2 Fuzzy soft modules

In this section we discuss some algebraic properties of fuzzy soft modules and prove some results on \(\alpha\)-level soft sets of fuzzy soft sets in module theory.

Definition 6.2.1[59] Let \((F, A)\) be a soft set over an \(R\)-module \(M\). \((F, A)\) is said to be a soft module over \(M\) if and only if \(F(x)\) is a submodule of \(M\) for all \(x \in A\).

Definition 6.2.2[14] Let \((f, A)\) be a fuzzy soft set over \(M\), then \((f, A)\) is said to be a fuzzy soft module over \(M\) if and only if for all \(a \in A\), \(f(a) = f_a\) is a fuzzy submodule of \(M\).

ie. \(\forall a \in A, \ x, y \in M, \ r \in R,\)
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\[ f_a(0) = 1 \]
\[ f_a(x + y) \geq f_a(x) \land f_a(y) \]
\[ f_a(rx) \geq f_a(x) \]

**Definition 6.2.3** [14] Let \((f, A)\) and \((g, B)\) be two fuzzy soft modules over \(M\). Then \((f, A)\) is called a **fuzzy soft submodule** of \((g, B)\) if

1. \(A \subset B\)
2. \(\forall a \in A, f_a\) is a fuzzy submodule of \(g_a\).

**Theorem 6.2.4** Let \((f, A)\) and \((g, B)\) be two fuzzy soft modules over \(M\). Then \((f, A) \sqcap (g, B)\) is a fuzzy soft module over \(M\) if \(A \cap B \neq \emptyset\).

**Proof.** Let \((f, A) \sqcap (g, B) = (h, C)\) where \(C = A \cap B \neq \emptyset\). Then we have \(h_c = f_c \cap g_c \ \forall \ c \in C\). Since \(f_c\) and \(g_c\) are fuzzy submodules of \(M\) \(\forall \ c \in C\), their intersection \(h_c\) is also fuzzy submodule of \(M\) \(\forall \ c \in C\).

\[ \therefore (h, C) = (f, A) \sqcap (g, B)\] is a fuzzy soft module over \(M\).

**Theorem 6.2.5** Let \((f, A)\) and \((g, B)\) be two fuzzy soft modules over \(M\). Then \((f, A) \sqcup (g, B)\) is a fuzzy soft module over \(M\).

**Proof.** By the definition we have \((f, A) \sqcup (g, B) = (h, A \times B)\) where \(h(a, b) = h_{a,b} = f_a \cap g_b \ \forall (a, b) \in A \times B\). Since \(f_a\) and \(g_b\) are fuzzy submodules of \(M\) \(\forall a \in A\) and \(\forall b \in B\) respectively, \(f_a \cap g_b = h_{a,b}\) is a fuzzy submodule of \(M\) \(\forall (a, b) \in A \times B\).
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\[ \because (f, A) \wedge (g, B) \text{ is a fuzzy soft module over } M. \]

**Theorem 6.2.6** Let \((f, A)\) and \((g, B)\) be two fuzzy soft modules over \(M\). If \(A \cap B = \emptyset\), then \((f, A) \cup (g, B)\) is a fuzzy soft module over \(M\).

**Proof.** By definition we have \((f, A) \cup (g, B) = (h, C)\) where \(C = A \cup B\) and \(\forall c \in C,

\[
h(c) = h_c = \begin{cases} 
f_c & \text{if } c \in A - B \\
g_c & \text{if } c \in B - A \\
f_c \cup g_c & \text{if } c \in A \cap B
\end{cases}
\]

Since \(A \cap B = \emptyset\), \(h(c) = h_c\) is either \(f_c\) or \(g_c\) \(\forall c \in C\), which is a fuzzy submodule of \(M\).

\[ \because (f, A) \cup (g, B) \text{ is a fuzzy soft module over } M. \]

**Theorem 6.2.7** Let \((f, A)\) be a fuzzy soft set over \(M\). Then \((f, A)\) is a fuzzy soft module over \(M\) if and only if \(\forall a \in A\) and for arbitrary \(\alpha \in (0, 1]\) with \((f_a)^\alpha \neq \emptyset\), the \(\alpha\)-level soft set \((f, A)^\alpha\) is a soft module over \(M\).

