CHAPTER 10

ON THE BOUNDS FOR THE ZEROS OF ENTIRE FUNCTIONS IN THE LIGHT OF SLOWLY CHANGING FUNCTIONS
CHAPTER

10

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10.1 Introduction, Definitions and Notations.

Let

\[ P(z) = a_0 + a_1 z + a_2 z^2 + a_3 z^3 + \ldots + a_{n-1} z^{n-1} + a_n z^n; |a_n| \neq 0 \]

be a polynomial of degree \( n \). Datt and Govil[8]; Govil and Rahaman[27]; Marden[38]; Mohammad[39]; Chattopadhyay, Das, Jain and Konwer[6]; Joyal, Labelle and Rahaman[30]; Jain[31,32]; Sun and Hsieh[49]; Zilovic, Roytman, Combettes and Swamy[54]; Das and Datta[10] etc. worked in the theory of the distribution of the zeros of polynomials and obtained some newly developed results.

In this chapter we intend to establish some sharper results concerning the theory of distribution of zeros of entire functions on the basis of slowly changing functions.

Though Definition 10.1.1 and Definition 10.1.2 have already been defined in Chapter 2 as Definition 2.1.1 and Definition 2.1.2 respectively, we state here again in order to keep a continuation of our discussion:

Some portion of the results of this chapter have been published in the International Journal of Mathematics And Its Applications(IJMAA), see [25] and some of the remaining portion of the results have been accepted for publication and to appear in the International Journal of Advances in Mathematics(IJAM), see [23] and some of the same have been communicated, see [22]
Definition 10.1.1 The order $\rho$ and lower order $\lambda$ of an entire function $f$ are defined as

$$
\rho = \limsup_{r \to \infty} \frac{\log[M(r, f)]}{\log r} \quad \text{and} \quad \lambda = \liminf_{r \to \infty} \frac{\log[M(r, f)]}{\log r},
$$

where $\log^{(k)} x = \log(\log^{(k-1)} x)$ for $k = 1, 2, 3, \ldots$ and $\log^{(0)} x = x$.

Let $L \equiv L(r)$ be a positive continuous function increasing slowly i.e., $L(ar) \sim L(r)$ as $r \to \infty$ for every positive constant $a$. Singh and Barker[44] defined it in the following way:

Definition 10.1.2 [44] A positive continuous function $L(r)$ is called a slowly changing function if for $\varepsilon (> 0)$,

$$
\frac{1}{k^\varepsilon} \leq \frac{L(kr)}{L(r)} \leq k^\varepsilon \quad \text{for } r > r(\varepsilon) \quad \text{and}
$$

uniformly for $k(\geq 1)$.

If further, $L(r)$ is differentiable, the above condition is equivalent to

$$
\lim_{r \to \infty} \frac{rL'(r)}{L(r)} = 0.
$$

Somasundaram and Thamizharasi [46] introduced the notions of $L$-order and $L$-lower order for entire functions defined in the open complex plane $\mathbb{C}$ as follows:

Definition 10.1.3 [46] The $L$-order $\rho^L$ and the $L$-lower order $\lambda^L$ of an entire function $f$ are defined as

$$
\rho^L = \limsup_{r \to \infty} \frac{\log[M(r, f)]}{\log[rL(r)]} \quad \text{and} \quad \lambda^L = \liminf_{r \to \infty} \frac{\log[M(r, f)]}{\log[rL(r)]}.
$$

When $\rho^L = 0$, in the line of Datta and Biswas[12] we may define the following two quantities:

$$
**\rho^L = \limsup_{r \to \infty} \frac{\log M(r, f)}{\log[rL(r)]} \quad \text{and} \quad **\lambda^L = \liminf_{r \to \infty} \frac{\log M(r, f)}{\log[rL(r)]}.
$$

The more generalised concept for $L$-order and $L$-lower order are $L^*$-order and $L^*$-lower order respectively. Their definitions are as follows:

Definition 10.1.4 The $L^*$-order $\rho^{L^*}$ and the $L^*$-lower order $\lambda^{L^*}$ of an entire function $f$ are defined as

$$
\rho^{L^*} = \limsup_{r \to \infty} \frac{\log[M(r, f)]}{\log[re^{L(r)}]} \quad \text{and} \quad \lambda^{L^*} = \liminf_{r \to \infty} \frac{\log[M(r, f)]}{\log[re^{L(r)}]}.
$$
10.2 Lemma.

In this section we present a lemma which will be needed in the sequel.

**Lemma 10.2.1** If \( f(z) \) is an entire function of order \( p \), then for every \( \varepsilon > 0 \) the inequality

\[
N(r) \leq [rL(r)]^{pL+\varepsilon}
\]

holds for all sufficiently large \( r \) where \( N(r) \) is the number of zeros of \( f(z) \) in \( |z| < rL(r) \).

**Proof.** Let us suppose that \( f(0) = 1 \). This supposition can be made without loss of generality because if \( f(z) \) has a zero of order \( 0 < m < \infty \) at the origin then we may consider \( g(z) = c \cdot \frac{f(z)}{z^m} \) where \( c \) is so chosen that \( g(0) = 1 \). Since the function \( g(z) \) and \( f(z) \) have the same order therefore it will be unimportant for our investigations that the number of zeros of \( g(z) \) and \( f(z) \) differ by \( m \).

We further assume that \( f(z) \) has no zeros on \( |z| = 2[rL(r)] \) and the zeros \( z_i \)'s of \( f(z) \) in \( |z| < rL(r) \) are in non decreasing order of their moduli so that \( |z_i| \leq |z_{i+1}| \). Also let \( \rho L \) supposed to be finite.

Now we shall make use of Jenson’s formula as state below

\[
\log |f(0)| = - \sum_{i=1}^{n} \log \frac{R}{|z_i|} + \frac{1}{2\pi} \int_{0}^{2\pi} \log |f(R e^{i\phi})| \, d\phi. \tag{10.2.1}
\]

Let us replace \( R \) by \( 2r \) and \( n \) by \( N(2r) \) in (10.2.1).

Hence

\[
\log |f(0)| = - \sum_{i=1}^{N(2r)} \log \frac{2r}{|z_i|} + \frac{1}{2\pi} \int_{0}^{2\pi} \log |f(2r e^{i\phi})| \, d\phi.
\]

Since \( f(0) = 1 \), therefore \( \log |f(0)| = \log 1 = 0 \). So

\[
\sum_{i=1}^{N(2r)} \log \frac{2r}{|z_i|} = \frac{1}{2\pi} \int_{0}^{2\pi} \log |f(2r e^{i\phi})| \, d\phi. \tag{10.2.2}
\]

L.H.S. \( = \sum_{i=1}^{N(2r)} \log \frac{2r}{|z_i|} \geq \sum_{i=1}^{N(r)} \log \frac{2r}{|z_i|} \geq N(r) \log 2 \) \( \tag{10.2.3} \)

because for all sufficiently large values of \( r \),

\[
\log \frac{2r}{|z_i|} \geq \log 2.
\]

R.H.S. \( = \frac{1}{2\pi} \int_{0}^{2\pi} \log |f(2r e^{i\phi})| \, d\phi \leq \frac{1}{2\pi} \int_{0}^{2\pi} \log M(2r) \, d\phi = \log M(2r) \). \( \tag{10.2.4} \)
Again by definition of $\rho^L$ and as $L(2r) \sim L(r)$, we get for every $\varepsilon > 0$ that

$$\log M(2r) \leq [2rL(2r)]^{\rho^L+\varepsilon/2},$$

i.e.,

$$\log M(2r) \leq [2rL(r)]^{\rho^L+\varepsilon/2}. \quad (10.2.5)$$

Hence by the help of (10.2.3), (10.2.4) and (10.2.5) we obtain from (10.2.2) that

$$N(r) \log 2 \leq [2rL(r)]^{\rho^L+\varepsilon/2}$$

$$N(r) \leq \frac{2^{\rho^L+\varepsilon/2}}{\log 2} \cdot \frac{[rL(r)]^{\rho^L+\varepsilon}}{[rL(r)]^{\varepsilon/2}} \leq [rL(r)]^{\rho^L+\varepsilon}.$$  

This proves the lemma. —

In the line of Lemma 10.2.1, we may state the following lemma:

**Lemma 10.2.2** If $f(z)$ is an entire function of $L^*$—order $\rho^{L^*}$, then for every $\varepsilon > 0$ the inequality

$$N(r) \leq [r e^{L(r)}]^{\rho^{L^*}+\varepsilon}$$

holds for all sufficiently large $r$ where $N(r)$ is the number of zeros of $f(z)$ in $|z| \leq [r e^{L(r)}]$.

**Proof.** With the initial assumptions as laid down in Lemma 10.2.1, let us suppose that $f(z)$ has no zeros on $|z| = 2[r e^{L(r)}]$ and the zeros $z_i$'s of $f(z)$ in $|z| < [r e^{L(r)}]$ are in non-decreasing order of their moduli so that $|z_i| \leq |z_{i+1}|$. Also let $\rho^{L^*}$ supposed to be finite.

In view of (10.2.1),(10.2.2),(10.2.3) and (10.2.4), by definition of $\rho^{L^*}$ and as $L(2r) \sim L(r)$, we get for every $\varepsilon > 0$ that

$$\log M(2r) \leq [2r e^{L(2r)}]^{\rho^{L^*}+\varepsilon/2}$$

i.e.,

$$\log M(2r) \leq [2r e^{L(r)}]^{\rho^{L^*}+\varepsilon/2}. \quad (10.2.6)$$

Hence by the help of (10.2.3), (10.2.4) and (10.2.6) we obtain from (10.2.2) that

$$N(r) \log 2 \leq [2r e^{L(r)}]^{\rho^{L^*}+\varepsilon/2}$$

$$N(r) \leq \frac{2^{\rho^{L^*}+\varepsilon/2}}{\log 2} \cdot \frac{[r e^{L(r)}]^{\rho^{L^*}+\varepsilon}}{[r e^{L(r)}]^{\varepsilon/2}} \leq [r e^{L(r)}]^{\rho^{L^*}+\varepsilon}.$$  

Thus the lemma is established. —

In the line of Lemma 10.2.1, we may state the following lemma without its proof.

**Lemma 10.2.3** If $f(z)$ is an entire function with $\rho^L = 0$, then for every $\varepsilon > 0$ the inequality

$$N(r) \leq [rL(r)]^{\rho^L+\varepsilon}$$

holds for all sufficiently large $r$ where $N(r)$ is the number of zeros of $f(z)$ in $|z| \leq [rL(r)]$. 


10.3 Theorems.

In this section we present the main results of the chapter.

Theorem 10.3.1 Let $P(z)$ be an entire function defined as

$$P(z) = a_0 + a_1 z + \ldots + a_n z^n + \ldots$$

with $L$-order $\rho^L$. Also for all sufficiently large $r$ in the disc $|z| \leq [rL(r)]$, $a_0 \neq 0$ and $a_{N(r)} \neq 0$. Also $a_n \to 0$ as $n > N(r)$. Then all the zeros of $P(z)$ lie in the ring shaped region

$$\frac{1}{t_0'} \leq |z| \leq t_0$$

where $t_0$ and $t_0'$ are the positive roots of the equations

$$g(t) \equiv |a_{N(r)}| t^{N(r)} - |a_{N(r)-1}| t^{N(r)-1} - \ldots - |a_0| = 0$$

and

$$h(t) \equiv |a_0| t^{N(r)} - |a_1| t^{N(r)-1} - \ldots - |a_{N(r)}| = 0$$

respectively in $|z| \leq [rL(r)]$ and $N(r)$ denotes the number of zeros of $P(z)$ in $|z| \leq [rL(r)]$ for sufficiently large $r$.

