CHAPTER 3

GROWTH OF COMPOSITE ENTIRE AND MEROMORPHIC FUNCTIONS IN THE LIGHT OF ZERO ORDER AND WEAK TYPE
3.1 Introduction, Definitions and Notations.

Let $f$ be a meromorphic function and $g$ be an entire function defined on the finite complex plane $\mathbb{C}$. In the sequel we use the following notation:

\[
\log^k x = \log \left( \log^{[k-1]} x \right) \quad \text{for} \quad k = 1, 2, 3, \ldots \quad \text{and} \\
\log^0 x = x.
\]

Also we write

\[
\exp^k y = \exp \left( \exp^{[k-1]} y \right) \quad \text{for} \quad k = 1, 2, 3, \ldots \quad \text{and} \\
\exp^0 y = y.
\]

Though Definition 3.1.1, Definition 3.1.2, Definition 3.1.3, Definition 3.1.4 and Definition 3.1.5 have already been defined in Chapter 2 as Definition 2.1.1, Definition 2.1.2, Definition

The results of this chapter have been published in the Universal Journal of Mathematics and Mathematical Sciences(UJMMS), see [20].
2.1.3, Definition 2.1.4 and Definition 2.1.5 respectively, we state here again in order to keep a continuation of our discussion:

**Definition 3.1.1** The order \( \rho_f \) and lower order \( \lambda_f \) of an entire function \( f \) are defined as

\[
\rho_f = \limsup_{r \to \infty} \frac{\log M(r, f)}{\log r} \quad \text{and} \quad \lambda_f = \liminf_{r \to \infty} \frac{\log M(r, f)}{\log r}.
\]

If \( f \) is meromorphic then

\[
\rho_f = \limsup_{r \to \infty} \frac{\log T(r, f)}{\log r} \quad \text{and} \quad \lambda_f = \liminf_{r \to \infty} \frac{\log T(r, f)}{\log r}.
\]

If \( \rho_f < \infty \) then \( f \) is of finite order. Also \( \rho_f = 0 \) means that \( f \) is of order zero. In this connection Liao and Yang [35] gave the following definition:

**Definition 3.1.2** [35]Let \( f \) be a meromorphic function of order zero. Then the quantities \( \rho_f^* \) and \( \lambda_f^* \) of a meromorphic function \( f \) are defined as :

\[
\rho_f^* = \limsup_{r \to \infty} \frac{\log T(r, f)}{\log^{(2)} r} \quad \text{and} \quad \lambda_f^* = \liminf_{r \to \infty} \frac{\log T(r, f)}{\log^{(2)} r}.
\]

If \( f \) is entire, then

\[
\rho_f^* = \limsup_{r \to \infty} \frac{\log M(r, f)}{\log^{(2)} r} \quad \text{and} \quad \lambda_f^* = \liminf_{r \to \infty} \frac{\log M(r, f)}{\log^{(2)} r}.
\]

Datta and Biswas [12] gave an alternative definition of zero order and zero lower order of a meromorphic function in the following way:

**Definition 3.1.3** [12]Let \( f \) be a meromorphic function of order zero. Then the quantities \( \rho_f^{**} \) and \( \lambda_f^{**} \) of \( f \) are defined by:

\[
\rho_f^{**} = \limsup_{r \to \infty} \frac{T(r, f)}{r \log r} \quad \text{and} \quad \lambda_f^{**} = \liminf_{r \to \infty} \frac{T(r, f)}{r \log r}.
\]

If \( f \) is an entire function then clearly

\[
\rho_f^{**} = \limsup_{r \to \infty} \frac{\log M(r, f)}{r \log^{(2)} r} \quad \text{and} \quad \lambda_f^{**} = \liminf_{r \to \infty} \frac{\log M(r, f)}{r \log^{(2)} r}.
\]

**Definition 3.1.4** The type \( \sigma_f \) and lower type \( \bar{\sigma}_f \) of a meromorphic function \( f \) are defined as

\[
\sigma_f = \limsup_{r \to \infty} \frac{T(r, f)}{r^{\rho_f}} \quad \text{and} \quad \bar{\sigma}_f = \liminf_{r \to \infty} \frac{T(r, f)}{r^{\rho_f}}, \quad 0 < \rho_f < \infty.
\]

When \( f \) is entire, it can be easily verified that

\[
\sigma_f = \limsup_{r \to \infty} \frac{\log M(r, f)}{r^{\rho_f}} \quad \text{and} \quad \bar{\sigma}_f = \liminf_{r \to \infty} \frac{\log M(r, f)}{r^{\rho_f}}, \quad 0 < \rho_f < \infty.
\]
Datta and Jha [11] gave the definition of weak type of a meromorphic function of finite positive lower order in the following way:

**Definition 3.1.5** [11] The weak type $\tau_f$ of a meromorphic function $f$ of finite positive lower order $\lambda_f$ is defined by

$$\tau_f = \liminf_{r \to \infty} \frac{T(r, f)}{r^{\lambda_f}}.$$  

For entire $f$,

$$\tau_f = \liminf_{r \to \infty} \frac{\log M(r, f)}{r^{\lambda_f}}, \quad 0 < \lambda_f < \infty.$$  

Similarly, one can define the growth indicator $\overline{\tau}_f$ of a meromorphic function $f$ of finite positive lower order $\lambda_f$ as

$$\overline{\tau}_f = \limsup_{r \to \infty} \frac{T(r, f)}{r^{\lambda_f}}.$$  

When $f$ is entire, it can be easily verified that

$$\overline{\tau}_f = \limsup_{r \to \infty} \frac{\log M(r, f)}{r^{\lambda_f}}, \quad 0 < \lambda_f < \infty.$$  