**Proof.** Let \((f, A)\) be a fuzzy soft module over \(M\). Then for each \(a \in A\), \(f(a) = f_a\) is a fuzzy submodule of \(M\).

Let \(a \in A\) and \(\alpha \in (0, 1]\) be arbitrary such that \((f_a)^\alpha \neq \emptyset\).

Then \(x, y \in (f_a)^\alpha\)

\[ \Rightarrow f_a(x) \geq \alpha \text{ and } f_a(y) \geq \alpha. \]

\[ \Rightarrow f_a(x - y) \geq f_a(x) \wedge f_a(y) \text{ (Since } f_a \text{ is a fuzzy submodule of } M.\)]

\[ \geq \alpha, \text{ which implies that } x - y \in (f_a)^\alpha. \]
Now for any \( r \in R \) and \( x \in (f_a)^\alpha \), \( f_a(rx) \geq f_a(x) \geq \alpha \), which implies that \( rx \in (f_a)^\alpha \).

\[ \therefore (f_a)^\alpha \text{ is a submodule of } M \text{ for each } a \in A. \]

ie. \( (f, A)^\alpha \) is a soft module over \( M \).

Conversely, let \( (f, A)^\alpha \) is a soft module over \( M \) for all \( \alpha \in (0, 1] \). \( (f_a)^\alpha \neq \emptyset \).

Then for each \( a \in A \), \( (f_a)^\alpha \) is a submodule of \( M \) \( \forall \alpha \in (0, 1] \).

\[ \therefore \text{We have } 0 \in (f_a)^1 \text{, i.e. } f_a(0) \geq 1, \text{ hence } f_a(0) = 1 \ \forall a \in A. \]

Let \( \alpha = f_a(x) \land f_a(y) \) for \( x, y \in M \), then \( x, y \in (f_a)^\alpha \) (since \( f_a(x) \geq \alpha \), \( f_a(y) \geq \alpha \))

\[ \therefore x + y \in (f_a)^\alpha \text{ (since } (f_a)^\alpha \text{ is submodule of } M) \]

Hence \( f_a(x + y) \geq \alpha = f_a(x) \land f_a(y) \ \forall a \in A. \)

Now for \( x \in M \), let \( \alpha = f_a(x) \). Then \( x \in (f_a)^\alpha \) implies that \( rx \in (f_a)^\alpha \) for any \( r \in R \), which implies that \( f_a(rx) \geq \alpha = f_a(x) \ \forall a \in A. \)

\[ \therefore f_a \text{ is a fuzzy submodule of } M \ \forall a \in A. \]

Hence \( (f, A) \) is a fuzzy soft module over \( M \).

**Theorem 6.2.8** Let \( (f, A) \) and \( (g, B) \) be two fuzzy soft sets over \( M \) such that \( (f, A) \sqsubseteq (g, B) \), then for any \( \alpha \in (0, 1] \), \( (f, A)^\alpha \sqsubseteq (g, B)^\alpha \).

**Proof.** Given that \( (f, A) \sqsubseteq (g, B) \). ie. \( A \subseteq B \) and \( f_a \sqsubseteq g_a \ \forall a \in A. \)
Now for any $\alpha \in (0, 1]$, $(f, A)^\alpha = (f^\alpha, A)$ is a soft set over $M$. Similarly $(g, B)^\alpha$ is also a soft set over $M$. To show that $(f, A)^\alpha \subseteq (g, B)^\alpha$, we have to show that $A \subseteq B$ and $(f_a)^\alpha \subseteq (g_a)^\alpha \ \forall \ a \in A$.

For any $a \in A$, let $x \in (f_a)^\alpha$ which implies that $f_a(x) \geq \alpha$.

Since $g_a(x) \geq f_a(x) \ \forall \ a \in A, x \in M$, we get $g_a(x) \geq \alpha$, i.e. $x \in (g_a)^\alpha$.

$\therefore (f_a)^\alpha \subseteq (g_a)^\alpha \ \forall \ a \in A$.