Proof. Since $P(z)$ is an entire function of $L$-order $\rho^L$, then from Lemma 10.2.1 we have for sufficiently large $r$ in the disc $|z| \leq [rL(r)]$,

$$N(r) \leq [rL(r)]^{\rho L + \epsilon} \text{ for } \epsilon > 0.$$ 

Also $a_0 \neq 0$ and $a_{N(r)} \neq 0$. Further $a_n \to 0$ as $n > N(r)$.

Hence we have

$$P(z) = a_0 + a_1 z + \ldots + a_n z^n + \ldots$$

$$\approx a_0 + a_1 z + \ldots + a_{N(r)} z^{N(r)}.$$ 

Therefore

$$|P(z)| \approx |a_0 + a_1 z + \ldots + a_{N(r)} z^{N(r)}|$$

$$\geq |a_{N(r)}| |z|^{N(r)} - |a_{N(r)-1}| |z|^{N(r)-1} \ldots - |a_0|$$

in the disc $|z| \leq [rL(r)]$ for sufficiently large $r$. In fact (10.3.1) can be deduced in the following way:

$$|a_0 + \ldots + a_{N(r)-1} z^{N(r)-1}| \leq |a_0| + \ldots + |a_{N(r)-1}| |z|^{N(r)-1}$$

i.e.,

$$- |a_0| \ldots - |a_{N(r)-1}| |z|^{N(r)-1} \leq - |a_0| + \ldots + a_{N(r)-1} z^{N(r)-1}.$$
Hence
\[ |a_N z^{N} + a_{N-1} z^{N-1} + \ldots + a_0| \leq |a_N| |z|^N - |a_{N-1} z^{N-1} + \ldots + a_0| \]
\[ \geq |a_N| |z|^N - |a_{N-1} z^{N-1} + \ldots + a_0| \]
\[ \geq |a_N| |z|^N - |a_{N-1} z^{N-1} + \ldots + a_0| \geq |a_N| |z|^N - |a_{N-1} z^{N-1} + \ldots + a_0| . \]

Now let us write
\[ g(t) \equiv |a_N| t^N - |a_{N-1} t^{N-1} - \ldots - a_0| . \] (10.3.2)

Since (10.3.2) has one change of sign, by Descartes’ rule of sign, the maximum number of positive root of (10.3.2) is one. Moreover
\[ g(0) = - |a_0| < 0 \]
and \( g(\infty) \) is a positive quantity.

Clearly \( t > t_0 \) implies \( g(t) > 0 \).

If not, let for some \( t_1 > t_0 \), \( g(t_1) < 0 \).

Then \( g(t) = 0 \) has another positive root in \((t_1, \infty)\) which gives a contradiction. Hence \( g(t) > 0 \) for \( t > t_0 \).

Therefore \( |P(z)| > 0 \) for \( |z| > t_0 \). So \( P(z) \) does not vanish in \( |z| > t_0 \) and therefore all the zeros of \( P(z) \) lie in \( |z| \leq t_0 \) where \( t_0 \) is the positive root of
\[ g(t) \equiv |a_N| t^N - |a_{N-1} t^{N-1} - \ldots - a_0| = 0 . \]

Now we give the proof of the other part of the theorem.

Let us consider
\[ Q(z) = z^N P \left( \frac{1}{z} \right) \quad \text{(10.3.3)} \]
for sufficiently large \( r \) in the disc \( |z| \leq |r L(r)| \). Now
\[ Q(z) = z^N P \left( \frac{1}{z} \right) \approx z^N \left[ a_0 + \frac{a_1}{z} + \ldots + a_N \frac{1}{z^N} \right] = a_0 z^N + a_1 z^{N-1} + \ldots + a_N . \] (10.3.4)

Again we have
\[ |a_1 z^{N-1} + \ldots + a_N| \leq |a_1| |z|^{N-1} + |a_2| |z|^N - 2 + \ldots + |a_N| \]
i.e.,
\[ -|a_1| |z|^N - 1 - \ldots - |a_N| \leq -|a_1 z^{N-1} + \ldots + a_N| . \]

So we get that
\[ |a_0 z^N + \ldots + a_N| \geq |a_0| |z|^N - |a_1 z^{N-1} + \ldots + a_N| \]
\[ \geq |a_0| |z|^N - |a_1| |z|^N - 1 - \ldots - |a_N| . \] (10.3.5)
Let us consider the equation
\[ h(t) \equiv |a_0| |t|^{N(r)} - |a_1| |t|^{N(r)-1} - \cdots - |a_{N(r)}| = 0. \] (10.3.6)

Since (10.3.6) has one change of sign, by Descartes’ rule of sign the maximum number of positive root of (10.3.6) is one. Moreover
\[ h(0) = -|a_{N(r)}| < 0 \]
and \( h(\infty) \) is a positive quantity. So \( h(t) \) has exactly one positive root.

Let \( t'_0 \) be the positive root of \( h(t) = 0 \). Clearly for \( t > t'_0 \) we get \( h(t) > 0 \).

If not, let \( t'_1 > t'_0 \). Then \( h(t'_1) < 0 \). Hence \( h(t) = 0 \) has another positive root in \((t'_1, \infty)\) which gives a contradiction.

Therefore \( h(t) > 0 \) for \( t > t'_0 \) and \( |Q(z)| > 0 \) for \( |z| > t'_0 \).

So \( Q(z) \) does not vanish in \( |z| > t'_0 \) and therefore all the zeros of \( Q(z) \) lie in \( |z| \leq t'_0 \). Let \( z = z_0 \) be any zero of \( P(z) = 0 \). Clearly \( z_0 \neq 0 \) as \( a_0 \neq 0 \).

Putting \( z = \frac{1}{z_0} \) in \( Q(z) \) we get that
\[ Q \left( \frac{1}{z_0} \right) = \left( \frac{1}{z_0} \right)^{N(r)} P(z_0) = \left( \frac{1}{z_0} \right)^{N(r)} 0 = 0. \]

So \( \frac{1}{z_0} \) is a zero of \( Q(z) \). Therefore \( \left| \frac{1}{z_0} \right| \leq t'_0 \) i.e., \( |z_0| \geq \frac{1}{t'_0} \). Since \( z_0 \) is any arbitrary zero of \( P(z) \),
all the zeros of \( P(z) \) lie in \( |z| \geq \frac{1}{t'_0} \).

Hence all the zeros of \( P(z) \) lie in the ring shaped region
\[ \frac{1}{t'_0} \leq |z| \leq t_0 \]
where \( t_0 \) and \( t'_0 \) are the positive roots of
\[ g(t) \equiv |a_{N(r)}| t^{N(r)} - |a_{N(r)-1}| t^{N(r)-1} - \cdots - |a_0| = 0 \]

and
\[ h(t) \equiv |a_0| t^{N(r)} - |a_1| t^{N(r)-1} - \cdots - |a_{N(r)}| = 0 \]
respectively for sufficiently large \( r \) in the disc \( |z| \leq [rL(r)] \).

This completes the proof of the theorem. ■

In the line of Theorem 10.3.1, we may state the following theorem in view of Lemma 10.2.2:

**Theorem 10.3.2** Let \( P(z) \) be an entire function defined by
\[ P(z) = a_0 + a_1 z + \cdots + a_n z^n + \cdots \]
with \( L^* \)-order \( \rho^* \). Also for all sufficiently large \( r \) in the disc \( |z| \leq [rL(r)] \), \( a_0 \neq 0 \) and \( a_{N(r)} \neq 0 \). Also \( a_n \to 0 \) as \( n > N(r) \). Then all the zeros of \( P(z) \) lie in the ring shaped region
\[ \frac{1}{t'_0} \leq |z| \leq t_0 \]
where $t_0$ and $t'_0$ are the positive roots of the equations
\[ g(t) \equiv |a_{N(r)}| t^{N(r)} - |a_{N(r)-1}| t^{N(r)-1} - \cdots - |a_0| = 0 \]
and
\[ h(t) \equiv |a_0| t^{N(r)} - |a_1| t^{N(r)-1} - \cdots - |a_{N(r)}| = 0 \]
respectively in $|z| \leq |re^{L(r)}|$ and $N(r)$ denotes the number of zeros of $P(z)$ in $|z| \leq |re^{L(r)}|$ for sufficiently large $r$.

The proof is omitted.

**Remark 10.3.1** The limit in Theorem 10.3.1 is attained by $P(z) = nz^2 + (n-1)z - 1$ for any positive real number $n \geq 2$. It can be easily seen that $M(r) = |n| r^2 = nr^2$ for large $r$ in $|z| = r$. Let $L(r) = \log r$.

So
\[
\rho^L = \limsup_{r \to \infty} \frac{\log^{[2]} M(r)}{\log [rL(r)]}
= \limsup_{r \to \infty} \frac{\log^{[2]} (nr^2)}{\log r + \log^{[2]} r}
= \limsup_{r \to \infty} \frac{\log^{[2]} r + O(1)}{\log r + \log^{[2]} r}
= \limsup_{r \to \infty} 1 + \frac{\log^{[2]} r}{\log r} = 0.
\]

Hence the $L$-order of the polynomial is $0$.

Also here
\[
** \rho^L = \limsup_{r \to \infty} \frac{\log(nr^2)}{\log [rL(r)]}
= \limsup_{r \to \infty} \frac{\log n + 2 \log r}{\log r + \log^{[2]} r}
= \limsup_{r \to \infty} 1 + \frac{\log^{[2]} r}{\log r}
= 2
\]

and in view of Lemma 10.2.3, $N(r) = 2 \leq (r \log r)^{2+\epsilon}$ for $\epsilon > 0$ and $r$ be sufficiently large in $|z| \leq |r \log r|$. The zeros of $P(z)$ is given by solving $P(z) = 0$ and $a_{N(r)} = n \neq 0$, $a_{N(r)+1} = a_{N(r)+2} = \cdots = 0$. Now
\[ nz^2 + (n-1)z - 1 = 0 \]
i.e., if $(nz - 1)(z + 1) = 0$
i.e., if $z = \frac{1}{n}, -1$. 
Let $z_1 = \frac{1}{n}$ and $z_2 = -1$. Then $z_1$ and $z_2$ are the zeros of

$$P(z) = nz^2 + (n - 1)z - 1 = 0.$$ 

Here $a_0 = 1$, $a_1 = n - 1$ and $a_2 = n$. Therefore $|a_0| = 1$, $|a_1| = n - 1$ and $|a_2| = n$ and so

$$f(t) \equiv |a_2|t^2 - |a_1|t - |a_0| = nt^2 - (n - 1)t - 1 = 0.$$ 

Hence $t = 1$ and $-\frac{1}{n}$.

Thus the positive root of $f(t) = 0$ is $t = t_0 = 1$.

Again to find the positive root of

$$g(t) \equiv |a_0|t^2 - |a_1|t - |a_2| = 0$$

we get that

$$t^2 - (n - 1)t - n = 0.$$ 

which implies $t = n$ and $t = -1$. Therefore the positive root of $g(t) = 0$ is $t'_0 = n$.

