CHAPTER 4

SOME INEQUALITIES CONCERNING THE GENERALISED GROWTH PROPERTIES OF COMPOSITE ENTIRE FUNCTIONS IN THE LIGHT OF THEIR MAXIMUM TERMS
4.1 Introduction, Definitions and Notations.

Let \( f \) be an entire function defined in the open complex plane \( \mathbb{C} \). The maximum term \( \mu (r, f) \) of \( f = \sum_{n=0}^{\infty} a_n z^n \) on \( |z| = r \) is defined by

\[
\mu (r, f) = \max_{n \geq 0} (|a_n| r^n).
\]

Though Definition 4.1.1, Definition 4.1.2 and Definition 4.1.3 have already been defined in Chapter 2 as Definition 2.1.1, Definition 2.1.2 and Definition 2.1.3 respectively, we state here again in order to keep a continuation of our discussion:

**Definition 4.1.1** The order \( \rho_f \) and lower order \( \lambda_f \) of an entire function \( f \) is defined as follows:

\[
\rho_f = \limsup_{r \to \infty} \frac{\log \log M (r, f)}{\log r} \quad \text{and} \quad \lambda_f = \liminf_{r \to \infty} \frac{\log \log M (r, f)}{\log r}
\]

The results of this chapter have been published in the *International Journal of Mathematical Manuscripts* (IJMM), see [16].
where
\[ \log^{[k]} x = \log \left( \log^{[k-1]} x \right) \text{ for } k = 1, 2, 3, \ldots \text{ and} \]
\[ \log^{[0]} x = x. \]

If \( \rho_f < \infty \) then \( f \) is of finite order. Also \( \rho_f = 0 \) means that \( f \) is of order zero. In this connection, Liao and Yang [35] gave the following definition:

**Definition 4.1.2** [35] Let \( f \) be an entire function of order zero. Then the quantities \( \rho_f^* \) and \( \lambda_f^* \) of an entire function \( f \) are defined as:
\[ \rho_f^* = \limsup_{r \to \infty} \frac{\log^{[2]} M(r, f)}{\log^{[2]} r} \text{ and } \lambda_f^* = \liminf_{r \to \infty} \frac{\log^{[2]} M(r, f)}{\log^{[2]} r}. \]

Datta and Biswas [12] gave an alternative definition of zero order and zero lower order of an entire function in the following way:

**Definition 4.1.3** [12] Let \( f \) be an entire function of order zero. Then the quantities \( \rho_f^{**} \) and \( \lambda_f^{**} \) of \( f \) are defined by
\[ \rho_f^{**} = \limsup_{r \to \infty} \frac{\log M(r, f)}{\log r} \text{ and } \lambda_f^{**} = \liminf_{r \to \infty} \frac{\log M(r, f)}{\log r}. \]

Sato [42] defined the generalised order \( \rho_f^{[l]} \) and generalised lower order \( \lambda_f^{[l]} \) of an entire function \( f \) respectively as follows:

**Definition 4.1.4** [42] Let \( l \) be an integer \( \geq 2 \). The generalised order \( \rho_f^{[l]} \) and generalised lower order \( \lambda_f^{[l]} \) of an entire function \( f \) are respectively defined as
\[ \rho_f^{[l]} = \limsup_{r \to \infty} \frac{\log^{[l]} M(r, f)}{\log r} \text{ and } \lambda_f^{[l]} = \liminf_{r \to \infty} \frac{\log^{[l]} M(r, f)}{\log r}. \]

For \( l = 1 \) and \( l = 2 \), Definition 4.1.4 respectively coincides with Definition 4.1.3 and Definition 4.1.1.

Since for \( 0 \leq r < R \),
\[ \mu(r, f) \leq M(r, f) \leq \frac{R}{R - r} \mu(R, f) \quad \text{cf. [47]} \]  \quad (4.1.1)

it is easy to see that
\[ \rho_f^{[l]} = \limsup_{r \to \infty} \frac{\log^{[l]} \mu(r, f)}{\log r}, \quad \lambda_f^{[l]} = \liminf_{r \to \infty} \frac{\log^{[l]} \mu(r, f)}{\log r} \]
where \( l \) is an integer \( \geq 1 \).

In our subsequent discussion, we will use the following notation:
\[ \exp^{[k]} y = \exp \left( \exp^{[k-1]} y \right) \text{ for } k = 1, 2, 3, \ldots \text{ and} \]
\[ \exp^{[0]} y = y. \]

In this chapter we investigate some aspects of the comparative growths of maximum terms of composition of two entire functions with their corresponding left and right factors.
4.2 Lemmas.

In this section we present some lemmas which will be needed in the sequel.

**Lemma 4.2.1** [47] Let $f$ and $g$ be any two entire functions. Then for all sufficiently large values of $r$, 
\[
\mu(r, f \circ g) \geq \frac{1}{4} \mu \left( \frac{1}{8} \mu \left( \frac{r}{4}, g \right) - |g(0)|, f \right).
\]

**Lemma 4.2.2** [47] Let $f$ and $g$ be any two entire functions. Then for every $\alpha > 0$ and $0 < r < R$, 
\[
\mu(r, f \circ g) \leq \frac{\alpha}{\alpha - 1} \mu \left( \frac{\alpha R}{R - r}, \mu(R, g), f \right).
\]

**Lemma 4.2.3** [12] Let $f$ be a meromorphic function and $g$ be entire such that $\rho_f < \infty$ and $\rho_g = 0$. Then $\rho_{f \circ g} < \infty$.

4.3 Theorems.

In this section we present the main results of the chapter.