For an entire function $f$ and a transcendental entire function $g$, Polya [41] proved that, if $\rho_{fog} < \infty$ then $\rho_f = 0$. Edrei and Fuchs [26] proved that if $f$ is a meromorphic function and $g$ is a transcendental entire function then $\lambda_{fog} < \infty$ implies that $\lambda_f = 0$. Under the same conditions, Gross [28] proved that if $\rho_{fog}$ is finite then $\rho_f = 0$. Now a question may be investigated regarding the order and lower order of $f \circ g$ where $f$ is of finite order and $g$ is of order zero. Datta and Biswas [12] worked on this question and proved that if $\rho_f < \infty$ and $\rho_g = 0$ then $\rho_{fog}$ is finite for meromorphic $f$ and entire $g$.

In this chapter we would like to investigate some growth properties of composite entire and meromorphic functions with both or any one of the left and right factor with order zero.

### 3.2 Lemmas.

In this section we present some lemmas which will be needed in the sequel.

**Lemma 3.2.1** [1] Let $f$ be a meromorphic function and $g$ be an entire function then for all sufficiently large values of $r$,

$$T(r, f \circ g) \leq \{1 + o(1)\} \frac{T(r, g)}{\log M(r, g)} T(M(r, g), f).$$  

**Lemma 3.2.2** [3] Let $f$ be a meromorphic function and $g$ be an entire function and suppose that $0 < \mu < \rho_g \leq \infty$. Then for a sequence of values of $r$ tending to infinity,

$$T(r, f \circ g) \geq T(\exp(r^\mu), f).$$  

**Lemma 3.2.3** [5] If $f$ and $g$ are any two entire functions then for all sufficiently large values of $r$,

$$M(r, f \circ g) \leq M(M(r, g), f).$$
3.3 Theorems.

In this section we present the main results of the chapter.

**Theorem 3.3.1** Let $f$ be a meromorphic function and $g$ be an entire function such that $0 < \lambda_f \leq \rho_f < \infty$ and $\rho_g^{**} < \infty$. Then

$$\lim_{r \to \infty} \frac{\log T(r, f \circ g)}{T(r^A, f^{(k)})} = 0,$$

where $k = 0, 1, 2, \ldots$ and $A$ is any positive real number.

**Proof.** In view of Lemma 3.2.1 and the inequality $T(r, g) \leq \log^+ M(r, g)$ we get for all sufficiently large values of $r$ that

$$T(r, f \circ g) \leq \{1 + o(1)\} T(M(r, g), f)$$

i.e., $\log T(r, f \circ g) \leq (\rho_f + \varepsilon) \log M(r, g) + O(1)$

i.e., $\log T(r, f \circ g) \leq (\rho_f + \varepsilon) (\rho_g^{**} + \varepsilon) \log r + O(1)$.

(3.3.1)

Again from the definition of $\lambda_f$ we have for arbitrary positive $\varepsilon$ and for all sufficiently large values of $r$,

$$\log T(r^A, f^{(k)}) \geq A(\lambda_f - \varepsilon) \log r$$

i.e., $T(r^A, f^{(k)}) \geq r^{A(\lambda_f - \varepsilon)}$.

(3.3.2)

Therefore it follows from (3.3.1) and (3.3.2) that for all sufficiently large values of $r$,

$$\frac{\log T(r, f \circ g)}{T(r^A, f^{(k)})} \leq \frac{(\rho_f + \varepsilon) (\rho_g^{**} + \varepsilon) \log r + O(1)}{r^{A(\lambda_f - \varepsilon)}}.$$  

(3.3.3)

As $\lambda_f > 0$, the theorem follows from (3.3.3). \qed

**Remark 3.3.1** The following example ensures the conclusion of Theorem 3.3.1.

**Example 3.3.1** Let $f = \frac{1}{z}$ and $g = z$. Further, let $A = 1$ and $k = 0$.

Then

$$\lambda_f = \rho_f = 0 \text{ and } \rho_g^{**} = 1.$$  

Now

$$\log T(r, f \circ g) = \log T(r, 1/z) = \log T(r, z) + O(1) = \log^2 r + O(1)$$

and

$$T(r^A, f^{(k)}) = T(r, 1/z) = T(r, z) + O(1) = \log r + O(1).$$

Therefore

$$\lim_{r \to \infty} \frac{\log T(r, f \circ g)}{T(r^A, f^{(k)})} = 0.$$
Remark 3.3.2 If we take $0 < ho_f < \infty$ or $0 < \lambda_f < \infty$ instead of $0 < \lambda_f \leq \rho_f < \infty$ in Theorem 3.3.1 and the other conditions remain the same, then in the line of Theorem 3.3.1 one can easily verify that
\[
\liminf_{r \to \infty} \frac{\log T(r, f \circ g)}{T(r^A, f^{(k)})} = 0,
\]
where $k = 0, 1, 2, \ldots$ and $A$ is any positive real number.

Remark 3.3.3 Also if we take $\lambda_g^{**} < \infty$ instead of $\rho_g^{**} < \infty$ in Theorem 3.3.1 and the other conditions remain the same, then in the line of Theorem 3.3.1 one can easily verify that
\[
\liminf_{r \to \infty} \frac{\log T(r, f \circ g)}{T(r^A, f^{(k)})} = 0,
\]
where $k = 0, 1, 2, \ldots$ and $A$ is any positive real number.