Hence according to Theorem 10.3.1, all the zeros of $P(z)$ lie in

$$\frac{1}{t'_0} \leq |z| \leq t_0$$

i.e., in

$$\frac{1}{n} \leq |z| \leq 1,$$

which has been shown in Figure 1 in the APPENDIX given at the end of the chapter.

**Theorem 10.3.3** Let $P(z)$ be an entire function defined by

$$P(z) = a_0 + a_1z + \ldots + a_nz^n + \ldots$$

with $L$-order $\rho^L$. Also for all sufficiently large $r$ in the disc $|z| \leq [rL(r)]$, $a_{N(r)} \neq 0$ and $a_0 \neq 0$. Further $a_n \to 0$ as $n > N(r)$. Then all the zeros of $P(z)$ lie in the ring shaped region

$$\frac{1}{1 + M'} < |z| < 1 + M$$

where $M = \max_{0 \leq k \leq N(r) - 1} \left| \frac{a_k}{a_{N(r)}} \right|$ and $M' = \max_{0 \leq k \leq N(r) - 1} \left| \frac{a_k}{a_0} \right|$. 

**Proof.** Since $P(z)$ is an entire function of $L$-order $\rho^L$, then by Lemma 10.2.1 for sufficiently large values of $r$ in the disc $|z| \leq [rL(r)]$ we have $N(r) \leq [rL(r)]^{\rho^L+\epsilon}$ for $\epsilon > 0$. Also $a_0 \neq 0$, $a_{N(r)} \neq 0$, and $a_n \to 0$ as $n > N(r)$. Hence we may write

$$P(z) = a_0 + a_1z + \ldots + a_nz^n + \ldots$$

$$\approx a_0 + a_1z + \ldots + a_{N(r)}z^{N(r)}.$$
Now

\[ |a_0 + a_1z + \ldots + a_{N(r)-1}z^{N(r)-1}| \]
\[ \leq |a_0| + \ldots + |a_{N(r)-1}| |z|^{N(r)-1} \]
\[ = |a_{N(r)}| \left\{ \frac{|a_0|}{|a_{N(r)}|} + \ldots + \frac{|a_{N(r)-1}|}{|a_{N(r)}|} |z|^{N(r)-1} \right\} \]
i.e.,

\[ |a_0 + a_1z + \ldots + a_{N(r)-1}z^{N(r)-1}| \]
\[ \leq |a_{N(r)}| M \left( |z|^{N(r)-1} + |z|^{N(r)-2} + \ldots + 1 \right) \]
\[ = |a_{N(r)}| M |z|^{N(r)} \left\{ \frac{1}{|z|} + \frac{1}{|z|^2} + \ldots + \frac{1}{|z|^{N(r)}} \right\} \]

where \(|z| \neq 0\). Therefore when \(|z| \neq 0\),

\[ - |a_0 + a_1z + \ldots + a_{N(r)-1}z^{N(r)-1}| \]
\[ \geq - |a_{N(r)}| M |z|^{N(r)} \left\{ \frac{1}{|z|} + \frac{1}{|z|^2} + \ldots + \frac{1}{|z|^{N(r)}} \right\}. \]

So for \(|z| \neq 0\)

\[ |a_{N(r)}| |z|^{N(r)} - |a_0 + a_1z + \ldots + a_{N(r)-1}z^{N(r)-1}| \]
\[ \geq |a_{N(r)}| |z|^{N(r)} - |a_{N(r)}| |z|^{N(r)} M \left\{ \frac{1}{|z|} + \frac{1}{|z|^2} + \ldots + \frac{1}{|z|^{N(r)}} \right\}. \] \hspace{1cm} (10.3.7)

Now

\[ |P(z)| \approx |a_0 + a_1z + \ldots + a_{N(r)}z^{N(r)}| \]
\[ \geq |a_{N(r)}| |z|^{N(r)} - |a_0 + a_1z + \ldots + a_{N(r)-1}z^{N(r)-1}|. \]

Using (10.3.7) we have

\[ |P(z)| \geq |a_{N(r)}| |z|^{N(r)} - |a_{N(r)}| |z|^{N(r)} M \left\{ \frac{1}{|z|} + \frac{1}{|z|^2} + \ldots + \frac{1}{|z|^{N(r)}} \right\} \]
\[ = |a_{N(r)}| |z|^{N(r)} \left\{ 1 - M \left( \frac{1}{|z|} + \frac{1}{|z|^2} + \ldots + \frac{1}{|z|^{N(r)}} \right) \right\} \text{ for } |z| \neq 0. \]
i.e., when \(|z| \neq 0\)

\[ |P(z)| > |a_{N(r)}| |z|^{N(r)} \left\{ 1 - M \left( \frac{1}{|z|} + \frac{1}{|z|^2} + \ldots + \frac{1}{|z|^{N(r)}} \right) \right\}. \]
Therefore

\[ |P(z)| > |a_{N(r)}| |z|^{N(r)} \left\{ 1 - M \sum_{j=1}^{\infty} \frac{1}{|z|^j} \right\} \text{ for } |z| \neq 0. \]

Now the geometric series \( \sum_{j=1}^{\infty} \frac{1}{|z|^j} \) is convergent when \( \frac{1}{|z|} < 1 \) i.e., when \( |z| > 1 \) and is equal to

\[ \frac{1}{|z|} \frac{1}{1 - \frac{1}{|z|}} = \frac{1}{|z| - 1}. \]

On \( |z| > 1 \), we can write

\[ |P(z)| > |a_{N(r)}| |z|^{N(r)} \left( 1 - \frac{M}{|z| - 1} \right). \]

Now on \( |z| > 1 \),

\[ |P(z)| > 0 \text{ if } |a_{N(r)}| |z|^{N(r)} \left( 1 - \frac{M}{|z| - 1} \right) \geq 0 \]

i.e., if \( 1 - \frac{M}{|z| - 1} \geq 0 \)

i.e., if \(|z| - 1 \geq M \)

i.e., if \(|z| \geq M + 1 \).

Therefore

\[ |z| \geq M + 1 > 1 \text{ as } M > 0. \]

Hence

\[ |P(z)| > 0 \text{ if } |z| \geq M + 1. \]

Therefore all the zeros of \( P(z) \) lie in \(|z| < M + 1 \).

Secondly, we give the proof of the lower bound. Let us consider

\[ Q(z) = z^{N(r)} P \left( \frac{1}{z} \right). \]

Therefore

\[ Q(z) = |z|^{N(r)} \left\{ a_0 + \frac{a_1}{|z|} + \ldots + \frac{a_{N(r)}}{|z|^{N(r)}} \right\} \]

\[ = a_0 |z|^{N(r)} + a_1 |z|^{N(r)-1} + \ldots + a_{N(r)}. \]
Now

\[ |a_1| |z|^{N(r)-1} + \ldots + a_{N(r)}| \leq |a_1| |z|^{N(r)-1} + \ldots + |a_{N(r)}| \]

\[ = |a_0| \left( \frac{|a_1|}{|a_0|} |z|^{N(r)-1} + \ldots + \frac{|a_{N(r)}|}{|a_0|} \right) \]

\[ \leq |a_0| M' \left( |z|^{N(r)-1} + \ldots + 1 \right) \]

\[ = |a_0| M' |z|^{N(r)} \left( \frac{1}{|z|} + \ldots + \frac{1}{|z|^{N(r)}} \right) \].

Therefore

\[ - |a_1| |z|^{N(r)-1} + \ldots + a_{N(r)}| \geq - |a_0| M' |z|^{N(r)} \left( \frac{1}{|z|} + \ldots + \frac{1}{|z|^{N(r)}} \right) . \]

So

\[ |Q(z)| \geq |a_0| |z|^{N(r)} - |a_1| z^{N(r)-1} + \ldots + a_{N(r)}| \]

\[ \geq |a_0| |z|^{N(r)} - |a_0| M' |z|^{N(r)} \left( \frac{1}{|z|} + \ldots + \frac{1}{|z|^{N(r)}} \right) \]

\[ = |a_0| |z|^{N(r)} \left\{ 1 - M' \left( \frac{1}{|z|} + \ldots + \frac{1}{|z|^{N(r)}} \right) \right\} \]

\[ \geq |a_0| |z|^{N(r)} \left\{ 1 - M' \left( \frac{1}{|z|} + \ldots + \frac{1}{|z|^{N(r)}} + \ldots \right) \right\} . \]

Hence using above we get that

\[ |Q(z)| > |a_0| |z|^{N(r)} \left\{ 1 - M' \sum_{j=1}^{\infty} \frac{1}{|z|^j} \right\} . \]

Now the geometric series \( \sum_{j=1}^{\infty} \frac{1}{|z|^j} \) is convergent when \( \frac{1}{|z|} < 1 \) i.e., \( |z| > 1 \) and is equal to

\[ \frac{1}{|z|} - \frac{1}{|z|} = \frac{1}{|z| - 1} . \]

On \( |z| > 1 \) we may write

\[ |Q(z)| \geq |a_0| |z|^{N(r)} \left( 1 - \frac{M'}{|z| - 1} \right) . \]
Now for $|z| > 1$,
\[ |Q(z)| > 0 \text{ if } |a_0| |z|^N(r) \left(1 - \frac{M'}{|z| - 1}\right) \geq 0 \]
i.e., if \[ 1 - \frac{M'}{|z| - 1} \geq 0 \]
i.e., if \[ |z| \geq 1 + M'. \]

Therefore $|z| \geq 1 + M' > 1$ as $M' > 0$.

Hence $|Q(z)| > 0$ for $|z| \geq 1 + M'$.

So all the zeros of $Q(z)$ lie in $|z| < 1 + M'$.

Let $z = z_0$ be any zero of $P(z)$. Therefore $P(z_0) = 0$. Clearly $z_0 \neq 0$ as $a_0 \neq 0$. Putting $z = \frac{1}{z_0}$ in $Q(z)$ we have
\[ Q \left( \frac{1}{z_0} \right) = \left( \frac{1}{z_0} \right)^n P(z_0) = 0. \]

Therefore $z = \frac{1}{z_0}$ is a root of $Q(z)$. So
\[ \left| \frac{1}{z_0} \right| < 1 + M', \]
which implies that
\[ |z_0| > \left| \frac{1}{1 + M'} \right|. \]

As $z_0$ is an arbitrary root of $P(z) = 0$, all the zeros of $P(z)$ lie in $|z| > \left| \frac{1}{1 + M'} \right|$. Hence all the zeros of $P(z)$ lie in the ring shaped region
\[ \frac{1}{1 + M'} < |z| < 1 + M. \]

This proves the theorem. ■

In the line of Theorem 10.3.3, the following theorem may be stated in view of Lemma 10.2.2:

**Theorem 10.3.4** Let $P(z)$ be an entire function defined by
\[ P(z) = a_0 + a_1 z + \ldots + a_n z^n + \ldots \]
with $L^*$-order $\rho^{L^*}$. Also for all sufficiently large $r$ in the disc $|z| \leq \lfloor r e^{L(r)} \rfloor$, $a_{N(r)} \neq 0$ and $a_0 \neq 0$. Further $a_n \to 0$ as $n > N(r)$. Then all the zeros of $P(z)$ lie in the ring shaped region
\[ \frac{1}{1 + M'} < |z| < 1 + M \]
where $M = \max_{0 \leq k \leq N(r) - 1} \left| \frac{a_k}{a_{N(r)}} \right|$ and $M' = \max_{0 \leq k \leq N(r) - 1} \frac{|a_k|}{|a_0|}$. 