**Theorem 4.3.1** Let $f$ and $g$ be any two entire functions with $0 < \lambda_f^{[l]} \leq \rho_f^{[l]} < \infty$ where $l$ is an integer such that $l \geq 1$ and $0 < A < \lambda_g$. Then for all sufficiently large values of $r$, 
\[
\log^{[l-1]} \mu(r, f \circ g) > \log^{[l-1]} \mu \left( \exp \left( r^A \right), f \right).
\]

**Proof.** In view of Lemma 4.2.1, we obtain for all sufficiently large values of $r$ that 
\[
\log^{[l]} \mu(r, f \circ g) \geq \log^{[l]} \left\{ \frac{1}{4} \mu \left( \frac{1}{8} \mu \left( \frac{r}{4}, g \right), f \right) \right\}
\]
i.e., 
\[
\log^{[l]} \mu(r, f \circ g) \geq \left( \lambda_f^{[l]} - \varepsilon \right) \log \left\{ \frac{1}{8} \mu \left( \frac{r}{4}, g \right) \right\} + O(1)
\]
i.e., 
\[
\log^{[l]} \mu(r, f \circ g) \geq \left( \lambda_f^{[l]} - \varepsilon \right) \log \mu \left( \frac{r}{4}, g \right) + O(1)
\]
i.e., 
\[
\log^{[l]} \mu(r, f \circ g) \geq \left( \lambda_f^{[l]} - \varepsilon \right) \left( \frac{r}{4} \right)^{\lambda_g - \varepsilon} + O(1). \tag{4.3.1}
\]

Again from the definition of $\rho_f^{[l]}$, we have for arbitrary positive $\varepsilon$ and for all sufficiently large values of $r$ that 
\[
\log^{[l]} \mu \left( \exp \left( r^A \right), f \right) \leq \left( \rho_f^{[l]} + \varepsilon \right) \log \exp \left( r^A \right)
\]
i.e., 
\[
\log^{[l]} \mu \left( \exp \left( r^A \right), f \right) \leq \left( \rho_f^{[l]} + \varepsilon \right) r^A. \tag{4.3.2}
\]
Therefore it follows from (4.3.1) and (4.3.2) for all sufficiently large values of $r$ that
\[
\frac{\log^{[l]} \mu(r, f \circ g)}{\log^{[l]} \mu(\exp(rA), f)} \geq \frac{\left(\lambda_f^{[l]} - \varepsilon\right) \left(\frac{r}{4}\right)^{\lambda_g - \varepsilon} + O(1)}{\left(\rho_f^{[l]} + \varepsilon\right)r^A}.
\]

(4.3.3)

As \( A < \lambda_g \) we can choose \( \varepsilon(>0) \) in such a way that
\[
A < \lambda_g - \varepsilon.
\]

(4.3.4)

Thus from (4.3.3) and (4.3.4) we get that
\[
\lim_{r \to \infty} \frac{\log^{[l]} \mu(r, f \circ g)}{\log^{[l]} \mu(\exp(rA), f)} = \infty.
\]

(4.3.5)

From (4.3.5) we obtain for all sufficiently large values of \( r \) and \( K > 1 \)
\[
\log^{[l]} \mu(r, f \circ g) > K \log^{[l]} \mu(\exp(rA), f)
\]
\( \text{i.e.}, \)
\[
\log^{[l]} \mu(r, f \circ g) > \log \left\{ \log^{[l-1]} \mu(\exp(rA), f) \right\}^{K}
\]

\( \text{i.e.}, \)
\[
\log^{[l]} \mu(r, f \circ g) > \log \log^{[l-1]} \mu(\exp(rA), f)
\]

\( \text{i.e.}, \)
\[
\log^{[l-1]} \mu(r, f \circ g) > \log^{[l-1]} \mu(\exp(rA), f).
\]

This completes the proof of the theorem. \( \blacksquare \)

**Remark 4.3.1** The following examples ensure the conclusion of Theorem 4.3.1.

**Example 4.3.1** Let \( f = z, g = \exp z \) and \( l = 1 \).

Then
\[
\lambda_f^{[l]} = \rho_f^{[l]} = 1, \lambda_g = 1 \text{ and } A = 1/2.
\]

Now
\[
\mu(r, f \circ g) \geq \frac{1}{2} M(r/2, f \circ g) = 1/2 M(r/2, \exp z) = 1/2 \exp \left(\frac{r}{2}\right)
\]

and therefore
\[
\log^{[l-1]} \mu(r, f \circ g) \geq 1/2 \exp \left(\frac{r}{2}\right).
\]

Again
\[
\mu(\exp(rA), f) = \mu(\exp(\frac{r}{2}), z) = \exp \left(\frac{r}{2}\right)
\]

and therefore
\[
\log^{[l-1]} \mu(\exp(rA), f) = \exp \left(\frac{r}{2}\right).
\]

Thus
\[
\log^{[l-1]} \mu(r, f \circ g) > \log^{[l-1]} \mu(\exp(rA), f).
\]
Example 4.3.2 Let $f = z, g = \exp z$ and $l = 2$.

Then

$$\lambda_f^{[l]} = \rho_f^{[l]} = 0, \lambda_g = 1 \text{ and } A = 1/2.$$ 

Now

$$\mu(r, f \circ g) \geq \frac{1}{2} M(r/2, f \circ g) = 1/2 M(r/2, \exp z) = 1/2 \exp \left( \frac{r}{2} \right)$$

and so

$$\log^{[l-1]} \mu(r, f \circ g) \geq \log \left( \frac{1}{2} \exp \left( \frac{r}{2} \right) \right) = \frac{r}{2} + O(1).$$

Again

$$\mu(\exp(r^A), f) = \mu(\exp \left( \frac{r^A}{2} \right), z) = \exp \left( \frac{r^A}{2} \right)$$

and so

$$\log^{[l-1]} \mu(\exp(r^A), f) = \log(\exp \left( \frac{r^A}{2} \right)) = r^A.$$

Therefore

$$\log^{[l-1]} \mu(r, f \circ g) > \log^{[l-1]} \mu(\exp(r^A), f).$$

Remark 4.3.2 If we take $0 < A < \rho_g$ instead of $0 < A < \lambda_g$ in Theorem 4.3.1 and the other conditions remain the same, then in the line of Theorem 4.3.1 one can easily verify

$$\log^{[l-1]} \mu(r, f \circ g) > \log^{[l-1]} \mu(\exp(r^A), f),$$

for a sequence of values of $r$ tending to infinity.