Theorem 3.3.2 Let $f$ be a meromorphic function such that $0 < \rho_f^{**} < \infty$ and $g$ be any entire function. Then
\[
\limsup_{r \to \infty} \frac{T(r, f \circ g)}{T(2r, g)} \leq 3 \{1 + o(1)\} \rho_f^{**}.
\]

Proof. As $T(r, g) \leq \log^+ M(r, g)$, it follows from Lemma 3.2.1 that for all sufficiently large values of $r$,
\[
T(r, f \circ g) \leq \{1 + o(1)\} T(M(r, g), f)
\]
i.e., $T(r, f \circ g) \leq \{1 + o(1)\} (\rho_f^{**} + \varepsilon) \log M(r, g) + O(1).
\]
(3.3.4)

Since $\log M(r, g) \leq 3T(2r, g)$, for all sufficiently large values of $r$ we get from (3.3.1) that
\[
T(r, f \circ g) \leq \{1 + o(1)\} (\rho_f^{**} + \varepsilon) \log M(r, g) + O(1)
\]
i.e., $T(r, f \circ g) \leq \{1 + o(1)\} (\rho_f^{**} + \varepsilon) 3T(2r, g) + O(1)$
i.e., $T(r, f \circ g) \leq \{1 + o(1)\} (\rho_f^{**} + \varepsilon) 3T(2r, g) + O(1)$
i.e., $\limsup_{r \to \infty} \frac{T(r, f \circ g)}{T(2r, g)} \leq 3 \{1 + o(1)\} \rho_f^{**}.$

This proves the theorem. 

Remark 3.3.4 The conclusion of Theorem 3.3.2 can be verified by the following examples.

Example 3.3.2 Let $f = \frac{1}{z}$ and $g = z$. Then $\rho_f^{**} = 1$.

Now
\[
T(r, f \circ g) = T(r, 1/z) = T(r, z) + O(1) = \log r + O(1)
\]
and
\[ T(2r, g) = T(2r, z) = \log(r) + O(1). \]

Therefore
\[
\limsup_{r \to \infty} \frac{T(r, f \circ g)}{T(2r, g)} = 1
\]
and
\[
3 \{1 + o(1)\} \rho_f^* = 3 \{1 + o(1)\}.
\]

So
\[
\limsup_{r \to \infty} \frac{T(r, f \circ g)}{T(2r, g)} < 3 \{1 + o(1)\} \rho_f^*.
\]

**Example 3.3.3** Let \( f = \frac{1}{z} \) and \( g = \exp z \). Then \( \rho_f^* = 1 \).

Now
\[
T(r, f \circ g) = T(r, 1/ \exp z) = T(r, \exp z) + O(1) = \frac{r}{\pi} + O(1)
\]
and
\[
T(2r, g) = T(2r, \exp z) = \frac{2r}{\pi}.
\]

Therefore
\[
\limsup_{r \to \infty} \frac{T(r, f \circ g)}{T(2r, g)} = 1/2
\]
and
\[
3 \{1 + o(1)\} \rho_f^* = 3 \{1 + o(1)\}.
\]

So
\[
\limsup_{r \to \infty} \frac{T(r, f \circ g)}{T(2r, g)} < 3 \{1 + o(1)\} \rho_f^*.
\]

**Example 3.3.4** Let \( f = \frac{1}{z} \) and \( g = \exp[z^2] \). Then \( \rho_f^* = 1 \).

Now
\[
T(r, f \circ g) = T(r, 1/ \exp[z^2])
\]
\[
= T(r, \exp[z^2] z) + O(1)
\]
\[
\sim \frac{\exp r}{(2\pi^2 r^3)^{1/2}} (r \to \infty)
\]
and

\[ T(2r, g) = T(2r, \exp^{[2]} z) \]
\[ \sim \frac{\exp(2r)}{(4\pi^3 r)^\frac{1}{2}} \quad (r \to \infty) \]

Therefore

\[ \limsup_{r \to \infty} \frac{T(r, f \circ g)}{T(2r, g)} = 0 \]

and

\[ 3 \{1 + o(1)\} \rho_f^{**} = 3 \{1 + o(1)\}. \]

Hence

\[ \limsup_{r \to \infty} \frac{T(r, f \circ g)}{T(2r, g)} < 3 \{1 + o(1)\} \rho_f^{**}. \]

**Theorem 3.3.3** Let \( f \) be a meromorphic function and \( g \) be an entire function such that \( 0 < \lambda_f \leq \rho_f < \infty \) and \( \rho_g^{**} \) is finite. Then for all sufficiently large values of \( r \)

\[ T(r, f \circ g) < T(\exp(r^\mu), f^{(k)}), \]

where \( k = 0, 1, 2, \ldots \) and \( \mu > 0 \).