The proof is omitted.

**Remark 10.3.2** Let us consider the polynomial

\[ P(z) = nz^2 + (n - 1)z - 1, n \geq 2. \]

Here \( a_0 = -1, a_1 = n - 1 \) and \( a_2 = n \).

Therefore

\[ |a_0| = 1, |a_1| = n - 1 \text{ and } |a_2| = n. \]

Also order \( \rho^L \) of \( P(z) \) is 0 and \( *\rho^L = 2 \), so \( N(r) = 2 \leq (\log r)^{2+\varepsilon} \) for sufficiently large \( r \).

Again \( M = \max \left\{ \frac{|a_0|}{|a_1|}, \frac{|a_1|}{|a_2|} \right\} = \max \left\{ \frac{1}{n}, \frac{n - 1}{n} \right\} = \frac{n - 1}{n} \)

and \( M' = \max \left\{ \frac{|a_1|}{|a_0|}, \frac{|a_2|}{|a_0|} \right\} = \max \left\{ \frac{n - 1}{n}, \frac{n}{1} \right\} = n. \)

The roots of \( P(z) = 0 \) are \( z_1 = \frac{1}{n} \) and \( z_2 = -1. \) So by Theorem 10.3.3, the roots of \( P(z) \) lies in

\[ \frac{1}{1 + M'} < |z| < 1 + M. \]

Hence

\[ \frac{1}{1 + n} < |z| < 1 + \frac{n - 1}{n} \]

i.e.,

\[ \frac{1}{1 + n} < |z| < 2 - \frac{1}{n}. \]

**Theorem 10.3.5** Let \( P(z) \) be an entire function defined by

\[ P(z) = a_0 + a_1 z + a_2 z^2 + \ldots + a_n z^n + \ldots \]

with \( L \)-order \( \rho^L \). Also for all sufficiently large \( r \) in the disc \( |z| \leq [rL(r)] \), \( a_{N(r)} \neq 0, a_0 \neq 0 \) and \( a_n \to 0 \) as \( n > N(r) \). For any \( p, q \) with \( p > 1, q > 1 \) and \( \frac{1}{p} + \frac{1}{q} = 1 \), all the zeros of \( P(z) \) lie in the annular region

\[ \left[ \frac{1}{1 + \left( \sum_{j=0}^{N(r)-1} \frac{a_{j} a_{N(r)}}{a_0} \right)^{\frac{1}{p}}} \right] \leq |z| \leq \left[ 1 + \left( \sum_{j=0}^{N(r)-1} \frac{a_{j} a_{N(r)}}{a_0} \right)^{\frac{1}{p}} \right]^\frac{1}{q}. \]

**Proof.** Given that \( a_0 \neq 0, a_{N(r)} \neq 0 \) and \( a_n \to 0 \) as \( n > N(r) \). Therefore for sufficiently large \( r \) in the disc \( |z| \leq [rL(r)] \) the existence of \( N(r) \) implies that

\[ P(z) = a_0 + a_1 z + a_2 z^2 + \ldots + a_n z^n + \ldots \]

\[ \approx a_0 + a_1 z + a_2 z^2 + \ldots + a_{N(r)} z^{N(r)}. \]
Now
\[
\begin{align*}
|a_0 + a_1 z + a_2 z^2 + \ldots + a_{N(r)-1} z^{N(r)-1}| & \\
\leq |a_0| + |a_1| |z| + \ldots + |a_{N(r)-1}| |z|^{N(r)-1} & \\
= |a_N(r)| \left\{ |a_0| \left| \frac{a_j}{a_N(r)} \right| + \ldots + \left| \frac{a_{N(r)-1}}{a_N(r)} \right| |z|^{N(r)-1} \right\} & \\
= \left| a_N(r) \right| \sum_{j=0}^{N(r)-1} \left| \frac{a_j}{a_N(r)} \right| |z|^j.
\end{align*}
\] (10.3.8)

Therefore using (10.3.8) we get that
\[
|P(z)| \approx \begin{align*}
& \left| a_0 + a_1 z + a_2 z^2 + \ldots + a_{N(r)} z^{N(r)} \right| \\
& \geq \left| a_N(r) \right| |z|^{N(r)} - \left| a_0 + a_1 z + a_2 z^2 + \ldots + a_{N(r)-1} z^{N(r)-1} \right| \\
& \geq \left| a_N(r) \right| |z|^{N(r)} - \left| a_N(r) \right| \sum_{j=0}^{N(r)-1} \left| \frac{a_j}{a_N(r)} \right| |z|^j
\end{align*}
\]
i.e.,
\[
|P(z)| \geq \left| a_N(r) \right| \left\{ |z|^{N(r)} - \sum_{j=0}^{N(r)-1} \left| \frac{a_j}{a_N(r)} \right| |z|^j \right\}.
\]

By Holder’s inequality we have
\[
\begin{align*}
\sum_{j=0}^{N(r)-1} \left| \frac{a_j}{a_N(r)} \right| |z|^j & \leq \left( \sum_{j=0}^{N(r)-1} \left| \frac{a_j}{a_N(r)} \right|^p \right)^{\frac{1}{p}} \left( \sum_{j=0}^{N(r)-1} \left| z \right|^{qj} \right)^{\frac{1}{q}}.
\end{align*}
\] (10.3.9)

In view of (10.3.9) we obtain that
\[
\begin{align*}
|P(z)| & \geq \left| a_N(r) \right| \left\{ |z|^{N(r)} - \left( \sum_{j=0}^{N(r)-1} \left| \frac{a_j}{a_N(r)} \right|^p \right)^{\frac{1}{p}} \left( \sum_{j=0}^{N(r)-1} \left| z \right|^{qj} \right)^{\frac{1}{q}} \right\} \\
& = \left| a_N(r) \right| \left\{ |z|^{N(r)} - |z|^{N(r)} \left( \sum_{j=0}^{N(r)-1} \left| \frac{a_j}{a_N(r)} \right|^p \right)^{\frac{1}{p}} \left( \sum_{j=0}^{N(r)-1} \left| z \right|^{jq} \right)^{\frac{1}{q}} \right\} \\
& = \left| a_N(r) \right| \left| z \right|^{N(r)} \left\{ \left[ 1 - \left( \sum_{j=0}^{N(r)-1} \left| \frac{a_j}{a_N(r)} \right|^p \right)^{\frac{1}{p}} \left( \sum_{j=0}^{N(r)-1} \left| z \right|^{jq} \right)^{\frac{1}{q}} \right] \right\} \\
& = \left| a_N(r) \right| \left| z \right|^{N(r)} \left\{ \left[ 1 - \left( \sum_{j=1}^{N(r)-1} \left| \frac{a_j}{a_N(r)} \right|^p \right)^{\frac{1}{p}} \left( \sum_{j=1}^{N(r)-1} \left| z \right|^{jq} \right)^{\frac{1}{q}} \right] \right\}.
\end{align*}
\]
Now the geometric series \( \sum_{j=1}^{\infty} \left( \frac{1}{|z|^q} \right)^j \) is convergent for
\[
\frac{1}{|z|^q} < 1
\]
i.e., for \( |z|^q > 1 \)
i.e., for \( |z| > 1 \)
and is convergent to
\[
\frac{1}{|z|^q} \cdot \frac{1}{1 - \frac{1}{|z|^q}} = \frac{1}{|z|^q - 1}.
\]

So
\[
\left( \sum_{j=1}^{\infty} \left( \frac{1}{|z|^q} \right)^j \right)^{\frac{1}{q}} \text{ converges to } \left( \frac{1}{|z|^q - 1} \right)^{\frac{1}{q}} \text{ for } |z| > 1.
\]

Therefore on \( |z| > 1 \),
\[
|P(z)| > \left| a_{N(r)} \right| |z|^{N(r)} \left\{ 1 - \left( \sum_{j=0}^{N(r)-1} \left| \frac{a_j}{a_{N(r)}} \right|^p \right)^{\frac{1}{p}} \left( \frac{1}{|z|^q - 1} \right)^{\frac{1}{q}} \right\}.
\]

Now if \( |P(z)| > 0 \), then we have
\[
\left| a_{N(r)} \right| |z|^{N(r)} \left\{ 1 - \left( \sum_{j=0}^{N(r)-1} \left| \frac{a_j}{a_{N(r)}} \right|^p \right)^{\frac{1}{p}} \left( \frac{1}{|z|^q - 1} \right)^{\frac{1}{q}} \right\} \geq 0
\]
i.e.,
\[
1 - \left( \sum_{j=0}^{N(r)-1} \left| \frac{a_j}{a_{N(r)}} \right|^p \right)^{\frac{1}{p}} \left( \frac{1}{|z|^q - 1} \right)^{\frac{1}{q}} \geq 0
\]
i.e.,
\[
(|z|^q - 1)^{\frac{1}{q}} \geq \left( \sum_{j=0}^{N(r)-1} \left| \frac{a_j}{a_{N(r)}} \right|^p \right)^{\frac{1}{p}}
\]
i.e.,
\[
|z|^q - 1 \geq \left( \sum_{j=0}^{N(r)-1} \left| \frac{a_j}{a_{N(r)}} \right|^p \right)^{\frac{2}{p}}
\]
i.e.,
\[
|z| \geq \left[ 1 + \left( \sum_{j=0}^{N(r)-1} \left| \frac{a_j}{a_{N(r)}} \right|^p \right)^{\frac{q}{p}} \right]^{\frac{1}{q}}.
\]
Clearly
\[
1 + \left( \sum_{j=0}^{N(r)-1} \left| \frac{a_j}{a(N(r))} \right|^p \right)^{\frac{1}{p}} > 1.
\]
Therefore \(|P(z)| > 0\) for
\[
|z| \geq 1 + \left( \sum_{j=0}^{N(r)-1} \left| \frac{a_j}{a(N(r))} \right|^p \right)^{\frac{1}{p}}.
\]
Therefore all the zeros of \(P(z)\) lie in
\[
|z| < 1 + \left( \sum_{j=0}^{N(r)-1} \left| \frac{a_j}{a(N(r))} \right|^p \right)^{\frac{1}{p}}.
\] (10.3.10)
For the lower bound let us take \(Q(z) = z^{N(r)}P\left(\frac{1}{z}\right)\). Therefore
\[
Q(z) = z^{N(r)} \left( \frac{1}{z} \right)^{N(r)} = z^{N(r)} \left\{ a_0 + \frac{a_1}{z} + \ldots + \frac{a_N(z)}{z^{N(r)}} \right\}
= a_0 z^{N(r)} + a_1 z^{N(r)-1} + \ldots + a_{N(r)}.
\]
Therefore
\[|Q(z)| \approx |a_0 z^{N(r)} + a_1 z^{N(r)-1} + \ldots + a_{N(r)}|.
\]
Now
\[
|a_1 z^{N(r)-1} + \ldots + a_{N(r)}| \leq |a_1| |z|^{N(r)-1} + \ldots + |a_{N(r)}|
= |a_0| \left\{ \left| \frac{a_1}{a_0} \right| |z|^{N(r)-1} + \ldots + \frac{|a_{N(r)}|}{|a_0|} \right\}
= |a_0| \sum_{j=0}^{N(r)-1} \left| \frac{a_{N(r)-j}}{a_0} \right| |z|^j.
\] (10.3.11)
Therefore using (10.3.11) we get that
\[
|Q(z)| \geq |a_0| |z|^{N(r)} - \sum_{j=0}^{N(r)-1} \left| \frac{a_{N(r)-j}}{a_0} \right| |z|^j
\]
\[
= |a_0| \left\{ |z|^{N(r)} - \sum_{j=0}^{N(r)-1} \left| \frac{a_{N(r)-j}}{a_0} \right| |z|^j \right\}.
\]
Now by Holder’s inequality we have

\[
\sum_{j=0}^{N(r)-1} \left| \frac{a_{N(r)-j}}{a_0} \right| |z|^j \leq \left( \sum_{j=0}^{N(r)-1} \left| \frac{a_{N(r)-j}}{a_0} \right|^p \right)^{\frac{1}{p}} \left( \sum_{j=0}^{N(r)-1} \left| z^j \right|^q \right)^{\frac{1}{q}}. \tag{10.3.12}
\]