Remark 4.3.3 Also if we consider $0 < \lambda_f^{[l]} < \infty$ or $0 < \rho_f^{[l]} < \infty$ instead of $0 < \lambda_f^{[l]} \leq \rho_f^{[l]} < \infty$ in Theorem 4.3.1 and the other conditions remain the same, then in the line of Theorem 4.3.1 one can easily verify for a sequence of values of $r$ tending to infinity that

$$\log^{[l-1]} \mu(r, f \circ g) > \log^{[l-1]} \mu(\exp(r^A), f).$$

In the line of Theorem 4.3.1 we may state the following theorem without its proof:

Theorem 4.3.2 Let $f$ and $g$ be any two entire functions with $0 < \lambda_f^{[l]} < \infty$ where $l \geq 1$ is an integer and $0 < A < \lambda_g \leq \rho_g < \infty$. Then for all sufficiently large values of $r$,

$$\log^{[l-1]} \mu(r, f \circ g) > \log \mu(\exp(r^A), g).$$

Remark 4.3.4 The condition $\lambda_f^{[l]} > 0$ is essential in Theorem 4.3.2 as we see in the following example.
Example 4.3.3 Let \( f = z \) and \( g = \exp z \). Further, let \( l = 2 \) and \( A = 1/2 \).

Then
\[
\lambda_f^{[l]} = \rho_f^{[l]} = 0 \quad \text{and} \quad \lambda_g = \rho_g = 1.
\]

Now
\[
\log^{[l-1]} \mu (r, f \circ g) = \mu (r, f \circ g) = \log \mu (r, \exp z)
\]
and
\[
\log \mu (\exp (r^A), f) = \log \mu \left( \exp \left( r^{1/2} \right), \exp z \right).
\]

Since \( \mu \) is an increasing function of \( r \), it follows that
\[
\log^{[l-1]} \mu (r, f \circ g) < \log \mu (\exp (r^A), g),
\]
which contradicts the conclusion of Theorem 4.3.2.

Remark 4.3.5 The following example verifies the conclusion of Theorem 4.3.2.

Example 4.3.4 Let \( f = z \) and \( g = \exp z \). Also let \( l = 1 \) and \( A = 1/3 \).

Then
\[
\lambda_f^{[l]} = \rho_f^{[l]} = 1 \quad \text{and} \quad \lambda_g = \rho_g = 1.
\]

Now
\[
\log^{[l-1]} \mu (r, f \circ g) = \mu (r, f \circ g) = \mu (r, \exp z) \geq \frac{1}{2} M \left( \frac{1}{2} r, \exp z \right) = \frac{1}{2} \exp \left( \frac{r}{2} \right)
\]
and
\[
\mu (\exp (r^A), f) = \mu \left( \exp \left( r^{1/3} \right), \exp z \right) \leq M \left( \exp \left( r^{1/3} \right), \exp z \right) = \exp^{[2]} \left( r^{1/3} \right).
\]

So
\[
\log \mu (\exp (r^A), f) \leq \exp \left( r^{1/3} \right).
\]

Therefore
\[
\log^{[l-1]} \mu (r, f \circ g) > \log \mu (\exp (r^A), g).
\]

Remark 4.3.6 If we take \( 0 < \rho_f^{[l]} < \infty \) instead of \( 0 < \lambda_f^{[l]} < \infty \) in Theorem 2 and the other conditions remain the same, then in the line of Theorem 2 one can easily verify for a sequence of values of \( r \) tending to infinity that
\[
\log^{[l-1]} \mu (r, f \circ g) > \log \mu (\exp (r^A), g).
\]
Remark 4.3.7 Also if we consider $0 < A < \lambda_g < \infty$ or $0 < A < \rho_g < \infty$ instead of $0 < A < \lambda_g \leq \rho_g < \infty$ in Theorem 4.3.2 and the other conditions remain the same, then in the line of Theorem 4.3.2 one can show for a sequence of values of $r$ tending to infinity that
\[
\log^{[l-1]} \mu (r, f \circ g) > \log \mu (\exp (r^A), g).
\]

Theorem 4.3.3 Let $f$ and $g$ be any two entire functions such that $0 < \lambda_f^l \leq \rho_f^l < \infty$ where $l$ is an integer with $l \geq 1$ and $\rho_f < A < \infty$. Then for all sufficiently large values of $r$,
\[
\log^{[l-1]} \mu (r, f \circ g) < \log^{[l-1]} \mu (\exp (r^A), f).
\]

Proof. By Lemma 4.2.2, we get for all sufficiently large values of $r$ that
\[
\log^{[l]} \mu (r, f \circ g) \leq \log^{[l]} \mu \left( \frac{\alpha R}{R - r} \mu (R, g), f \right) + O(1)
\]
\[i.e., \quad \log^{[l]} \mu (r, f \circ g) \leq \left( \rho_f^l + \varepsilon \right) \log \left( \frac{\alpha R}{R - r} \mu (R, g) \right) + O(1)
\]
\[i.e., \quad \log^{[l]} \mu (r, f \circ g) \leq \left( \rho_f^l + \varepsilon \right) \log \mu (R, g) + O(1)
\]
\[i.e., \quad \log^{[l]} \mu (r, f \circ g) \leq \left( \rho_f^l + \varepsilon \right) R^{(\rho_f^l+\varepsilon)} + O(1).
\]

(4.3.6)

Again from the definition of $\lambda_f^l$ we have for arbitrary positive $\varepsilon$ and for all sufficiently large values of $r$ that
\[
\log^{[l]} \mu (\exp (r^A), f) \geq \left( \lambda_f^l - \varepsilon \right) \log \exp (r^A)
\]
\[i.e., \quad \log^{[l]} \mu (\exp (r^A), f) \geq \left( \lambda_f^l - \varepsilon \right) (r^A).
\]