**Proof.** In view of Lemma 3.2.1 and the inequality \( T(r, g) \leq \log^+ M(r, g) \), we get for all sufficiently large values of \( r \) that

\[ T(r, f \circ g) \leq \{1 + o(1)\} T(M(r, g), f) \]

\[ \text{i.e.,} \quad \log T(r, f \circ g) \leq (\rho_f + \varepsilon) \log M(r, g) + O(1) \]

\[ \text{i.e.,} \quad \log T(r, f \circ g) \leq (\rho_f + \varepsilon) (\rho_g^{**} + \varepsilon) \log r + O(1). \tag{3.3.5} \]

Again from the definition of \( \lambda_f \) and for any \( \mu > 0 \) we obtain for all sufficiently large values of \( r \) that

\[ \log T(\exp(r^\mu), f^{(k)}) \geq (\lambda_f - \varepsilon) \log \exp(r^\mu) \]

\[ \text{i.e.,} \quad \log T(\exp(r^\mu), f^{(k)}) \geq (\lambda_f - \varepsilon) r^\mu. \tag{3.3.6} \]

Now from (3.3.5) and (3.3.6) it follows that for all sufficiently large values of \( r \),

\[ \frac{\log T(\exp(r^\mu), f^{(k)})}{\log T(r, f \circ g)} \geq \frac{(\lambda_f - \varepsilon) r^\mu}{(\rho_f + \varepsilon) (\rho_g^{**} + \varepsilon) \log r + O(1)} \]

\[ \text{i.e.,} \quad \lim_{r \to \infty} \frac{\log T(\exp(r^\mu), f^{(k)})}{\log T(r, f \circ g)} = \infty. \tag{3.3.7} \]
From (3.3.7) we obtain that for all sufficiently large values of $r$ and $K > 1$,

$$\log T(\exp(r^\mu), f^{(k)}) > K \log T(r, f \circ g)$$

i.e.,

$$\log T(\exp(r^\mu), f^{(k)}) > \log \{T(r, f \circ g)\}^K$$

i.e.,

$$\log T(\exp(r^\mu), f^{(k)}) > \log T(r, f \circ g)$$

i.e.,

$$T(\exp(r^\mu), f^{(k)}) > T(r, f \circ g).$$

This proves the theorem. ■

**Remark 3.3.5** The following example verifies the conclusion of Theorem 3.3.3.

**Example 3.3.5** Let $f = \frac{1}{z}, g = z, \mu = 1$ and $k = 0$.

Then

$$\rho_f^{**} = 1 \text{ and } \lambda_f = \rho_f = 0.$$  

Now

$$T(r, f \circ g) = T(r, 1/z) = \log r + O(1)$$

and

$$T(\exp(r^\mu), f^{(k)}) = T(\exp r, 1/z) = \log (\exp r) + O(1) = r + O(1).$$

Therefore

$$T(r, f \circ g) < T(\exp(r^\mu), f^{(k)}).$$

**Theorem 3.3.4** Let $f$ be an entire function of finite order and $g$ be an entire function such that $0 < \lambda_g^{**} \leq \rho_g^{**} < \infty$. Then for all sufficiently large values of $r$,

$$T(r, f \circ g) < M(\exp(r^\mu), g),$$

where $\mu > 0$.

**Proof.** From the definition of $\lambda_g^{**}$ we have for arbitrary positive $\varepsilon$ and for all sufficiently large values of $r$,

$$\log M(\exp(r^\mu), g) \geq (\lambda_g^{**} - \varepsilon) \log \exp(r^\mu)$$

i.e.,

$$\log M(\exp(r^\mu), g) \geq (\lambda_g^{**} - \varepsilon) r^\mu. \quad (3.3.8)$$

Again in view of Lemma 3.2.3 it follows that for all sufficiently large values of $r$,

$$\log^2 M(r, f \circ g) \leq (\rho_f + \varepsilon) \log M(r, g)$$

i.e.,

$$\log^2 M(r, f \circ g) \leq (\rho_f + \varepsilon) (\rho_g^{**} + \varepsilon) \log r. \quad (3.3.9)$$
Now from (3.3.8) and (3.3.9) it follows that for all sufficiently large values of $r$,

$$\frac{\log M(\exp(r^\mu), g)}{\log^2 M(r, f \circ g)} \geq \frac{\left(\lambda_y^{**} - \varepsilon\right) r^\mu}{\left(\rho_f + \varepsilon\right) \left(\rho_f^{**} + \varepsilon\right) \log r}$$

i.e., \(\lim_{r \to \infty} \frac{\log M(\exp(r^\mu), g)}{\log^2 M(r, f \circ g)} = \infty.\) \hfill (3.3.10)

From (3.3.10) we obtain that for all sufficiently large values of $r$ and $K > 1$

$$\log M(\exp(r^\mu), g) > K \log^2 M(r, f \circ g)$$

i.e., \(\log M(\exp(r^\mu), g) > \log \{\log M(r, f \circ g)\}^K\)

i.e., \(\log M(\exp(r^\mu), g) > \log \log M(r, f \circ g)\)

i.e., \(M(\exp(r^\mu), g) > \log M(r, f \circ g).\) \hfill (3.3.11)

Since \(T(r, g) \leq \log^+ M(r, g),\) the theorem follows from (3.3.11).

**Remark 3.3.6** The following examples ensure the conclusion of Theorem 3.3.4.

**Example 3.3.6** Let \(f = \exp^2 z, g = z\) and \(\mu = 1.\)

Then

\(\lambda_f = \rho_f = \infty\) and \(\lambda_y^{**} = \rho_f^{**} = 1.\)

Now

\[ T(r, f \circ g) = T(r, \exp^2 z) \]

\[ \sim \frac{\exp r}{\left(2\pi^3 r^2\right)^{\frac{1}{2}}} (r \to \infty) \]

and

\(M(\exp(r^\mu), g) = M(\exp r, z) = \exp r.\)

Therefore

\(T(r, f \circ g) < M(\exp(r^\mu), g),\)

**Example 3.3.7** Let \(f = g = z\) and \(\mu = 1.\)

Then

\(\lambda_f = \rho_f = 0\) and \(\lambda_y^{**} = \rho_f^{**} = 1.\)

Now
\[
T(r, f \circ g) = T(r, z) = \log r
\]
and
\[
M(\exp(r^{\mu}), g) = M(\exp r, z) = \exp r.
\]