Using (10.3.12) we obtain from above that

\[
|Q(z)| \geq |a_0| \left\{ \left| z \right|^{N(r)} - \left( \sum_{j=0}^{N(r)-1} \left| \frac{a_{N(r)-j}}{a_0} \right|^p \right)^{\frac{1}{p}} \left( \sum_{j=0}^{N(r)-1} \left| z^j \right|^q \right)^{\frac{1}{q}} \right\}
\]

\[
= |a_0| \left\{ \left| z \right|^{N(r)} - \left( \sum_{j=0}^{N(r)-1} \left| \frac{a_{N(r)-j}}{a_0} \right|^p \right)^{\frac{1}{p}} \left( \sum_{j=0}^{N(r)-1} \left| z^j \right|^q \right)^{\frac{1}{q}} \right\}
\]

\[
= |a_0| \left\{ 1 - \left( \sum_{j=0}^{N(r)-1} \left| \frac{a_{N(r)-j}}{a_0} \right|^p \right)^{\frac{1}{p}} \left( \sum_{j=0}^{N(r)-1} \left| z^j \right|^q \right)^{\frac{1}{q}} \right\}
\]

\[
= |a_0| \left\{ \left( \sum_{j=0}^{N(r)-1} \left| \frac{a_{N(r)-j}}{a_0} \right|^p \right)^{\frac{1}{p}} \left( \sum_{j=0}^{N(r)-1} \left| z^j \right|^q \right)^{\frac{1}{q}} \right\}
\]

Therefore

\[
|Q(z)| > |a_0| \left\{ 1 - \left( \sum_{j=0}^{N(r)-1} \left| \frac{a_{N(r)-j}}{a_0} \right|^p \right)^{\frac{1}{p}} \left( \sum_{j=0}^{N(r)-1} \left| z^j \right|^q \right)^{\frac{1}{q}} \right\}.
\]

Now the geometric series \( \sum_{j=1}^{\infty} \left( \frac{1}{|z|^q} \right)^j \) is convergent for

\[
\frac{1}{|z|^q} < 1
\]

i.e., for \( |z|^q > 1 \).
Therefore for $|z| > 1$ and the series is convergent to

$$
\frac{1}{|z|^q} \frac{1}{1 - \frac{1}{|z|^q}} = \frac{1}{|z|^q - 1}.
$$

So

$$
\left( \sum_{j=1}^{\infty} \left( \frac{1}{|z|^q} \right)^j \right)^{\frac{1}{q}}
$$

is convergent to $\left( \frac{1}{|z|^q - 1} \right)^{\frac{1}{q}}$ for $|z| > 1$.

Therefore on $|z| > 1$,

$$
|Q(z)| > |a_0| |z|^{N(r)} \left\{ 1 - \left( \sum_{j=0}^{N(r)-1} \left| \frac{a_{N(r)-j}}{a_0} \right|^p \right)^{\frac{1}{p}} \left( \frac{1}{|z|^q - 1} \right)^{\frac{1}{q}} \right\}.
$$

Now if $|Q(z)| > 0$, then

$$
|a_0| |z|^{N(r)} \left\{ 1 - \left( \sum_{j=0}^{N(r)-1} \left| \frac{a_{N(r)-j}}{a_0} \right|^p \right)^{\frac{1}{p}} \left( \frac{1}{|z|^q - 1} \right)^{\frac{1}{q}} \right\} \geq 0
$$

i.e.,

$$
1 - \left( \sum_{j=0}^{N(r)-1} \left| \frac{a_{N(r)-j}}{a_0} \right|^p \right)^{\frac{1}{p}} \left( \frac{1}{|z|^q - 1} \right)^{\frac{1}{q}} \geq 0
$$

i.e.,

$$
1 \geq \left( \sum_{j=0}^{N(r)-1} \left| \frac{a_{N(r)-j}}{a_0} \right|^p \right)^{\frac{1}{p}} \left( \frac{1}{|z|^q - 1} \right)^{\frac{1}{q}}
$$

i.e.,

$$
(|z|^q - 1)^{\frac{1}{q}} \geq \left( \sum_{j=0}^{N(r)-1} \left| \frac{a_{N(r)-j}}{a_0} \right|^p \right)^{\frac{1}{p}}
$$

i.e.,

$$
|z|^q - 1 \geq \left( \sum_{j=0}^{N(r)-1} \left| \frac{a_{N(r)-j}}{a_0} \right|^p \right)^{\frac{q}{p}}
$$

i.e.,

$$
|z| \geq \left[ 1 + \left( \sum_{j=0}^{N(r)-1} \left| \frac{a_{N(r)-j}}{a_0} \right|^p \right)^{\frac{q}{p}} \right]^{\frac{1}{q}}.
$$

Clearly,

$$
\left[ 1 + \left( \sum_{j=0}^{N(r)-1} \left| \frac{a_{N(r)-j}}{a_0} \right|^p \right)^{\frac{q}{p}} \right]^{\frac{1}{q}} > 1.
$$
Therefore \(|Q(z)| > 0\) if

\[
|z| \geq \left[ 1 + \left( \sum_{j=0}^{N(r)-1} \left| \frac{a_{N(r)-j}}{a_0} \right|^p \right)^{\frac{1}{p}} \right]^{\frac{1}{q}}.
\]

Therefore all the zeros \(Q(z)\) lie in

\[
|z| < \left[ 1 + \left( \sum_{j=0}^{N(r)-1} \left| \frac{a_{N(r)-j}}{a_0} \right|^p \right)^{\frac{1}{p}} \right]^{\frac{1}{q}}.
\]

Let \(z = z_0\) be any other zero of \(P(z)\). Therefore \(P(z_0) = 0\). Clearly \(z_0 \neq 0\) as \(a_0 \neq 0\).

Putting \(z = \frac{1}{z_0}\) in \(Q(z)\) we have

\[
Q\left( \frac{1}{z_0} \right) = \left( \frac{1}{z_0} \right)^{N(r)} \cdot P\left( \frac{1}{z_0} \right) = \left( \frac{1}{z_0} \right)^{N(r)} \cdot 0 = 0
\]

Therefore \(z = \frac{1}{z_0}\) is a zero of \(Q(z)\). So

\[
\left| \frac{1}{z_0} \right| < \left[ 1 + \left( \sum_{j=0}^{N(r)-1} \left| \frac{a_{N(r)-j}}{a_0} \right|^p \right)^{\frac{1}{p}} \right]^{\frac{1}{q}},
\]

i.e., \(|z_0| > \frac{1}{\left[ 1 + \left( \sum_{j=0}^{N(r)-1} \left| \frac{a_{N(r)-j}}{a_0} \right|^p \right)^{\frac{1}{p}} \right]^{\frac{1}{q}}}.
\]

As \(z_0\) is an arbitrary zero of \(P(z)\) so all the zeros of \(P(z)\) lie in

\[
|z| > \frac{1}{\left[ 1 + \left( \sum_{j=0}^{N(r)-1} \left| \frac{a_{N(r)-j}}{a_0} \right|^p \right)^{\frac{1}{p}} \right]^{\frac{1}{q}}}.
\]

Hence combining (10.3.10) and (10.3.13) we may say that all the zeros of \(P(z)\) lie in the ring shaped region

\[
\left[ 1 + \left( \sum_{j=0}^{N(r)-1} \left| \frac{a_{N(r)-j}}{a_0} \right|^p \right)^{\frac{1}{p}} \right]^{\frac{1}{q}} < |z| < \left[ 1 + \left( \sum_{j=0}^{N(r)-1} \left| \frac{a_j}{a_{N(r)}} \right|^p \right)^{\frac{1}{p}} \right]^{\frac{1}{q}}.
\]

This proves the theorem. ■

The next theorem can be carried out by using Lemma 10.2.2 and in the way of Theorem 10.3.5 and therefore its proof is omitted.
Theorem 10.3.6 Let \( P(z) \) be an entire function defined by
\[
P(z) = a_0 + a_1 z + a_2 z^2 + \ldots + a_n z^n + \ldots
\]
with \( L^\ast \)-order \( \rho^{L^\ast} \). Also for all sufficiently large \( r \) in the disc \( |z| \leq \text{re}^{L(r)} \), \( a_{N(r)} \neq 0, a_0 \neq 0 \) and \( a_n \to 0 \) as \( n > N(r) \). For any \( p, q \) with \( p > 1, q > 1 \) and \( \frac{1}{p} + \frac{1}{q} = 1 \), all the zeros of \( P(z) \) lie in the annular region
\[
1 < |z| \left( 1 + \left( \sum_{j=0}^{N(r)-1} \left| \frac{a_{N(r)-j}}{a_0} \right|^p \right)^{\frac{1}{p}} \right)^{\frac{1}{q}} \leq \left( 1 + \left( \sum_{j=0}^{N(r)-1} \left| \frac{a_{N(r)-j}}{a_0} \right|^p \right) \right)^{\frac{1}{q}}.
\]

Corollary 10.3.1 In particular if we take \( p = 2, q = 2 \) in Theorem 10.3.5 then we get that all the zeros of the polynomial
\[
P(z) = a_0 + a_1 z + a_2 z^2 + \ldots + a_n z^n
\]
lie in the ring shaped region
\[
1 < |z| \left( 1 + \left( \sum_{j=0}^{n} \left| \frac{a_j}{a_n} \right|^2 \right) \right)^{\frac{1}{2}} \leq \left( 1 + \left( \sum_{j=0}^{n} \left| \frac{a_j}{a_n} \right|^2 \right) \right)^{\frac{1}{2}}.
\]

Corollary 10.3.2 In particular if we take \( p = 2, q = 2 \) in Theorem 10.3.6 then we get that all the zeros of the polynomial
\[
P(z) = a_0 + a_1 z + a_2 z^2 + \ldots + a_n z^n
\]
lie in the ring shaped region
\[
1 < |z| \left( 1 + \left( \sum_{j=0}^{n} \left| \frac{a_j}{a_n} \right|^2 \right) \right)^{\frac{1}{2}} \leq \left( 1 + \left( \sum_{j=0}^{n} \left| \frac{a_j}{a_n} \right|^2 \right) \right)^{\frac{1}{2}}.
\]

Theorem 10.3.7 Let \( P(z) \) be an entire function defined by
\[
P(z) = a_0 + a_1 z + a_2 z^2 + \ldots + a_n z^n + \ldots
\]
with \( L \)-order \( \rho^L \). Also for all sufficiently large \( r \) in the disc \( |z| \leq |rL(r)|, |a_{N(r)}| \neq 0, |a_0| \neq 0 \) and also \( a_n \to 0 \) as \( n > N(r) \). Then all the zeros of \( P(z) \) lie in the ring shaped region
\[
\frac{1}{t_0} \leq |z| \leq t_0
\]
where $t_0$ is the greatest positive root of
\[ g(t) \equiv \left| a_{N(r)} \right| t^{N(r)+1} - \left( \left| a_{N(r)} \right| + M \right) t^{N(r)} + M = 0 \]
and $t'_0$ is the greatest positive root of
\[ f(t) \equiv \left| a_0 \right| t^{N(r)+1} - \left( \left| a_0 \right| + M' \right) t^{N(r)} + M' = 0 \]
where $M = \max \left\{ \left| a_0 \right|, \left| a_1 \right|, \ldots, \left| a_{N(r)-1} \right| \right\}$
and $M' = \max \left\{ \left| a_1 \right|, \left| a_2 \right|, \ldots, \left| a_{N(r)} \right| \right\}$.