(4.3.7)

Therefore it follows from (4.3.6) and (4.3.7) for all sufficiently large values of $r$ that
\[
\frac{\log^{[l]} \mu (\exp (r^A), f)}{\log^{[l]} \mu (r, f \circ g)} \geq \frac{\left( \lambda_f^l - \varepsilon \right) (r^A)}{\left( \rho_f^l + \varepsilon \right) R^{(\rho_f^l+\varepsilon)} + O(1)}.
\]

(4.3.8)

As $\rho_g < A$, we can choose $\varepsilon(>0)$ in such a way that
\[
\rho_g + \varepsilon < A.
\]

(4.3.9)

Thus from (4.3.8) and (4.3.9) we get that
\[
\lim_{r \to \infty} \frac{\log^{[l]} \mu (\exp (r^A), f)}{\log^{[l]} \mu (r, f \circ g)} = \infty.
\]

(4.3.10)

From (4.3.10) we obtain for all sufficiently large values of $r$ and $K > 1$
\[
\log^{[l]} \mu (\exp (r^A), f) > K \log^{[l]} \mu (r, f \circ g)
\]
\[i.e., \quad \log^{[l]} \mu (\exp (r^A), f) > \log \left\{ \log^{[l-1]} \mu (r, f \circ g) \right\}^K
\]
\[i.e., \quad \log^{[l]} \mu (\exp (r^A), f) > \log \log^{[l-1]} \mu (r, f \circ g)
\]
\[i.e., \quad \log^{[l-1]} \mu (\exp (r^A), f) > \log^{[l-1]} \mu (r, f \circ g).
\]

This completes the proof of the theorem. ■
Remark 4.3.8 The conclusion of Theorem 4.3.3 can be checked by the following examples.

Example 4.3.5 Let $f = z$ and $g = z$. Also let $l = 2$ and $A = 1$.

Then
\[ \lambda_f^{[l]} = \rho_f^{[l]} = 0 \quad \text{and} \quad \lambda_g = \rho_g = 0. \]

Now
\[ \log^{[l^{-1}]} \mu (r, f \circ g) = \log \mu (r, f \circ g) = \log \mu (r, z) \]
and
\[ \log^{[l^{-1}]} \mu (\exp (r^A), f) = \log \mu (\exp r, z). \]

Since $\mu$ is an increasing function of $r$, it follows that
\[ \log^{[l^{-1}]} \mu (r, f \circ g) < \log^{[l^{-1}]} \mu (\exp (r^A), f). \]

Example 4.3.6 Let $f = z$ and $g = \exp r$. Further, let $l = 2$ and $A = 2$.

Then
\[ \lambda_f^{[l]} = \rho_f^{[l]} = 0 \quad \text{and} \quad \lambda_g = \rho_g = 1. \]

Now
\[ \log^{[l^{-1}]} \mu (r, f \circ g) = \log \mu (r, f \circ g) = \log \mu (r, \exp z) \leq \log M(r, \exp z) = \log (\exp r) = r \]
and
\[ \log^{[l^{-1}]} \mu (\exp (r^A), f) = \log \mu (\exp (r^2), z) = \log (\exp (r^2)) = r^2. \]

Therefore
\[ \log^{[l^{-1}]} \mu (r, f \circ g) < \log^{[l^{-1}]} \mu (\exp (r^A), f). \]

Example 4.3.7 Let $f = z$ and $g = z$. Further, let $l = 1$ and $A = 1$.

Then
\[ \lambda_f^{[l]} = \rho_f^{[l]} = 1, \lambda_g = \rho_g = 0. \]

Now
\[ \log^{[l^{-1}]} \mu (r, f \circ g) = \mu (r, f \circ g) = \mu (r, z) = r \]
and
\[ \log^{[l-1]} \mu (\exp (r^A), f) = \mu (\exp (r), z) = \exp r. \]

Therefore
\[ \log^{[l-1]} \mu (r, f \circ g) < \log^{[l-1]} \mu (\exp (r^A), f). \]

**Example 4.3.8** Let \( f = z, g = \exp z, l = 1 \) and \( A = 2 \).

Then
\[ \lambda_f^{[l]} = \rho_f^{[l]} = 1 \quad \text{and} \quad \lambda_g = \rho_g = 1. \]

Now
\[ \log^{[l-1]} \mu (r, f \circ g) = \mu (r, f \circ g) = \mu (r, \exp z) \leq M(r, \exp z) = \exp r \]
and
\[ \log^{[l-1]} \mu (\exp (r^A), f) = \mu (\exp (r^2), z) = \exp (r^2). \]

Therefore
\[ \log^{[l-1]} \mu (r, f \circ g) < \log^{[l-1]} \mu (\exp (r^A), f). \]

**Remark 4.3.9** If we take \( 0 < \rho_f^{[l]} < \infty \) or \( 0 < \lambda_f^{[l]} < \infty \) instead of \( 0 < \lambda_f^{[l]} \leq \rho_f^{[l]} < \infty \) in Theorem 4.3.3 and the other conditions remain the same, then in the line of Theorem 4.3.3 one can easily verify that
\[ \log^{[l-1]} \mu (r, f \circ g) < \log^{[l-1]} \mu (\exp (r^A), f), \]
for a sequence of values of \( r \) tending to infinity.