Therefore
\[
T(r, f \circ g) < M(\exp(r^{\mu}), g).
\]

As an application of Theorem 3.3.3, we may prove the following four theorems:

**Theorem 3.3.5** Let \( f \) be a meromorphic function and \( g \) be an entire function of order zero such that \( \rho_g^* < \infty \), \( \lambda_f > 0 \) and \( \sigma_f > 0 \). Then
\[
\limsup_{r \to \infty} \frac{\log T(r, f \circ g)}{T(r, f)} \leq \frac{\rho_f}{\sigma_f}.
\]

**Proof.** Since \( \rho_f > 0 \), taking \( \mu = \rho_f \) in Theorem 3.3.3 we obtain for all sufficiently large values of \( r \) that
\[
\log T(r, f \circ g) < \log \{\exp r^{\rho_f}, f\}
\]
i.e., \( \log T(r, f \circ g) < (\rho_f + \varepsilon) \log \exp r^{\rho_f} \)
i.e., \( \log T(r, f \circ g) < (\rho_f + \varepsilon) r^{\rho_f} \). (3.3.12)

Again we have for arbitrary positive \( \varepsilon \) and for all sufficiently large values of \( r \),
\[
T(r, f) \geq (\sigma_f - \varepsilon) r^{\rho_f}.
\] (3.3.13)

Therefore from (3.3.12) and (3.3.13) it follows that for all sufficiently large values of \( r \),
\[
\frac{\log T(r, f \circ g)}{T(r, f)} \leq \frac{(\rho_f + \varepsilon) r^{\rho_f}}{(\sigma_f - \varepsilon) r^{\rho_f}}
\]
i.e., \( \limsup_{r \to \infty} \frac{\log T(r, f \circ g)}{T(r, f)} \leq \frac{\rho_f}{\sigma_f} \).

Thus the theorem is established. \( \blacksquare \)

**Remark 3.3.7** The conclusion of Theorem 3.3.5 can be verified by the following example.

**Example 3.3.8** Let \( f = \frac{1}{\sin z} \) and \( g = z \).
Then

$$\rho_g^{**} = 1, \lambda_f = \rho_f = 1 \text{ and } \bar{\sigma}_f = 1.$$  

Now

$$\log T(r, f \circ g) = \log T(r, 1/\sin z)$$

$$= \log T(r, \sin z) + O(1)$$

$$= \log r + \log \left\{ 1 + r^{-1} \left( \log \left( \frac{1 + \exp(-2r)}{2} \right) \right) \right\} + O(1)$$

and

$$T(r, f) = r + \log \left\{ \frac{1 + \exp(-2r)}{2} \right\}.$$  

Therefore

$$\limsup_{r \to \infty} \frac{\log T(r, f \circ g)}{T(r, f)} = 0 \text{ and } \frac{\rho_f}{\sigma_f} = 1.$$  

Hence

$$\limsup_{r \to \infty} \frac{\log T(r, f \circ g)}{T(r, f)} \leq \frac{\rho_f}{\sigma_f}.$$  

**Theorem 3.3.6** Let $f$ be a meromorphic function and $g$ be an entire function of order zero such that $\rho_g^{**} < \infty$, $\lambda_f > 0$ and $\bar{\sigma}_f > 0$. Then

$$\liminf_{r \to \infty} \frac{\log T(r, f \circ g)}{T(r, f)} \leq \min \left\{ \frac{\rho_f}{\sigma_f}, \frac{\lambda_f}{\bar{\sigma}_f} \right\}.$$  

Theorem 3.3.6 can easily be proved in the line of Theorem 3.3.5. Hence its proof is omitted.

**Remark 3.3.8** The conclusion of Theorem 3.3.6 can be checked by means of the following example.

**Example 3.3.9** Let $f = \frac{1}{\sin z}$ and $g = z$.

Then

$$\rho_f^{**} = 1, \lambda_f = \rho_f = 1 \text{ and } \sigma_f = \bar{\sigma}_f = 1.$$  

Now
\[
\log T(r, f \circ g) = \log T(r, 1/\sin z)
\]
\[
= \log T(r, \sin z) + O(1)
\]
\[
= \log r + \log \left[ 1 + r^{-1} \left( \log \frac{1 + \exp(-2r)}{2} \right) \right] + O(1)
\]

and

\[
T(r, f) = r + \log \left( \frac{1 + \exp(-2r)}{2} \right).
\]

Therefore

\[
\frac{\log T(r, f \circ g)}{T(r, f)} = 0, \quad \frac{\rho_f}{\sigma_f} = \frac{\lambda_f}{\bar{\sigma}_f} = 1
\]

and so

\[
\min \left\{ \frac{\rho_f}{\sigma_f}, \frac{\lambda_f}{\bar{\sigma}_f} \right\} = 1.
\]

Thus

\[
\liminf_{r \to \infty} \frac{\log T(r, f \circ g)}{T(r, f)} \leq \min \left\{ \frac{\rho_f}{\sigma_f}, \frac{\lambda_f}{\bar{\sigma}_f} \right\}.
\]

In view of Theorem 3.3.4, the following two corollaries may also be proved in the line of Theorem 3.3.5 and Theorem 3.3.6 respectively:

**Corollary 3.3.1** Let \( f \) and \( g \) be any two entire functions such that \( 0 < \lambda_g^{**} \leq \rho_g^{**} < \infty \) and \( \bar{\sigma}_f > 0 \). Then

\[
\limsup_{r \to \infty} \frac{\log T(r, f \circ g)}{T(r, f)} \leq \frac{\rho_g^{**}}{\sigma_f}.
\]