**Proof.**

Now
\[ P(z) \approx a_0 + a_1 z + a_2 z^2 + \ldots + a_{N(r)} z^{N(r)} \]
because $N(r)$ exists for $|z| \leq |rL(r)|$; $r$ is sufficiently large and $a_n \to 0$ as $n > N(r)$. Then all the zeros of $P(z)$ lie in the ring shaped region given in Theorem 10.3.4 which we are to prove.

Now
\[
|P(z)| \approx \left| a_0 + a_1 z + a_2 z^2 + \ldots + a_{N(r)} z^{N(r)} \right| \\
\geq \left| a_{N(r)} \right| |z|^{N(r)} - \left| a_0 + a_1 z + a_2 z^2 + \ldots + a_{N(r)-1} z^{N(r)-1} \right|.
\]

Also
\[
\left| a_0 + a_1 z + a_2 z^2 + \ldots + a_{N(r)-1} z^{N(r)-1} \right| \\
\leq \left| a_0 \right| + \ldots + \left| a_{N(r)-1} \right| |z|^{N(r)-1} \\
\leq M \left( 1 + |z| + \ldots + |z|^{N(r)-1} \right) \\
= M \frac{|z|^{N(r)} - 1}{|z| - 1} \text{ if } |z| \neq 1.
\]

Therefore using (10.3.14) we obtain that
\[
|P(z)| \geq \left| a_{N(r)} \right| |z|^{N(r)} - M \frac{|z|^{N(r)} - 1}{|z| - 1}.
\]

Hence
\[
|P(z)| \geq 0 \text{ if } \left| a_{N(r)} \right| |z|^{N(r)} - M \frac{|z|^{N(r)} - 1}{|z| - 1} > 0
\]
i.e., if $\left| a_{N(r)} \right| |z|^{N(r)} > M \frac{|z|^{N(r)} - 1}{|z| - 1}$
i.e., if $\left| a_{N(r)} \right| |z|^{N(r)+1} - \left| a_{N(r)} \right| |z|^{N(r)} > M \left( |z|^{N(r)} - 1 \right)$
i.e., if $\left| a_{N(r)} \right| |z|^{N(r)+1} - \left| a_{N(r)} \right| |z|^{N(r)} - M |z|^{N(r)} + M > 0$
i.e., if $\left| a_{N(r)} \right| |z|^{N(r)+1} - \left( \left| a_{N(r)} \right| + M \right) |z|^{N(r)} + M > 0.$
Therefore on $|z| \neq 1$,

$$|P(z)| \geq 0 \text{ if } |a_{N(r)}| |z|^{N(r)+1} - (|a_{N(r)}| + M) |z|^{N(r)} + M > 0.$$  

Now let us consider

$$g(t) \equiv |a_{N(r)}| t^{N(r)+1} - (|a_{N(r)}| + M) t^{N(r)} + M = 0. \quad (10.3.15)$$

Clearly the maximum number of changes in sign in (10.3.15) is two. So the maximum number of positive roots of $g(t) = 0$ is two and by Descartes’ rule of sign if it is less, less by two. Clearly $t = 1$ is one positive root of (10.3.15). So $g(t) = 0$ must have another positive root $t_1$ (say).

Let us take $t_0 = \max \{1, t_1\}$. Clearly for $t > t_0$, $g(t) > 0$. If not, for some $t = t_2 > t_0$, $g(t_2) < 0$.

Now $g(t_2) < 0$ and $g(\infty) > 0$ imply that $g(t) = 0$ has another positive root in $(t_2, \infty)$ which gives a contradiction.

Therefore for $t > t_0$, $g(t) > 0$ and so $t_0 > 1$.

Hence $|P(z)| \geq 0$ for $|z| > t_0$.

Therefore all the zeros of $P(z)$ lie in the disc $|z| \leq t_0$. \quad (10.3.16)

Again let us consider

$$Q(z) = z^{N(r)} P \left( \frac{1}{z} \right) \approx z^{N(r)} \left\{ a_0 + \frac{a_1}{z} + \ldots + \frac{a_{N(r)}}{z^{N(r)}} \right\} = a_0 z^{N(r)} + a_1 z^{N(r)-1} + \ldots + a_{N(r)}$$

i.e., $|Q(z)| \geq |a_0| |z|^{N(r)} - |a_1 z^{N(r)-1} + \ldots + a_{N(r)}|$ for $|z| \neq 1$.

Now

$$|a_1 z^{N(r)-1} + \ldots + a_{N(r)}| \leq |a_1| |z|^{N(r)-1} + \ldots + |a_{N(r)}|$$

$$\leq M' \left( |z|^{N(r)-1} + \ldots + 1 \right)$$

$$= M' \left( \frac{|z|^{N(r)} - 1}{|z| - 1} \right) \text{ for } |z| \neq 1. \quad (10.3.17)$$

Using (10.3.17) we get that

$$|Q(z)| \geq |a_0| |z|^{N(r)} - |a_1 z^{N(r)-1} + \ldots + a_{N(r)}|$$

$$\geq |a_0| |z|^{N(r)} - M' \left( \frac{|z|^{N(r)} - 1}{|z| - 1} \right) \text{ for } |z| \neq 1.$$
Therefore for $|z| \neq 1$,

$$|Q(z)| \geq 0 \text{ if } |a_0||z|^{N(r)} - M' \left( \frac{|z|^{N(r)} - 1}{|z| - 1} \right) > 0$$

i.e., if $|a_0||z|^{N(r)} > M' \left| \frac{|z|^{N(r)} - 1}{|z| - 1} \right|

i.e., if $|a_0||z|^{N(r)+1} - |a_0||z|^{N(r)} - M'||z|^{N(r)} + M' > 0$

i.e., if $|a_0||z|^{N(r)+1} - (|a_0|+M')|z|^{N(r)} + M' > 0$.

So for $|z| \neq 1$,

$$|Q(z)| \geq 0 \text{ if } |a_0||z|^{N(r)+1} - (|a_0|+M')|z|^{N(r)} + M' > 0.$$ 

Let us consider

$$f(t) \equiv |a_0| t^{N(r)+1} - (|a_0|+M') t^{N(r)} + M' = 0.$$ 

Since the maximum number of changes of sign in $f(t)$ is two, the maximum number of positive roots of $f(t) = 0$ is two and by Descartes’ rule of sign if it is less, less by two. Clearly $t = 1$ is one positive root of $f(t) = 0$. So $f(t) = 0$ must have another positive root.

Let us take $t_0' = \max \{ 1, t_2 \}$ . Clearly for $t > t_0'$, $f(t) > 0$. If not, for some $t_3 > t_0'$, $f(t_3) < 0$.

Now $f(t_3) < 0$ and $f(\infty) > 0$ implies that $f(t) = 0$ have another positive root in the interval $(t_3, \infty)$ which is a contradiction.

Therefore for $t > t_0'$, $f(t) > 0$.

Also $t_0' \geq 1$. So $|Q(z)| \geq 0$ for $|z| > t_0'$.

Therefore $Q(z)$ does not vanish in $|z| > t_0'$.

Hence all the zeros of $Q(z)$ lie in $|z| \leq t_0'$.

Let $z = z_0$ be a zero of $P(z)$. Therefore $P(z_0) = 0$. Clearly $z_0 \neq 0$ as $a_0 \neq 0$.

Putting $z = \frac{1}{z_0}$ in $Q(z)$ we get that

$$Q \left( \frac{1}{z_0} \right) = \left( \frac{1}{z_0} \right)^{N(r)} P(z_0) = \left( \frac{1}{z_0} \right)^{N(r)} \cdot 0 = 0.$$ 

Therefore $Q \left( \frac{1}{z_0} \right) = 0$. So $z = \frac{1}{z_0}$ is a root of $Q(z) = 0$. Hence $\left| \frac{1}{z_0} \right| \leq t_0'$ implies that $|z_0| \geq \frac{1}{t_0'}$.

As $z_0$ is an arbitrary root of $P(z) = 0$.

Therefore all the zeros of $P(z)$ lie in $|z| \geq \frac{1}{t_0'}$. (10.3.18)

From (10.3.16) and (10.3.18), we get that all the zeros of $P(z)$ lie in the proper ring shaped region

$$\frac{1}{t_0'} \leq |z| \leq t_0$$

where $t_0$ and $t_0'$ are the greatest positive roots of the equations

$$g(t) \equiv |a_N(r)| t^{N(r)+1} - \left( |a_N(r)| + M \right) t^{N(r)} + M = 0$$
and

\[ f(t) \equiv |a_0| t^{N(r)+1} - (|a_0| + M') t^{N(r)} + M' = 0 \]

where \( M \) and \( M' \) are given in the statement of Theorem 10.3.4.

This completes the proof of the theorem. \( \blacksquare \)

**Theorem 10.3.8** Let \( P(z) \) be an entire function defined by

\[ P(z) = a_0 + a_1 z + a_2 z^2 + \ldots + a_n z^n + \ldots \]

with \( L^* \)-order \( \rho L^* \). Also for all sufficiently large \( r \) in the disc \( |z| \leq \left[ r e^{L(r)} \right] \), \( |a_{N(r)}| \neq 0 \), \( |a_0| \neq 0 \), and also \( a_n \to 0 \) as \( n > N(r) \). Then all the zeros of \( P(z) \) lie in the ring shaped region

\[ \frac{1}{t'_0} \leq |z| \leq t_0 \]

where \( t_0 \) is the greatest positive root of

\[ g(t) \equiv |a_{N(r)}| t^{N(r)+1} - (|a_{N(r)}| + M) t^{N(r)} + M = 0 \]

and \( t'_0 \) is the greatest positive root of

\[ f(t) \equiv |a_0| t^{N(r)+1} - (|a_0| + M') t^{N(r)} + M' = 0 \]

where \( M = \max \{ |a_0|, |a_1|, \ldots, |a_{N(r)-1}| \} \)

and \( M' = \max \{ |a_1|, |a_2|, \ldots, |a_{N(r)}| \} \).

We omit the proof of Theorem 10.3.8 because it can be carried out in the line of Theorem 10.3.7 and by using Lemma 10.2.2.