**Remark 4.3.10** Also if we take \( \lambda_g < A < \infty \) instead of \( \rho_g < A < \infty \) in Theorem 4.3.3 and the other conditions remain the same, then in the line of Theorem 4.3.3 one can easily verify for a sequence of values of \( r \) tending to infinity,
\[ \log^{[l-1]} \mu (r, f \circ g) < \log^{[l-1]} \mu (\exp (r^A), f). \]

In the line of Theorem 4.3.3 we may state the following theorem without its proof:

**Theorem 4.3.4** Let \( f \) and \( g \) be any two entire functions such that \( \rho_f^{[l]} < \infty \) where \( l(\geq 1) \) is an integer and \( 0 < \lambda_g \leq \rho_g < A < \infty \). Then for all sufficiently large values of \( r \),
\[ \log^{[l-1]} \mu (r, f \circ g) < \log \mu (\exp (r^A), g). \]

**Remark 4.3.11** The condition \( \rho_f^{[l]} < \infty \) is necessary in Theorem 4.3.4 which is evident from the following example.
Example 4.3.9 Let $f = \exp^{[2]} z, g = \exp z, l = 2$ and $A = 2$.

Then

$$\lambda^{[l]}_f = \rho^{[l]}_f = \infty \text{ and } \lambda_g = \rho_g = 1.$$ 

Now

$$\log^{[l-1]} \mu (r, f \circ g) = \log \mu (r, f \circ g)$$
$$= \log \mu (r, \exp^{[3]} z)$$
$$\leq \log M(r, \exp^{[3]} z) = \log (\exp^{[3]} r) = \exp^{[2]} r$$

and

$$\log^{[l-1]} \mu \{ \exp (r^A), g \} = \log \mu \{ \exp (r^2), \exp z \}$$
$$\geq \log \left\{ \frac{1}{2} M(\exp \left( \frac{r^2}{2} \right), \exp z) \right\}$$
$$= \log \left( \exp^{[2]} \left( \frac{r^2}{4} \right) \right) + O(1) = \exp \left( \frac{r^2}{4} \right) + O(1).$$

Therefore

$$\log^{[l-1]} \mu (r, f \circ g) \not\leq \log^{[l-1]} \mu \{ \exp (r^A), g \},$$

which is contrary to Theorem 4.3.4.

Remark 4.3.12 The following example ensures the conclusion of Theorem 4.3.4.

Example 4.3.10 Let $f = \exp z, g = z, l = 2$ and $A = 1$.

Then

$$\lambda^{[l]}_f = \rho^{[l]}_f = 1 \text{ and } \lambda_g = \rho_g = 0.$$ 

Now

$$\log^{[l-1]} \mu (r, f \circ g) = \log \mu (r, f \circ g)$$
$$= \log \mu (r, \exp z)$$
$$\leq \log M(r, \exp z) = \log (\exp r) = r$$

and

$$\log^{[l-1]} \mu \{ \exp (r^A), g \} = \log \mu (\exp r, z) = \log (\exp r) = r.$$ 

Thus

$$\log^{[l-1]} \mu (r, f \circ g) \leq \log^{[l-1]} \mu \{ \exp (r^A), g \}. $$
Remark 4.3.13 If we take $0 < \lambda_g < A < \infty$ or $0 < \rho_g < A < \infty$ instead of $0 < \lambda_g \leq \rho_g < A < \infty$ in Theorem 4.3.4 and the other conditions remain the same, then in the line of Theorem 4.3.4 one can easily verify for a sequence of values of $r$ tending to infinity that

$$\log^{[l-1]} \mu(r, f \circ g) < \log \mu \left( \exp \left( r^A \right), g \right).$$

Remark 4.3.14 Also if we take $\lambda_f < \infty$ instead of $\rho_f < \infty$ in Theorem 4.3.4 and the other conditions remain the same, then in the line of Theorem 4.3.4 one can easily verify for a sequence of values of $r$ tending to infinity that

$$\log^{[l-1]} \mu(r, f \circ g) < \log \mu \left( \exp \left( r^A \right), g \right).$$

Theorem 4.3.5 Let $f$ and $g$ be any two entire functions such that $\rho_f^{[l]}$ is finite and $0 \leq \lambda_g^{[m]} \leq \rho_g^{[m]} < \infty$ where $l$ and $m$ are any two integers with $l \geq 1$ and $m \geq 2$. Then for all sufficiently large values of $r$,

$$\log^{[m-1]} \mu(\exp(r^A), g) > \log^{[m+l-2]} \mu(r, f \circ g),$$

where $A > 0$.

Proof. In view of Lemma 4.2.2, we get for all sufficiently large values of $r$ that

$$\log^{[l]} \mu(r, f \circ g) \leq \log^{[l]} \mu \left( \frac{\alpha R}{R - r} \mu(R, g), f \right) + O(1)$$

i.e.,

$$\log^{[l]} \mu(r, f \circ g) \leq \left( \rho_f^{[l]} + \varepsilon \right) \log \left( \frac{\alpha R}{R - r} \mu(R, g) \right) + O(1)$$

i.e.,

$$\log^{[l+m-1]} \mu(r, f \circ g) \leq \log^{[m]} \mu(R, g) + O(1)$$

i.e.,

$$\log^{[l+m-1]} \mu(r, f \circ g) \leq \left( \rho_g^{[m]} + \varepsilon \right) \log R + O(1). \tag{4.3.11}$$

Again from the definition of $\lambda_g^{[m]}$ and for any $A > 0$, we obtain for all sufficiently large values of $r$ that

$$\log^{[m]} \mu(\exp(r^A), g) \geq (\lambda_g^{[m]} - \varepsilon) \log \exp(r^A)$$

i.e.,

$$\log^{[m]} \mu(\exp(r^A), g) \geq (\lambda_g^{[m]} - \varepsilon) r^A. \tag{4.3.12}$$