Hence $(f, A)^\alpha \subseteq (g, B)^\alpha$.

**Corollary 6.2.9** Let $(f, A)$ and $(g, B)$ be two fuzzy soft sets over $M$. Then $(f, A) = (g, B) \Rightarrow (f, A)^\alpha = (g, B)^\alpha \ \forall \ \alpha \in (0, 1]$.

**Theorem 6.2.10** Let $(f, A)$ be a fuzzy soft set over $M$ and let $\alpha, \beta \in (0, 1]$ with $\alpha \leq \beta$. Then $(f, A)^\beta \subseteq (f, A)^\alpha$.

**Proof.** Let $\alpha, \beta \in (0, 1]$ with $\alpha \leq \beta$. We have $(f, A)^\beta = (f^\beta, A)$ and

$$(f, A)^\alpha = (f^\alpha, A).$$

For any $a \in A$, let $x \in (f_a)^\beta$. i.e. $f_a(x) \geq \beta$.

Since $\beta \geq \alpha$, we get $f_a(x) \geq \alpha$, i.e. $x \in (f_a)^\alpha$.

$\therefore (f_a)^\beta \subseteq (f_a)^\alpha \ \forall \ a \in A$.

Hence $(f, A)^\beta \subseteq (f, A)^\alpha$. 
6.3 Homomorphism of Fuzzy soft modules

In this section we give some characteristic properties of fuzzy soft function in module theory and prove some theorems on fuzzy soft homomorphism between fuzzy soft modules.

**Definition 6.3.1**[9] Let \((f, A)\) and \((g, B)\) be two fuzzy soft sets over \(X\) and \(Y\) respectively. And let \(\varphi: X \to Y\) and \(\psi: A \to B\) be two functions. Then the pair \((\varphi, \psi)\) is called a fuzzy soft function from \(X\) to \(Y\), which means \((\varphi, \psi)\) is a fuzzy soft function from the fuzzy soft set \((f, A)\) over \(X\) to the fuzzy soft set \((g, B)\) over \(Y\).

**Remark.** In the case of taking \((f, A)\) and \((g, B)\) as soft sets over \(X\) and \(Y\) respectively, the pair \((\varphi, \psi)\) is called a soft function from \(X\) to \(Y\).

**Definition 6.3.2**[9] Let \((f, A)\) and \((g, B)\) be two fuzzy soft sets over \(X\) and \(Y\) respectively. Let \((\varphi, \psi)\) be a fuzzy soft function from \(X\) to \(Y\).

1. The image of \((f, A)\) under the fuzzy soft function \((\varphi, \psi)\) denoted by \((\varphi, \psi)(f, A) = (\varphi(f), \psi(A))\) is a fuzzy soft set over \(Y\) where

\[
(\varphi(f))_b(y) = \begin{cases} 
\bigvee_{\varphi(x)=y} V_{\psi(a)=b} f_a(x) & \text{if } \varphi^{-1}(y) \neq \emptyset \\
0 & \text{otherwise}
\end{cases}
\]

\[\forall b \in \psi(A), \forall y \in Y.\]
(2) The **pre-image** of \((g, B)\) under the fuzzy soft function \((\varphi, \psi)\) is denoted by 

\[
(\varphi, \psi)^{-1}(g, B) = (\varphi^{-1}(g), \psi^{-1}(B))
\]

The pre-image of \((g, B)\) under the fuzzy soft function \((\varphi, \psi)\) is the fuzzy soft set over \(X\) defined by 

\[
(\varphi, \psi)^{-1}(g, B) = (\varphi^{-1}(g), \psi^{-1}(B))
\]

\[
(\varphi^{-1}(g))_a(x) = g_{\psi(a)}(\varphi(x)) \quad \forall \ a \in \psi^{-1}(B), \ \forall \ x \in X.
\]