**Remark 10.3.3** The limit in Theorem 10.3.7 is attained by \( P(z) = z^2 - z - 1 \). Let \( L(r) = \log r \). Here

\[ \rho^L = \limsup_{r \to \infty} \frac{\log |a_2(r^2)|}{\log [r L(r)]} \]

\[ = \limsup_{r \to \infty} \frac{\log 2 + \log |a_2(r)|}{\log r + \log |a_2(r)|} \]

\[ = \limsup_{r \to \infty} \frac{\log 2 + \log |a_2(r)|}{\log r} \]

\[ = 0 \]

and
\[ **\rho^L = \limsup_{r \to \infty} \frac{\log(r^2)}{\log[rL(r)]} \]
\[ = \limsup_{r \to \infty} \frac{2 \log r}{\log r + \log^2 r} \]
\[ = \limsup_{r \to \infty} \frac{2}{\log^2 r \log r} = 2. \]

Further in view of Lemma 10.2.3, \( N(r) = 2 \leq (\log r)^{2+\epsilon} \) for \( \epsilon > 0 \) and sufficiently large \( r \). Also \( a_0 = -1, a_1 = -1, a_2 = 1 \) and all \( a_n = 0, n > 2 \).

Therefore

\[ M = \max \{|a_0|, |a_1|\} = 1 \text{ and } M' = \max \{|a_1|, |a_2|\} = 1 \]

and

\[ g(t) \equiv |a_2| t^3 - (|a_2| + M) t^2 + M = 0 \]
\[ \text{i.e., } g(t) \equiv t^3 - (1 + 1) t^2 + 1 = 0 \]
\[ \text{i.e., } g(t) \equiv t^3 - 2t^2 + 1 = 0. \]

Again

\[ f(t) \equiv |a_0| t^3 - (|a_0| + M') t^2 + M' = 0 \]
\[ \text{i.e., } f(t) \equiv 1 \cdot t^3 - (1 + 1) t^2 + 1 = 0 \]
\[ \text{i.e., } f(t) \equiv t^3 - 2t^2 + 1 = 0. \]

So \( f(t) = 0 \) and \( g(t) = 0 \) represent the same equation. Therefore the maximum number of positive roots of \( f(t) = 0 \) and \( g(t) = 0 \) are same. Now

\[ g(t) = 0 \]
\[ \text{implies that } t^3 - 2t^2 + 1 = 0 \]
\[ \text{i.e., } (t - 1)(t^2 - t - 1) = 0. \]

Therefore

\[ t = 1 \text{ and } t = \frac{1 \pm \sqrt{(-1)^2 - 4.1.(-1)}}{2.1} = \frac{1 \pm \sqrt{5}}{2}. \]

Hence the positive roots of \( g(t) = 0 \) are 1 and \( \frac{1 + \sqrt{5}}{2} \). So

\[ t_0 = \max \left\{ 1, \frac{1 + \sqrt{5}}{2} \right\} = \frac{1 + \sqrt{5}}{2}. \]

Also the maximum positive root of \( f(t) = 0 \) is

\[ t_0' = \max \left\{ 1, \frac{1 + \sqrt{5}}{2} \right\} = \frac{1 + \sqrt{5}}{2}. \]
So in view of Theorem 10.3.4 all the zeros of \( P(z) \) lie in

\[
\frac{1}{t_0} \leq |z| \leq t_0
\]

i.e., \( \frac{1}{1 + \sqrt{5}/2} \leq |z| \leq \frac{1 + \sqrt{5}}{2} \)

i.e., \( \sqrt{5} - 1 \leq |z| \leq \frac{1 + \sqrt{5}}{2} \),

which has been shown in Figure 2 in the APPENDIX given at the end of the chapter.

Now the zeros of \( P(z) \) are given by solving \( z^2 - z - 1 = 0 \). Therefore \( z = \frac{1 \pm \sqrt{5}}{2} \).

Let us denote the zeros of \( P(z) \) by \( z_1 = \frac{1 + \sqrt{5}}{2} \) and \( z_2 = \frac{1 - \sqrt{5}}{2} = -\frac{\sqrt{5}-1}{2} \). Clearly \( z_1 \) lies on the upper boundary and \( z_2 \) lies on the lower boundary. So the best possible result is given by

\[
P(z) = z^2 - z - 1.
\]

**Theorem 10.3.9** Let \( P(z) \) be an entire function defined by

\[
P(z) = a_0 + a_1 z + a_2 z^2 + \ldots + a_n z^n + \ldots
\]

with L-order \( \rho_L \), \( a_{N(r)} \neq 0 \), \( a_0 \neq 0 \) and also \( a_n \to 0 \) for \( n > N(r) \) for the disc \( |z| \leq \rho L(r) \) when \( r \) is sufficiently large. Further for \( \rho_L > 0 \),

\[
|a_0| (\rho_L)^{N(r)} \geq |a_1| (\rho_L)^{N(r)-1} \geq \ldots \geq |a_{N(r)-1}| \rho_L \geq |a_{N(r)}|.
\]

Then all the zeros of \( P(z) \) lie in the ring shaped region

\[
\frac{1}{\rho_L(1 + \frac{|a_1|}{|a_0|})} < |z| < \frac{1}{\rho_L} \left( 1 + \frac{|a_0|}{|a_{N(r)}|} (\rho_L)^{N(r)} \right).
\]

**Proof.** For the given entire function

\[
P(z) = a_0 + a_1 z + a_2 z^2 + \ldots + a_n z^n + \ldots
\]

with \( a_n \to 0 \) as \( n > N(r) \), where \( r \) is sufficiently large, \( N(r) \) exists and \( N(r) \leq \rho L(r) \rho^{r+\epsilon} \).

Therefore

\[
P(z) \approx a_0 + a_1 z + a_2 z^2 + \ldots + a_{N(r)} z^{N(r)}
\]

as \( a_0 \neq 0, a_{N(r)} \neq 0 \) and \( a_n \to 0 \) for \( n > N(r) \).

Let us consider

\[
R(z) = (\rho_L)^{N(r)} P \left( \frac{z}{\rho_L} \right)
\]

\[
\approx (\rho_L)^{N(r)} \left( a_0 + a_1 \frac{z}{\rho_L} + a_2 \left( \frac{z}{\rho_L} \right)^2 + \ldots + a_{N(r)} \frac{z^{N(r)}}{(\rho_L)^{N(r)}} \right)
\]

\[
= \left( a_0 (\rho_L)^{N(r)} + a_1 (\rho_L)^{N(r)} - 1 z + \ldots + a_{N(r)} z^{N(r)} \right).
\]
Therefore
\[ |R(z)| \geq |a_{N(r)}| |z|^r - |a_0| (\rho^r)^r |z|^r \left( \frac{1}{|z|} + \ldots + \frac{1}{|z|^r} \right) \]
	\[ \geq |a_{N(r)}| |z|^r - |a_0| (\rho^r)^r |z|^r \left( \frac{1}{|z|} + \ldots + \frac{1}{|z|^r} + \ldots \right) \]
	\[ = |z|^r \left[ |a_{N(r)}| - |a_0| (\rho^r)^r \sum_{k=1}^{\infty} \frac{1}{|z|^k} \right] . \]

Clearly \( \sum_{k=1}^{\infty} \frac{1}{|z|^k} \) is a geometric series which is convergent for \( \frac{1}{|z|} < 1 \) i.e., for \( |z| > 1 \) and converges to
\[ \frac{1}{|z|} \left( 1 - \frac{1}{|z|} \right) = \frac{1}{|z| - 1} . \]

Therefore
\[ \sum_{k=1}^{\infty} \frac{1}{|z|^k} = \frac{1}{|z| - 1} \text{ if } |z| > 1. \]

Hence we get from above that for \( |z| > 1 \)
\[ |R(z)| > |z|^r \left( |a_{N(r)}| - (\rho^r)^r |a_0| \frac{1}{|z| - 1} \right) . \]
Now for \(|z| > 1\),

\[ |R(z)| > 0 \text{ if } |z|^{N(r)} \left( |a_{N(r)}| - (\rho^L)^{N(r)} |a_0| \frac{1}{|z| - 1} \right) \geq 0 \]

i.e., if \( |a_{N(r)}| - (\rho^L)^{N(r)} |a_0| \frac{1}{|z| - 1} \geq 0 \)

i.e., if \( |a_{N(r)}| \geq (\rho^L)^{N(r)} \frac{|a_0|}{|z| - 1} \)

i.e., if \(|z| - 1 \geq (\rho^L)^{N(r)} \frac{|a_0|}{|a_{N(r)}|} \)

i.e., if \(|z| \geq 1 + (\rho^L)^{N(r)} \frac{|a_0|}{|a_{N(r)}|} > 1 \).

Therefore

\[ |R(z)| > 0 \text{ if } |z| \geq 1 + (\rho^L)^{N(r)} \frac{|a_0|}{|a_{N(r)}|}. \]

So all the zeros of \( R(z) \) lie in

\[ |z| < 1 + \frac{|a_0|}{|a_{N(r)}|} (\rho^L)^{N(r)}. \]

Let \( z_0 \) be an arbitrary zero of \( P(z) \). Therefore \( P(z_0) = 0 \). Clearly \( z_0 \neq 0 \) as \( a_0 \neq 0 \). Putting \( z = \rho^L z_0 \) in \( R(z) \) we have

\[ R(\rho^L z_0) = (\rho^L)^{N(r)} P(z_0) = (\rho^L)^{N(r)} 0 = 0. \]

Hence \( z = \rho^L z_0 \) is a zero of \( R(z) \). Therefore

\[ |\rho^L z_0| < 1 + \frac{|a_0|}{|a_{N(r)}|} (\rho^L)^{N(r)} \]

i.e., \( |z_0| < \frac{1}{\rho^L} \left( 1 + \frac{|a_0|}{|a_{N(r)}|} (\rho^L)^{N(r)} \right). \)

Since \( z_0 \) is any zero of \( P(z) \), therefore all the zeros of \( P(z) \) lie in

\[ |z| < \frac{1}{\rho^L} \left( 1 + \frac{|a_0|}{|a_{N(r)}|} (\rho^L)^{N(r)} \right). \quad (10.3.21) \]

Again let us consider

\[ F(z) = (\rho^L)^{N(r)} z^{N(r)} P \left( \frac{1}{\rho^L z} \right). \]

Now

\[ F(z) = (\rho^L)^{N(r)} z^{N(r)} P \left( \frac{1}{\rho^L z} \right) \approx (\rho^L)^{N(r)} z^{N(r)} \left\{ a_0 + \frac{a_1}{\rho^L z} + \ldots + \frac{a_{N(r)}}{(\rho^L z)^{N(r)}} \right\} \]

\[ = a_0 (\rho^L)^{N(r)} z^{N(r)} + a_1 (\rho^L)^{N(r)-1} z^{N(r)-1} + \ldots + a_{N(r)}. \]
Therefore

\[ |F(z)| \geq |a_0| (\rho L)^N(r) |z|^N(r) - |a_1 (\rho L)^N(r) - 1 \cdot z^{N(r) - 1} + \ldots + a_{N(r)}| . \]

Again

\[
|a_1 (\rho L)^N(r) - 1 \cdot z^{N(r) - 1} + \ldots + a_{N(r)}|
\leq |a_1| (\rho L)^{N(r) - 1} |z|^N(r) - 1 + \ldots + |a_{N(r)}|
\leq |a_1| (\rho L)^{N(r) - 1} \left( |z|^N(r) - 1 + \ldots + |z| + 1 \right) \text{ provided } |z| \neq 0.
\]