Now from (4.3.11) and (4.3.12), it follows for all sufficiently large values of $r$ that

$$\frac{\log^{[m]} \mu(\exp(r^A), g)}{\log^{[l+m-1]} \mu(r, f \circ g)} \geq \frac{(\lambda_g^{[m]} - \varepsilon) r^A}{(\rho_g^{[m]} + \varepsilon) \log R + O(1)}$$

i.e.,

$$\lim_{r \to \infty} \frac{\log^{[m]} \mu(\exp(r^A), g)}{\log^{[l+m-1]} \mu(r, f \circ g)} = \infty. \tag{4.3.13}$$
From (4.3.13) we obtain for all sufficiently large values of \( r \) and for \( K > 1 \)

\[
\log^{[m]} \mu(\exp(r^A), g) > K \log^{[m+l-1]} \mu(r, f \circ g)
\]

i.e.,

\[
\log^{[m]} \mu(\exp(r^A), g) > \log \left( \log^{[m+l-2]} \mu(r, f \circ g) \right)^K
\]

i.e.,

\[
\log^{[m]} \mu(\exp(r^A), g) > \log \log^{[m+l-2]} \mu(r, f \circ g)
\]

i.e.,

\[
\log^{[m-1]} \mu(\exp(r^A), g) > \log^{[m+l-2]} \mu(r, f \circ g).
\]

This proves the theorem. \( \blacksquare \)

**Remark 4.3.15** The condition "\( \rho_f^{[l]} \) is finite" in Theorem 4.3.5 is essential as we see in the following example.

**Example 4.3.11** Let \( f = \exp^{[2]} z, g = \exp z, l = 2, m = 2 \) and \( A = 1 \).

Then

\[
\lambda_f^{[l]} = \rho_f^{[l]} = \infty \quad \text{and} \quad \lambda_g^{[m]} = \rho_g^{[m]} = 1.
\]

Now

\[
\log^{[m-1]} \mu \left( \exp \left( r^A \right), g \right) = \log \mu(\exp r, g)
\]

\[
= \log \mu(\exp r, \exp z)
\]

\[
\geq \log M(\exp r/2, \exp z) + O(1)
\]

\[
= \log \left[ \exp \left( \frac{\exp r}{2} \right) \right] + O(1) = \frac{\exp r}{2} + O(1).
\]

and

\[
\log^{[m+l-2]} \mu(r, f \circ g) = \log^{[2]} \mu \left( r, \exp^{[3]} z \right)
\]

\[
\leq \log^{[2]} M(r, \exp^{[3]} z) \log^{[2]} (\exp^{[3]} r) = \exp r.
\]

Therefore

\[
\log^{[m-1]} \mu(\exp(r^A), g) \neq \log^{[m+l-2]} \mu(r, f \circ g),
\]

which contradicts the conclusion of Theorem 4.3.5.

**Remark 4.3.16** The following example verifies the conclusion of Theorem 4.3.5.

**Example 4.3.12** Let \( f = \exp z \) and \( g = \exp z \). Also let \( l = 2, m = 2 \) and \( A = 1 \).
Then
\[ \lambda_f^{[l]} = \rho_f^{[l]} = 1 \text{ and } \lambda_g^{[m]} = \rho_g^{[m]} = 1. \]

Now
\[
\log^{[m-1]} \mu \left( \exp \left( r^A \right), g \right) = \log \mu \left( \exp r, g \right) \\
= \log \mu \left( \exp r, \exp z \right) \\
\geq \log M \left( \frac{\exp r}{2}, \exp z \right) + O(1) \\
= \log \left( \exp \left( \frac{\exp r}{2} \right) \right) + O(1) = \frac{1}{2} \exp r + O(1)
\]
and
\[
\log^{[m+l-2]} \mu \left( r, f \circ g \right) = \log^{[2]} \mu \left( r, \exp^{[2]} z \right) \\
\leq \log^{[2]} M \left( r, \exp^{[2]} z \right) \\
= \log^{[2]} \left( \exp^{[2]} r \right) = r.
\]

Therefore
\[
\log^{[m-1]} \mu \left( \exp(r^A), g \right) > \log^{[m+l-2]} \mu \left( r, f \circ g \right).
\]

**Remark 4.3.17** If we take \(0 < \lambda_g^{[m]} < \infty\) or \(0 < \rho_g^{[m]} < \infty\) instead of \(0 < \lambda_g^{[m]} \leq \rho_g^{[m]} < \infty\) in Theorem 4.3.5 and the other conditions remain the same, then in the line of Theorem 4.3.5 one can easily verify for a sequence of values of \(r\) tending to infinity,
\[
\log^{[m-1]} \mu \left( r, f \circ g \right) < \log^{[m+l-2]} \mu \left( \exp(r^A), g \right).
\]

**Remark 4.3.18** Also if we take \(\lambda_f^{[l]} < \infty\) instead of \(\rho_f^{[l]} < \infty\) in Theorem 4.3.5 and the other conditions remain the same, then in the line of Theorem 4.3.5 one can easily verify for a sequence of values of \(r\) tending to infinity that
\[
\log^{[m-1]} \mu \left( r, f \circ g \right) < \log^{[m+l-2]} \mu \left( \exp(r^A), g \right).
\]

In the line of Theorem 4.3.5, we may state the following theorem without its proof.

**Theorem 4.3.6** Let \(f\) be an entire function of order zero and \(g\) be an entire function such that \(0 < \lambda_f^{[l]} \leq \rho_f^{[l]} < \infty\) and \(\rho_g^{[m]}\) is finite where \(l(\geq 1)\) and \(m(\geq 2)\) are integers. Then for all sufficiently large values of \(r\),
\[
\log^{[m+l-2]} \mu \left( r, f \circ g \right) < \log^{[l-1]} \mu \left( \exp(r^A), f \right),
\]
where \(A > 0\).
Remark 4.3.19 The conclusion of Theorem 4.3.6 can be verified by the following example.

Example 4.3.13 Let $f = z$ and $g = \exp z$. Also let $l = 1, m = 2$ and $A = 1$.