**Remark 6.3.3**

(1) Let \(\varphi : X \to Y\) be a function, \(I : A \to A\) be the identity function and \((f, A)\) be a fuzzy soft set over \(X\). Then \((\varphi, I_A)(f, A) = (\varphi(f), A)\) is a fuzzy soft set over \(Y\), where for each \(a \in A\), \((\varphi(f))_a\) is a fuzzy subset of \(Y\) defined by 

\[
(\varphi(f))_a(y) = \begin{cases} 
\{f_a(x) : y = \varphi(x)\} & \text{if } \varphi^{-1}(y) \neq \emptyset \\
0 & \text{otherwise}
\end{cases} \quad \forall \ y \in Y.
\]

(2) Let \(\varphi : X \to Y\) be a function and \((g, A)\) be a fuzzy soft set over \(Y\). Then \((\varphi, I_A)^{-1}(g, A) = (\varphi^{-1}(g), A)\) is a fuzzy soft set over \(X\) where for each \(a \in A\), \((\varphi^{-1}(g))_a\) is a fuzzy subset of \(X\) defined by 

\[
(\varphi^{-1}(g))_a(x) = g_{\varphi(a)}(\varphi(x)) \quad \forall \ x \in X.
\]

**Theorem 6.3.4**

Let \(\varphi : X \to Y\) be a function. Let \((f, A)\) and \((g, A)\) be fuzzy soft sets over \(X\) and \(Y\) respectively. Then for any \(\alpha \in (0, 1]\).

(i) \((\varphi, I_A)(f, A)\)^\alpha \subseteq ((\varphi, I_A)(f, A))^\alpha\)

(ii) \((\varphi, I_A)^{-1}(g, A)^\alpha = ((\varphi, I_A)^{-1}(g, A))^\alpha\)
Proof. (i) We know that \((\varphi, I_A)(f, A)^\alpha = (\varphi, I_A)(f^\alpha, A) = (\varphi(f^\alpha), A)\) is a soft set over \(Y\) (Here \((\varphi, I_A)\) is a soft function from \(X\) to \(Y\)) defined by,

for each \(a \in A\), \((\varphi(f^\alpha))_a\) is a subset of \(Y\) where

\[
(\varphi(f^\alpha))_a = \varphi((f_a)^\alpha) = \{ \varphi(x) : x \in (f_a)^\alpha \}
\]

\[= \{ \varphi(x) : f_a(x) \geq \alpha \} \tag{1} \]

Now \(((\varphi, I_A)(f, A))^\alpha = (\varphi(f), A)^\alpha = ((\varphi(f))^\alpha, A)\) is a soft set over \(Y\) defined by,

for each \(a \in A\), \(((\varphi(f))^\alpha_a = ((\varphi(f))_a)^\alpha\) is a subset of \(Y\), where

\[
((\varphi(f))_a)^\alpha = \{ y \in Y : (\varphi(f))_a(y) \geq \alpha \} \tag{2} \]

Now let \(y \in (\varphi(f^\alpha))_a = \varphi((f_a)^\alpha)\)

\[\Rightarrow y = \varphi(x) \text{ for some } x \in (f_a)^\alpha\]

\[\Rightarrow y = \varphi(x) \text{ for some } x \text{ with } f_a(x) \geq \alpha\]

\[\Rightarrow \vee \{ f_a(x) : y = \varphi(x) \} \geq \alpha\]

\[\Rightarrow (\varphi(f))_a(y) \geq \alpha\]

\[\Rightarrow y \in ((\varphi(f))_a)^\alpha\]

\[\therefore (\varphi(f^\alpha))_a \subseteq ((\varphi(f))_a)^\alpha \forall a \in A.\]

Hence \(((\varphi, I_A)(f, A))^\alpha \subseteq ((\varphi, I_A)(f, A))^{\alpha}.\)

(ii) We have \((\varphi, I_A)^{-1}(g, A)^\alpha = (\varphi, I_A)^{-1}(g^\alpha, A) = (\varphi^{-1}(g^\alpha), A)\) is a soft set over \(X\) defined by

\[ (\varphi^{-1}(g^\alpha))_a = \varphi^{-1}((g_a)^\alpha) \] is a subset of \(X\) where
\[ \varphi^{-1}((g_a)^\alpha) = \{ x \in X : \varphi(x) \in (g_a)^\alpha \} \]
\[ = \{ x \in X : g_a(\varphi(x)) \geq \alpha \} \]
\[ = \{ x \in X : (\varphi^{-1}(g))_a(x) \geq \alpha \} \]
\[ = ((\varphi^{-1}(g))_a)^\alpha \]

i.e. \((\varphi^{-1}(g^a))_a = ((\varphi^{-1}(g))_a)^\alpha \ \forall \alpha \in A\)

Hence \((\varphi, I_A)^{-1}(g, A)^\alpha = ((\varphi, I_A)^{-1}(g, A))^\alpha\).