**Corollary 3.3.2** Let \( f \) and \( g \) be any two entire functions such that \( 0 < \lambda_g^{**} \leq \rho_g^{**} < \infty \) and \( \bar{\sigma}_f > 0 \). Then

\[
\liminf_{r \to \infty} \frac{\log T(r, f \circ g)}{T(r, f)} \leq \min \left\{ \frac{\rho_g^{**}}{\sigma_f}, \frac{\lambda_g^{**}}{\bar{\sigma}_f} \right\}.
\]

Using the notion of weak type, we may state the following two theorems without their proofs:

**Theorem 3.3.7** Let \( f \) be a meromorphic function and \( g \) be an entire function of order zero such that \( \rho_g^{**} < \infty \), \( \rho_f < \infty \) and \( \tau_f > 0 \). Then

\[
\limsup_{r \to \infty} \frac{\log T(r, f \circ g)}{T(r, f)} \leq \frac{\rho_f}{\tau_f}.
\]
Remark 3.3.9 The following example ensures the conclusion of Theorem 3.3.7.

Example 3.3.10 Let \( f = \frac{1}{\sin z} \) and \( g = z \).

Then

\[
\rho_f^* = 1, \lambda_f = \rho_f = 1 \text{ and } \tau_f = 1.
\]

Now

\[
\log T(r, f \circ g) = \log T(r, 1/ \sin z) = \log T(r, \sin z) + O(1)
= \log r + \log \left[ 1 + r^{-1} \left( \log \frac{1 + \exp(-2r)}{2} \right) \right] + O(1)
\]

and

\[
T(r, f) = r + \log \left( \frac{1 + \exp(-2r)}{2} \right).
\]

So

\[
\limsup_{r \to \infty} \frac{\log T(r, f \circ g)}{T(r, f)} = 0 \text{ and } \frac{\rho_f}{\tau_f} = 1.
\]

Therefore

\[
\limsup_{r \to \infty} \frac{\log T(r, f \circ g)}{T(r, f)} \leq \frac{\rho_f}{\tau_f}.
\]

Theorem 3.3.8 Let \( f \) be a meromorphic function and \( g \) be an entire function of order zero such that \( 0 < \lambda_g^* \leq \rho_g^* < \infty, \rho_f < \infty \) and \( \tau_f > 0 \). Then

\[
\liminf_{r \to \infty} \frac{\log T(r, f \circ g)}{T(r, f)} \leq \min \left\{ \frac{\rho_f}{\tau_f}, \frac{\lambda_f}{\tau_f} \right\}.
\]

Remark 3.3.10 The conclusion of Theorem 3.3.8 can be verified by the following example.

Example 3.3.11 Let \( f = \frac{1}{\sin z} \) and \( g = z \).

Then

\[
\rho_f^* = 1, \lambda_f = \rho_f = 1 \text{ and } \tau_f = \bar{\tau}_f = 1.
\]

Now
\[ \log T(r, f \circ g) = \log T(r, 1/\sin z) = \log T(r, \sin z) + O(1) \]
\[ = \log r + \log \left[ 1 + r^{-1} \left\{ \log \left( \frac{1 + \exp(-2r)}{2} \right) \right\} \right] + O(1) \]

and

\[ T(r, f) = r + \log \left\{ \frac{1 + \exp(-2r)}{2} \right\}. \]

Therefore

\[ \liminf_{r \to \infty} \log T(r, f \circ g) T(r, f) = 0, \quad \rho_f = 1 \quad \text{and} \quad \lambda_f = 1 \quad \text{and so} \]

\[ \min\left\{ \frac{\rho_f}{\sigma_f}, \frac{\lambda_f}{\tau_f} \right\} = 1. \]

Thus

\[ \liminf_{r \to \infty} \frac{\log T(r, f \circ g)}{T(r, f)} \leq \min\left\{ \frac{\rho_f}{\sigma_f}, \frac{\lambda_f}{\tau_f} \right\}. \]

**Corollary 3.3.3** Let \( f \) and \( g \) be any two entire functions with \( 0 < \lambda_g^* \leq \rho_g^* < \infty \) and \( \tau_f > 0 \). Then

\[ \limsup_{r \to \infty} \frac{\log T(r, f \circ g)}{T(r, f)} \leq \frac{\rho_g^*}{\tau_f}. \]

**Corollary 3.3.4** Let \( f \) and \( g \) be any two entire functions with \( 0 < \lambda_g^* \leq \rho_g^* < \infty \) and \( \tau_f > 0 \). Then

\[ \liminf_{r \to \infty} \frac{\log T(r, f \circ g)}{T(r, f)} \leq \min\left\{ \frac{\rho_g^*}{\tau_f}, \frac{\lambda_g^*}{\tau_f} \right\}. \]

Corollary 3.3.3 and Corollary 3.3.4 can easily be proved with the help of Theorem 3.3.4 and in the line of Theorem 3.3.7 and Theorem 3.3.8 respectively. Hence its proof is omitted.