Then

$$\lambda_f^{|l|} = \rho_f^{|l|} = 1 \text{ and } \lambda_g^{|m|} = \rho_g^{|m|} = 1.$$ 

Now

$$\log^{[m+l-2]} \mu(r, f \circ g) = \log \mu(r, \exp z) \leq \log M(r, \exp z) = \log (\exp r) = r$$

and

$$\log^{[l-1]} \mu(\exp(r^A), f) = \mu(\exp r, f) = \mu(\exp r, z) = \exp r.$$

Therefore

$$\log^{[m+l-2]} \mu(r, f \circ g) < \log^{[l-1]} \mu(\exp(r^A), f).$$

Remark 4.3.20 If we consider $\rho_g^{|m|} \leq \infty$ instead of $\rho_g^{|m|} < \infty$ in Theorem 4.3.6, the sign "<" in the conclusion of Theorem 4.3.6 may also be replaced by the "$\leq$" as we see in the subsequent example.

Example 4.3.14 Let $f = z$ and $g = \exp[2] z$. Further, let $l = 1, m = 2$ and $A = 1$.

Then

$$\lambda_f^{|l|} = \rho_f^{|l|} = 1 \text{ and } \lambda_g^{|m|} = \rho_g^{|m|} = \infty.$$  

Now

$$\log^{[m+l-2]} \mu(r, f \circ g) = \log \mu(r, \exp[2] z)$$

$$\leq \log M(r, \exp[2] z) = \log (\exp[2] r) = \exp r$$

and

$$\log^{[l-1]} \mu(\exp(r^A), f) = \mu(\exp r, f) = \mu(\exp r, z) = \exp r.$$

Therefore

$$\log^{[m+l-2]} \mu(r, f \circ g) \leq \log^{[l-1]} \mu(\exp(r^A), f).$$

Remark 4.3.21 If we take $0 < \lambda_f^{|l|} < \infty$ or $0 < \rho_f^{|l|} < \infty$ instead of $0 < \lambda_f^{|l|} \leq \rho_f^{|l|} < \infty$ in Theorem 4.3.6 and the other conditions remain the same, then in the line of Theorem 4.3.6 one can easily verify for a sequence of values of $r$ tending to infinity,

$$\log^{[m+l-2]} \mu(r, f \circ g) < \log^{[l-1]} \mu(\exp(r^A), f).$$
Remark 4.3.22 Further if we take \(\lambda_g^{[m]} < \infty\) instead of \(\rho_g^{[m]} < \infty\) in Theorem 4.3.6 and the other conditions remain the same, then in the line of Theorem 4.3.6 one can easily verify for a sequence of values of \(r\) tending to infinity that
\[
\log^{[m+i-2]} \mu(r, f \circ g) < \log^{[l-1]} \mu(\exp(r^A), f).
\]

The conclusion of Theorem 4.3.3 can also be deduced under some different conditions as we see in the following theorem:

**Theorem 4.3.7** Let \(f\) and \(g\) be any two entire functions such that \(\rho_g^{**} < \infty\) and \(0 < \lambda^{[l]} \leq \rho_f^{[l]} < \infty\) where \(l\) is an integer with \(l \geq 1\). Then for all sufficiently large values of \(r\),
\[
\log^{[l-1]} \mu(r, f \circ g) < \log^{[l-1]} \mu(\exp(r^A), f),
\]
where \(A > 0\).

**Proof.** In view of Lemma 4.2.2 and Lemma 4.2.3 we get for all sufficiently large values of \(r\) that
\[
\log^{[l]} (r, f \circ g) \leq \log^{[l]} \left( \frac{\alpha R}{R-r} \mu(R, g), f \right) + O(1)
\]
i.e.,
\[
\log^{[l]} (r, f \circ g) \leq \left( \rho_f^{[l]} + \varepsilon \right) \log \left( \frac{\alpha R}{R-r} \mu(R, g) \right) + O(1)
\]
i.e.,
\[
\log^{[l]} \mu(r, f \circ g) \leq \left( \rho_f^{[l]} + \varepsilon \right) \log \mu(R, g) + O(1)
\]
i.e.,
\[
\log^{[l]} \mu(r, f \circ g) \leq \left( \rho_f^{[l]} + \varepsilon \right) (\rho_g^{**} + \varepsilon) \log R + O(1).
\]

(4.3.14)

Now from (4.3.14) and (4.3.7) it follows for all sufficiently large values of \(r\) that
\[
\frac{\log^{[l]} \mu(\exp(r^A), f)}{\log^{[l]} \mu(r, f \circ g)} \geq \frac{\left( \lambda_f^{[l]} - \varepsilon \right) (r^A)}{\left( \rho_f^{[l]} + \varepsilon \right) (\rho_g^{**} + \varepsilon) \log r + O(1)}
\]
i.e.,
\[
\lim_{r \to \infty} \frac{\log^{[l]} \mu(\exp(r^A), f)}{\log^{[l]} \mu(r, f \circ g)} = \infty.
\]
(4.3.15)

From (4.3.15) we obtain for all sufficiently large values of \(r\) and \(K > 1\)
\[
\log^{[l]} \mu(\exp(r^A), f) > K \log^{[l]} \mu(r, f \circ g)
\]
i.e.,
\[
\log^{[l]} \mu(\exp(r^A), f) > \log \left\{ \log^{[l-1]} \mu(r, f \circ g) \right\}^K
\]
i.e.,
\[
\log^{[l-1]} \mu(\exp(r^A), f) > \log \log^{[l-1]} \mu(r, f \circ g)
\]
i.e.,
\[
\log^{[l-1]} \mu(\exp(r^A), f) > \log^{[l-1]} \mu(r, f \circ g).
\]

This proves the theorem. ■
Remark 4.3.23 The following examples ensure the conclusion of Theorem 4.3.7.

Example 4.3.15 Let $f = z, g = z, l = 1$ and $A = 1$.