**Definition 6.3.5** [9] Let \((f, A)\) and \((g, B)\) be fuzzy soft sets over the R-modules \(M\) and \(N\) respectively and \((\varphi, \psi)\) be a fuzzy soft function from \(M\) to \(N\). If \(\varphi\) is a module homomorphism from \(M\) to \(N\), then \((\varphi, \psi)\) is said to be a fuzzy soft homomorphism from \(M\) to \(N\). If \(\varphi\) is an isomorphism from \(M\) to \(N\) and \(\psi\) is a one-to-one mapping from \(A\) onto \(B\), then \((\varphi, \psi)\) is said to be a fuzzy soft isomorphism.

**Theorem 6.3.6** Let \(M\) and \(N\) be two \(R\)-modules. Let \((f, A)\) be a fuzzy soft module over \(M\) and \((\varphi, \psi)\) be a fuzzy soft homomorphism from \(M\) to \(N\). Then \((\varphi, \psi)(f, A)\) is a fuzzy soft module over \(N\).

**Proof.** We have \((\varphi, \psi)(f, A) = (\varphi(f), \psi(A))\) is a fuzzy soft set over \(N\) defined by
\[
(\varphi(f))_b(y) = \begin{cases} 
\bigvee_{\varphi(x) = y} \bigvee_{\psi(a) = b} f_a(x) & \text{if } \varphi^{-1}(y) \neq \emptyset \\
0 & \text{otherwise}
\end{cases}
\]
∀ b ∈ ψ(A), ∀ y ∈ N.

Let b ∈ ψ(A),

\[(φ(f))_b(0) = V_{φ(x)=0} V_{ψ(a)=b} f_a(x)\]

\[= V_{ψ(a)=b} f_a(0) = 1 \quad (\text{Since } φ \text{ is a homomorphism and } f_a \text{ is a fuzzy submodule of } M \ \forall a \in A)\]

Let y_1, y_2 ∈ N and b ∈ ψ(A). If φ⁻¹(y_1) = ∅ or φ⁻¹(y_2) = ∅ then clearly,

\[\left(φ(f)\right)_b(y_1 + y_2) ≥ \left(φ(f)\right)_b(y_1) ∧ \left(φ(f)\right)_b(y_2).\]

If φ⁻¹(y_1) ≠ ∅ and φ⁻¹(y_2) ≠ ∅, then

\[\left(φ(f)\right)_b(y_1 + y_2) = V_{φ(x)=y_1+y_2} V_{ψ(a)=b} f_a(x)\]

\[≥ V_{ψ(a)=b} f_a(x_1 + x_2), \text{ by taking } x = x_1 + x_2 \text{ where } x_1 ∈ φ⁻¹(y_1) \text{ and } x_2 ∈ φ⁻¹(y_2)\]

\[≥ V_{ψ(a)=b} (f_a(x_1) ∧ f_a(x_2))\]

\[= (V_{ψ(a)=b} f_a(x_1)) ∧ (V_{ψ(a)=b} f_a(x_2)), \text{ this is true for all } x_1, x_2 ∈ M \text{ such that } φ(x_1) = y_1, \ φ(x_2) = y_2.\]

\[∴ \left(φ(f)\right)_b(y_1 + y_2) ≥ \left(V_{φ(x_1)=y_1} V_{ψ(a)=b} f_a(x_1)\right) ∧ \left(V_{φ(x_2)=y_2} V_{ψ(a)=b} f_a(x_2)\right)\]

\[= \left(φ(f)\right)_b(y_1) ∧ \left(φ(f)\right)_b(y_2).\]