**Theorem 3.3.9** Let \( f \) be a meromorphic function and \( g \) be an entire function such that \( 0 < \lambda_f \leq \rho_f < \infty \) and \( \rho_g^* < \infty \). Then

\[ \limsup_{r \to \infty} \frac{\log T(r, f \circ g)}{\log T(r^A, f^{(k)})} \leq \frac{\rho_f \rho_g^*}{A \lambda_f}, \]

where \( k = 0, 1, 2, \ldots \) and \( A \) is any positive real number.
Proof. From the definition of $\lambda_f$ we have for arbitrary positive $\varepsilon$ and for all sufficiently large values of $r$,

$$\log T \left( r^A, f^{(k)} \right) \geq A (\lambda_f - \varepsilon) \log r.$$

(3.3.14)

Now combining (3.3.1) and (3.3.14) we get for all sufficiently large values of $r$ that

$$\frac{\log T (r, f \circ g)}{\log T (r^A, f^{(k)})} \leq \frac{(\rho_f + \varepsilon) (\rho_g^{**} + \varepsilon) \log r + O (1)}{A (\lambda_f - \varepsilon) \log r}.$$

i.e.,

$$\limsup_{r \to \infty} \frac{\log T (r, f \circ g)}{\log T (r^A, f^{(k)})} \leq \frac{\rho_f \rho_g^{**}}{A \lambda_f}. $$

This completes the proof of the theorem. ■

Remark 3.3.11 The following example ensures the conclusion of Theorem 3.3.9.

Example 3.3.12 Let $f = \frac{1}{\sin z}$ and $g = z$. Further, let $A = 1, k = 0$.

Then

$$\rho_g^{**} = 1$$

and $\lambda_f = \rho_f = 1$.

Now

$$\log T (r, f \circ g) = \log T (r, 1/\sin z)$$

$$= \log T (r, \sin z) + O(1)$$

$$= \log r + \log \left[ 1 + r^{-1} \left\{ \log \left( 1 + \exp(-2r) \right) \right\} \right] + O(1)$$

and

$$\log T (r^A, f^{(k)}) = \log T (r, 1/\sin z)$$

$$= \log T (r, \sin z) + O(1)$$

$$= \log r + \log(1 + r^{-1} (\log \frac{1 + \exp(-2r)}{2})) + O(1).$$

Therefore

$$\limsup_{r \to \infty} \frac{\log T (r, f \circ g)}{\log T (r^A, f^{(k)})} = 1$$

and

$$\frac{\rho_f \rho_g^{**}}{A \lambda_f} = 1.$$

Hence

$$\limsup_{r \to \infty} \frac{\log T (r, f \circ g)}{\log T (r^A, f^{(k)})} = \frac{\rho_f \rho_g^{**}}{A \lambda_f}.$$

Similarly, we may state the following theorem without its proof for the right factor $g$ of the composite function $f \circ g$:
**Theorem 3.3.10** Let \( f \) be meromorphic and \( g \) be entire such that \( \rho_f < \infty \) and \( 0 < \lambda_g^{**} \leq \rho_g^{**} < \infty \). Then
\[
\limsup_{r \to \infty} \frac{\log T(r, f \circ g)}{\log T(r^A, f^{(k)})} \leq \frac{\rho_f \rho_g^{**}}{A \lambda_g^{**}}
\]
where \( A \) is any positive real number.

**Remark 3.3.12** The following example verifies the conclusion of Theorem 3.3.10.

**Example 3.3.13** Let \( f = \frac{1}{\sin z} \), \( g = z \) and \( A = 1 \).

Then
\[
\rho_f = 1 \quad \text{and} \quad \lambda_g^{**} = \rho_g^{**} = 1.
\]

Now
\[
\log T(r, f \circ g) = \log T(r, 1/\sin z)
\]
\[
= \log T(r, \sin z) + O(1)
\]
\[
= \log r + \log \left[ 1 + r^{-1} \left\{ \log \left( \frac{1 + \exp(-2r)}{2} \right) \right\} \right] + O(1)
\]
and
\[
T(r^A, g) = r.
\]

Therefore
\[
\limsup_{r \to \infty} \frac{\log T(r, f \circ g)}{T(r^A, g)} = 0, \quad \text{and} \quad \frac{\rho_f \rho_g^{**}}{A \lambda_g^{**}} = 1.
\]

Hence
\[
\limsup_{r \to \infty} \frac{\log T(r, f \circ g)}{T(r^A, g)} \leq \frac{\rho_f \rho_g^{**}}{A \lambda_g^{**}}.
\]

**Remark 3.3.13** Under the same conditions of Theorem 3.3.9 if \( f \) is of regular growth then
\[
\limsup_{r \to \infty} \frac{\log T(r, f \circ g)}{\log T(r^A, f^{(k)})} \leq \frac{\rho_g^{**}}{A}.
\]

**Remark 3.3.14** In Theorem 3.3.9 if we take \( \lambda_g^{**} < \infty \) instead of \( \rho_g^{**} < \infty \) and the other conditions remain the same then it can be shown that
\[
\liminf_{r \to \infty} \frac{\log T(r, f \circ g)}{\log T(r^A, f^{(k)})} \leq \frac{\rho_f \lambda_g^{**}}{A \lambda_f}.
\]

In addition, if \( f \) is of regular growth then
\[
\liminf_{r \to \infty} \frac{\log T(r, f \circ g)}{\log T(r^A, f^{(k)})} \leq \frac{\lambda_g^{**}}{A}.
\]
Remark 3.3.15 If we take $0 < \rho_f < \infty$ or $0 < \lambda_f < \infty$ instead of $0 < \lambda_f \leq \rho_f < \infty$ in Theorem 3.3.9 and the other conditions remain the same, then one can easily verify that

$$\liminf_{r \to \infty} \frac{\log T(r, f \circ g)}{\log T(r^A, f^{(k)})} \leq \frac{\rho^{**}_g}{A}.$$ 