Then

$$\lambda_l^{[f]} = \rho_l^{[f]} = 1 \text{ and } \rho_g^{**} = 1.$$ 

Now

$$\log^{[l-1]} \mu (r, f \circ g) = \mu (r, z) = r$$

and

$$\log^{[l-1]} \mu \left( \exp \left( r^A \right) , f \right) = \mu \left( \exp r, z \right) = \exp r.$$ 

Thus

$$\log^{[l-1]} \mu (r, f \circ g) < \log^{[l-1]} \mu \left( \exp \left( r^A \right) , f \right).$$

Example 4.3.16 Let $f = \exp z, g = z, l = 2$ and $A = 1$.

Then

$$\lambda_l^{[f]} = \rho_l^{[f]} = 1 \text{ and } \rho_g^{**} = 1.$$ 

Now

$$\log^{[l-1]} \mu (r, f \circ g) = \log \mu (r, \exp z) \leq \log M(r, \exp z) = \log (\exp r) = r$$

and

$$\log^{[l-1]} \mu \left( \exp \left( r^A \right) , f \right) = \log \mu \left( \exp r, \exp z \right) \geq \log M \left( \frac{\exp r}{2} , \exp z \right) + O(1) \geq \log \exp \left( \frac{\exp r}{2} \right) + O(1) = \frac{\exp r}{2} + O(1).$$

Therefore

$$\log^{[l-1]} \mu (r, f \circ g) < \log^{[l-1]} \mu \left( \exp \left( r^A \right) , f \right).$$

Remark 4.3.24 If we take $0 < \lambda_l^{[f]} < \infty$ or $0 < \rho_l^{[f]} < \infty$ instead of $0 < \lambda_l^{[f]} \leq \rho_l^{[f]} < \infty$ in Theorem 4.3.7 and the other conditions remain the same, then in the line of Theorem 4.3.7 one can easily verify for a sequence of values of $r$ tending to infinity,

$$\log^{[l-1]} \mu (r, f \circ g) < \log^{[l-1]} \mu \left( \exp \left( r^A \right) , f \right).$$
Remark 4.3.25 Further if we take $\lambda_g^* < \infty$ instead of $\rho_g^* < \infty$ in Theorem 4.3.7 and the other conditions remain the same, then in the line of Theorem 4.3.7 one can easily show for a sequence of values of $r$ tending to infinity that
\[
\log^{[l-1]} \mu(r, f \circ g) < \log^{[l-1]} \mu(\exp(r^A), f).
\]

In the line of Theorem 4.3.7 we may state the following theorem without its proof:

**Theorem 4.3.8** Let $f$ be an entire function such that $\rho_f^{[l]} < \infty$ where $l \geq 1$ is an integer and $g$ be an entire function with $0 < \lambda_g^{**} \leq \rho_g^{**} < \infty$. Then for all sufficiently large values of $r$,
\[
\log^{[l-1]} \mu(r, f \circ g) < \mu(\exp(r^A), g),
\]
where $A > 0$.

**Remark 4.3.26** The following examples verify the conclusion of Theorem 4.3.8.

**Example 4.3.17** Let $f = z, g = z, l = 1$ and $A = 1$.

Then
\[
\lambda_f^{[l]} = \rho_f^{[l]} = 1 \text{ and } \lambda_g^{**} = \rho_g^{**} = 1.
\]

Now
\[
\log^{[l-1]} \mu(r, f \circ g) = \mu(r, z) = r
\]
and
\[
\mu(\exp(r^A), g) = \mu(\exp(r, z)) = \exp r.
\]
Therefore
\[
\log^{[l-1]} \mu(r, f \circ g) < \mu(\exp(r^A), g).
\]

**Example 4.3.18** Let $f = \exp z$ and $g = z$. Also let $l = 2$ and $A = 1$.

Then
\[
\lambda_f^{[l]} = \rho_f^{[l]} = 1 \text{ and } \lambda_g^{**} = \rho_g^{**} = 1.
\]

Now
\[
\log^{[l-1]} \mu(r, f \circ g) = \log \mu(r, \exp z) \leq \log M(r, \exp z) = r
\]
and
\[
\mu(\exp(r^A), g) = \mu(\exp(r), z) = \exp r.
\]
Therefore
\[
\log^{[l-1]} \mu(r, f \circ g) < \mu(\exp(r^A), g).
\]
Remark 4.3.27 If we consider $\rho_f^l \leq \infty$ instead of $\rho_f^l < \infty$ in Theorem 4.3.8, the sign "<" in the conclusion of Theorem 4.3.8 may also be replaced by "\leq" as we see in the following example.

Example 4.3.19 Let $f = \exp^2 z$ and $g = z$. Further, let $l = 2$ and $A = 1$.

Then

$$\lambda_f^l = \rho_f^l = \infty \text{ and } \lambda_g^{**} = \rho_g^{**} = 1.$$ 

Now

$$\log^{[l-1]} \mu (r, f \circ g) = \log \mu (r, \exp^2 z) \leq \log M (r, \exp^2 z) = \exp r$$

and

$$\mu (\exp (r^A), g) = \mu (\exp (r), z) = \exp r.$$

Therefore

$$\log^{[l-1]} \mu (r, f \circ g) \leq \mu (\exp (r^A), g).$$

Remark 4.3.28 If we take $0 < \lambda_g^{**} < \infty$ or $0 < \rho_g^{**} < \infty$ instead of $0 < \lambda_g^{**} \leq \rho_g^{**} < \infty$ in Theorem 4.3.7 and the other conditions remain the same, then in the line of Theorem 4.3.7 one can easily verify for a sequence of values of $r$ tending to infinity,

$$\log^{[l-1]} \mu (r, f \circ g) \leq \mu (\exp (r^A), g).$$

Remark 4.3.29 Also if we take $\lambda_f^l < \infty$ instead of $\rho_f^l < \infty$ in Theorem 4.3.7 and the other conditions remain the same, then in the line of Theorem 4.3.7 one can easily prove for a sequence of values of $r$ tending to infinity that

$$\log^{[l-1]} \mu (r, f \circ g) \leq \mu (\exp (r^A), g).$$

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