Now let y ∈ N, r ∈ R, b ∈ ψ(A). If φ⁻¹(y) = ∅, then clearly

\[\left(φ(f)\right)_b(ry) ≥ \left(φ(f)\right)_b(y).\]
If $\varphi^{-1}(y) \neq \emptyset$,
then
$$\left(\varphi(f)\right)_b(ry) = V_{\varphi(x) = ry} V_{\psi(a) = b} f_a(x)$$
$$\geq V_{\varphi(x_1) = ry} V_{\psi(a) = b} f_a(rx_1)$$
$$\geq V_{x_1 \in \varphi^{-1}(y)} V_{\psi(a) = b} f_a(rx_1)$$
(Since $\varphi$ is a homomorphism)
$$\geq V_{\varphi(x_1) = y} V_{\psi(a) = b} f_a(x_1)$$ (Since $f_a(rx_1) \geq f_a(x_1)$)
$$= \left(\varphi(f)\right)_b(y)$$

$\therefore \left(\varphi(f)\right)_b$ is a fuzzy submodule of $N \forall b \in \psi(A)$.

Hence $(\varphi, \psi)(f, A) = (\varphi(f), \psi(A))$ is a fuzzy soft module over $N$.

**Theorem 6.3.7** Let $M$ and $N$ be two $R$-modules. Let $(g, B)$ be a fuzzy soft module over $N$ and $(\varphi, \psi)$ be a fuzzy soft homomorphism from $M$ to $N$. Then $(\varphi, \psi)^{-1}(g, B)$ is a fuzzy soft module over $M$.

**Proof.** We have $(\varphi, \psi)^{-1}(g, B) = (\varphi^{-1}(g), \psi^{-1}(B))$ where
$$(\varphi^{-1}(g))_a(x) = g_{\psi(a)}(\varphi(x)) \forall a \in \psi^{-1}(B), \forall x \in M.$$ For any $a \in \psi^{-1}(B)$, $(\varphi^{-1}(g))_a(0) = g_{\psi(a)}(\varphi(0)) = g_{\psi(a)}(0) = 1$ (Since $\varphi$ is a homomorphism from $M$ to $N$ and $g_{\psi(a)}$ is a fuzzy submodule of $N$.)

Let $x_1, x_2 \in M$, $(\varphi^{-1}(g))_a(x_1 + x_2) = g_{\psi(a)}(\varphi(x_1 + x_2))$
$$= g_{\psi(a)}(\varphi(x_1) + \varphi(x_2))$$
$$\geq g_{\psi(a)}(\varphi(x_1)) \land g_{\psi(a)}(\varphi(x_2))$$
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\[(\varphi^{-1}(g))_a(x_1) \land (\varphi^{-1}(g))_a(x_2) \quad \forall a \in \psi^{-1}(B).\]

Now let \( r \in R, x \in M \), \((\varphi^{-1}(g))_a(rx) = g_{\psi(a)}(\varphi(rx))

\[= g_{\psi(a)}(r\varphi(x))\]

\[\geq g_{\psi(a)}(\varphi(x))\]

\[= (\varphi^{-1}(g))_a(x) \quad \forall a \in \psi^{-1}(B).\]

\[\therefore (\varphi^{-1}(g))_a \text{ is a fuzzy submodule of } M \forall a \in \psi^{-1}(B).\]

Hence \((\varphi, \psi)^{-1}(g, B) = (\varphi^{-1}(g), \psi^{-1}(B))\) is a fuzzy soft module over \(M\).

**Corollary 6.3.8** Let \( \varphi : M \to N \) be an \( R \)-module homomorphism and \((f, A), (g, A)\) be fuzzy soft modules over \(M\) and \(N\) respectively. Then \((\varphi(f), A)\) is a fuzzy soft module over \(N\) and \((\varphi^{-1}(g), A)\) is a fuzzy soft module over \(M\).

**Proof.** It is clear by taking \((\varphi, I_A)\) as fuzzy soft homomorphism from \(M\) to \(N\).