Remark 3.3.16 Under the same conditions of Theorem 3.3.10 if $g$ be such that $\lambda^{**}_g = \rho^{**}_g$, then

$$\limsup_{r \to \infty} \frac{\log T(r, f \circ g)}{T(r^A, g)} \leq \frac{\rho_f}{A}.$$ 

Remark 3.3.17 In Theorem 3.3.10 if we take $\lambda_f < \infty$ instead of $\rho_f < \infty$ and the other conditions remain the same then it can be shown that

$$\liminf_{r \to \infty} \frac{\log T(r, f \circ g)}{T(r^A, g)} \leq \frac{\lambda_f \rho^{**}_g}{A \lambda^{**}_g}.$$ 

In addition, if $g$ be such that $\lambda^{**}_g = \rho^{**}_g$, then

$$\liminf_{r \to \infty} \frac{\log T(r, f \circ g)}{T(r^A, g)} \leq \frac{\lambda_f}{A}.$$ 

Remark 3.3.18 If we take $0 < \lambda^{**}_g < \infty$ or $0 < \rho^{**}_g < \infty$ instead of $0 < \lambda^{**}_g \leq \rho^{**}_g < \infty$ in Theorem 3.3.10 and the other conditions remain the same, then one can easily verify that

$$\liminf_{r \to \infty} \frac{\log T(r, f \circ g)}{T(r^A, g)} \leq \frac{\rho^{**}_g}{A}.$$ 

In the line of Theorem 3.3.9 the following corollary may be deduced:

Corollary 3.3.5 Let $f$ be a meromorphic function and $g$ be an entire function such that $0 < \lambda^{**}_f \leq \rho^{**}_f < \infty$ and $\rho^{**}_g < \infty$. Then

$$\limsup_{r \to \infty} \frac{T(r, f \circ g)}{T(r^A, f)} \leq \frac{(1 + o(1)) \rho^{**}_f \rho^{**}_g}{A \lambda^{**}_f},$$

where $A$ is any positive real number.

The proof is omitted.

Corollary 3.3.6 Let $f$ be a meromorphic function and $g$ be an entire function such that $0 < \lambda^{**}_g \leq \rho^{**}_g < \infty$ and $\rho^{**}_f < \infty$. Then

$$\limsup_{r \to \infty} \frac{T(r, f \circ g)}{T(r^A, g)} \leq \frac{(1 + o(1)) \rho^{**}_f \rho^{**}_g}{A \lambda^{**}_g},$$

where $A$ is any positive real number.

The proof is omitted.
Remark 3.3.19 Under the same conditions of Corollary 3.3.5 if $f$ be such that $\rho_f^{**} = \lambda_f^{**}$, then
\[
\limsup_{r \to \infty} \frac{T(r, f \circ g)}{T(r^A, f)} \leq \frac{1 + o(1)}{A} \rho_f^{**}.
\]

Remark 3.3.20 In Corollary 3.3.5 if we take $\lambda_g^{**} > 0$ instead of $\rho_g^{**} > 0$ and the other conditions remain the same then it can be shown that
\[
\liminf_{r \to \infty} \frac{T(r, f \circ g)}{T(r^A, f)} \leq \frac{1 + o(1)}{A} \lambda_f^{**} \lambda_g^{**}.
\]

In addition, if $f$ be such that $\rho_f^{**} = \lambda_f^{**}$, then
\[
\liminf_{r \to \infty} \frac{T(r, f \circ g)}{T(r^A, f)} \leq \frac{1 + o(1)}{A} \lambda_f^{**}.
\]

Remark 3.3.21 If we take $0 < \lambda_f^{**} < \infty$ or $0 < \rho_f^{**} < \infty$ instead of $0 < \lambda_f^{**} \leq \rho_f^{**} < \infty$ in Corollary 3.3.5 and the other conditions remain the same, then one can easily verify that
\[
\liminf_{r \to \infty} \frac{T(r, f \circ g)}{T(r^A, f)} \leq \frac{1 + o(1)}{A} \rho_f^{**}.
\]

Remark 3.3.22 Under the same conditions of Corollary 3.3.6 if $g$ be such that $\lambda_g^{**} = \rho_g^{**}$, then
\[
\limsup_{r \to \infty} \frac{\log T(r, f \circ g)}{T(r^A, g)} \leq \frac{1 + o(1)}{A} \rho_f^{**}.
\]

Remark 3.3.23 In Corollary 3.3.6 if we take $\lambda_f^{**} < \infty$ instead of $\rho_f^{**} < \infty$ and the other conditions remain the same then it can be shown that
\[
\liminf_{r \to \infty} \frac{\log T(r, f \circ g)}{T(r^A, g)} \leq \frac{1 + o(1)}{A} \lambda_f^{**} \rho_g^{**}.
\]

In addition, if $g$ be such that $\lambda_g^{**} = \rho_g^{**}$, then
\[
\liminf_{r \to \infty} \frac{\log T(r, f \circ g)}{T(r^A, g)} \leq \frac{1 + o(1)}{A} \lambda_f^{**}.
\]

Remark 3.3.24 If we take $0 < \lambda_g^{**} < \infty$ or $0 < \rho_g^{**} < \infty$ instead of $0 < \lambda_g^{**} \leq \rho_g^{**} < \infty$ in Corollary 3.3.6 and the other conditions remain the same, then one can easily verify that
\[
\liminf_{r \to \infty} \frac{\log T(r, f \circ g)}{T(r^A, g)} \leq \frac{1 + o(1)}{A} \rho_g^{**}.
\]