So

\[
|a_1 (\rho L)^N(r) - 1 \cdot z^{N(r) - 1} + \ldots + a_{N(r)}|
\leq |a_1| (\rho L)^{N(r) - 1} |z|^N(r) \left( \frac{1}{|z|} + \ldots + \frac{1}{|z|^N(r)} \right) .
\]

So for \( |z| \neq 0, \)

\[
|F(z)| \geq |a_0| (\rho L)^N(r) |z|^N(r) - |a_1| (\rho L)^{N(r) - 1} |z|^N(r) \left( \frac{1}{|z|} + \ldots + \frac{1}{|z|^N(r)} \right)
= (\rho L)^{N(r) - 1} |z|^N(r) \left[ |a_0| \rho - |a_1| \left( \frac{1}{|z|} + \ldots + \frac{1}{|z|^N(r)} \right) \right] .
\]

Therefore for \( |z| \neq 0, \)

\[
|F(z)| \geq (\rho L)^{N(r) - 1} |z|^N(r) \left[ |a_0| \rho - |a_1| \left( \sum_{k=1}^{\infty} \frac{1}{|z|^k} \right) \right] . \quad (10.3.22)
\]

The geometric series \( \sum_{k=1}^{\infty} \frac{1}{|z|^k} \) is convergent for \( \frac{1}{|z|} < 1 \)

\[ i.e., \text{ for } |z| > 1 \]

and converges to \( \frac{1}{|z| - 1} \)

\[
\sum_{k=1}^{\infty} \frac{1}{|z|^k} = \frac{1}{|z| - 1} \text{ if } |z| > 1 . \quad (10.3.23)
\]
Using (10.3.22) and (10.3.23) we have for $|z| > 1$,

$$|F(z)| > (\rho^L)^{N(r)-1} |z|^{N(r)} \left[ |a_0| \rho^L - \frac{|a_1|}{|z| - 1} \right].$$

Hence for $|z| > 1$,

$$|F(z)| > 0 \text{ if } |z|^{N(r)} (\rho^L)^{N(r)-1} \left[ |a_0| \rho^L - \frac{|a_1|}{|z| - 1} \right] \geq 0$$

i.e., if $|a_0| \rho^L - \frac{|a_1|}{|z| - 1} \geq 0$

i.e., if $|a_0| \rho^L \geq \frac{|a_1|}{|z| - 1}$

i.e., if $|z| \geq 1 + \frac{|a_1|}{|a_0| \rho^L} > 1$.

Therefore

$$|F(z)| > 0 \text{ for } |z| > 1 + \frac{|a_1|}{|a_0| \rho^L}.$$ 

So $F(z)$ does not vanish in

$$|z| \geq 1 + \frac{|a_1|}{|a_0| \rho^L}.$$ 

Equivalently all the zeros of $F(z)$ lie in

$$|z| < 1 + \frac{|a_1|}{|a_0| \rho^L}.$$ 

Let $z = z_0$ be any zero of $P(z)$. Therefore $P(z_0) = 0$. Clearly $a_0 \neq 0$ and $z_0 \neq 0$.

Now let us put $z = \frac{1}{\rho^L z_0}$ in $F(z)$. So we have

$$F \left( \frac{1}{\rho^L z_0} \right) = (\rho^L)^{N(r)} \left( \frac{1}{\rho^L z_0} \right)^{N(r)} P(z_0)$$

$$= \left( \frac{1}{z_0} \right)^{N(r)} \cdot 0 = 0.$$ 

Therefore $z = \frac{1}{\rho^L z_0}$ is a root of $F(z)$.

Hence

$$\left| \frac{1}{\rho^L z_0} \right| < 1 + \frac{|a_1|}{|a_0| \rho^L}$$

i.e.,

$$\left| \frac{1}{z_0} \right| < \rho^L \left( 1 + \frac{|a_1|}{|a_0| \rho^L} \right)$$

i.e.,

$$|z_0| > \frac{1}{\rho^L \left( 1 + \frac{|a_1|}{|a_0| \rho^L} \right)}.$$
As \( z_0 \) is an arbitrary zero of \( P(z) \), all the zeros of \( P(z) \) lie on

\[
|z| > \frac{1}{\rho^L \left(1 + \frac{|a_1|}{|a_0| \rho^L}\right)}.
\]

From (10.3.21) and (10.3.24), we get that all the zeros of \( P(z) \) lie on the proper ring shaped region

\[
\frac{1}{\rho^L \left(1 + \frac{|a_1|}{|a_0| \rho^L}\right)} < |z| < \frac{1}{\rho^L} \left(1 + \frac{|a_0|}{|a_{N(r)}| (\rho^L)^{N(r)}\right)}
\]

where

\[
|a_0| (\rho^L)^{N(r)} \geq |a_1| (\rho^L)^{N(r)-1} \geq \ldots \geq |a_{N(r)}|
\]

for \( \rho^L > 0 \).

This proves the theorem. ■

In the line of Theorem 10.3.10, we may state the following theorem in view of Lemma 10.2.2:

**Theorem 10.3.10** Let \( P(z) \) be an entire function defined by

\[
P(z) = a_0 + a_1 z + a_2 z^2 + \ldots + a_n z^n + \ldots
\]

with \( L^* \)-order \( \rho^{L^*} \), \( a_{N(r)} \neq 0 \), \( a_0 \neq 0 \) and also \( a_n \to 0 \) for \( n > N(r) \) for the disc \( |z| \leq \left| r e^{L(r)} \right| \) when \( r \) is sufficiently large. Further for \( \rho^{L^*} > 0 \),

\[
|a_0| (\rho^{L^*})^{N(r)} \geq |a_1| (\rho^{L^*})^{N(r)-1} \geq \ldots \geq |a_{N(r)}-1| \rho^{L^*} \geq |a_{N(r)}|.
\]

Then all the zeros of \( P(z) \) lie in the ring shaped region

\[
\frac{1}{\rho^{L^*} \left(1 + \frac{|a_1|}{|a_0| \rho^{L^*}}\right)} < |z| < \frac{1}{\rho^{L^*} \left(1 + \frac{|a_0|}{|a_{N(r)}| (\rho^{L^*})^{N(r)}}\right)}.
\]

The proof is omitted.

**Corollary 10.3.3** From Theorem 10.3.9 we can easily conclude that all the zeros of

\[
P(z) = a_0 + a_1 z + \ldots + a_n z^n
\]

of degree \( n \), \( |a_n| \neq 0 \) with the property \( |a_0| \geq |a_1| \geq \ldots \geq |a_n| \) lie in the proper ring shaped region

\[
\frac{1}{\left(1 + \frac{|a_1|}{|a_0|}\right)} < |z| < \left(1 + \frac{|a_0|}{|a_n|}\right)
\]

just on putting \( \rho^L = 1 \).
Corollary 10.3.4 From Theorem 10.3.10 we can easily conclude that all the zeros of
\[ P(z) = a_0 + a_1 z + \ldots + a_n z^n \]
of degree \( n, |a_n| \neq 0 \) with the property \( |a_0| \geq |a_1| \geq \ldots \geq |a_n| \) lie in the proper ring shaped region
\[ \frac{1}{1 + \frac{|a_1|}{|a_0|}} < |z| < \left( 1 + \frac{|a_0|}{|a_n|} \right)^{1/r} \]
just on putting \( \rho^{L^*} = 1 \).

Theorem 10.3.11 Let \( P(z) \) be an entire function with \( L \)-order \( \rho^L \). For sufficiently large values of \( r \) in the disc \( |z| \leq [rL(r)] \), the Taylor’s series expansion of \( P(z) \)
\[ P(z) = a_0 + a_{p_1} z^{p_1} + a_{p_2} z^{p_2} + \ldots + a_{p_m} z^{p_m} + a_{N(r)} z^{N(r)}; a_0 \neq 0 \]
be such that \( 1 \leq p_1 < p_2 \ldots \ldots \leq p_m \leq N(r) - 1, p_i \)'s are integers and also for \( \rho^L > 0 \),
\[ |a_0| (\rho^L)^{N(r)} \geq |a_{p_1}| (\rho^L)^{N(r) - p_1} \geq \ldots \ldots \geq |a_{p_m}| (\rho^L)^{N(r) - p_m}. \]
Then all the zeros of \( P(z) \) lie in the proper ring shaped region
\[ \frac{1}{\rho^L t_0'} < |z| < \frac{t_0}{\rho^L} \]
where \( t_0 \) and \( t_0' \) are the unique positive roots of the equations
\[ g(t) \equiv |a_{N(r)}| t^{N(r) - p_m} - |a_{N(r)}| t^{N(r) - p_m - 1} - |a_0| (\rho^L)^{N(r)} = 0 \text{ and} \]
\[ f(t) \equiv |a_0| (\rho^L)^{p_1} t^{p_1} - |a_0| (\rho^L)^{p_1} t^{p_1 - 1} - |a_{p_1}| = 0 \]
respectively.

Proof. Let
\[ P(z) = a_0 + a_{p_1} z^{p_1} + \ldots + a_{p_m} z^{p_m} + a_{N(r)} z^{N(r)}; |a_{N(r)}| \neq 0. \tag{10.3.25} \]
Also for some \( \rho^L > 0 \),
\[ |a_0| (\rho^L)^{N(r)} \geq |a_{p_1}| (\rho^L)^{N(r) - p_1} \geq \ldots \ldots \geq |a_{N(r)}|. \]
Let us consider that
\[ R(z) = (\rho^L)^{N(r)} P \left( \frac{z}{\rho^L} \right) \]
\[ = (\rho^L)^{N(r)} \left\{ a_0 + a_{p_1} \frac{z^{p_1}}{(\rho^L)^{p_1}} + \ldots + a_{p_m} \frac{z^{p_m}}{(\rho^L)^{p_m}} + a_{N(r)} \frac{z^{N(r)}}{(\rho^L)^{N(r)}} \right\} \]
\[ = a_0 (\rho^L)^{N(r)} + a_{p_1} (\rho^L)^{N(r) - p_1} z^{p_1} + \ldots + a_{p_m} (\rho^L)^{N(r) - p_m} z^{p_m} + a_{N(r)} z^{N(r)}. \]
Therefore
\[ |R(z)| \geq \left| a_{N(r)} z^{N(r)} \right| - \left| a_0 \left( \rho^L \right)^{N(r) - p_1} z^{p_1} + \ldots + a_{p_m} \left( \rho^L \right)^{N(r) - p_m} z^{p_m} \right|. \] (10.3.26)

Now for \( |z| \neq 0 \),
\[ \left| a_0 \left( \rho^L \right)^{N(r)} + a_{p_1} \left( \rho^L \right)^{N(r) - p_1} z^{p_1} + \ldots + a_{p_m} \left( \rho^L \right)^{N(r) - p_m} z^{p_m} \right| \leq |a_0| \left( \rho^L \right)^{N(r)} + |a_{p_1}| \left( \rho^L \right)^{N(r) - p_1} |z|^{p_1} + \ldots + |a_{p_m}| \left( \rho^L \right)^{N(r) - p_m} |z|^{p_m} \]
\[ \leq |a_0| \left( \rho^L \right)^{N(r)} (1 + |z|^{p_1} + \ldots + |z|^{p_m}) \]
\[ = |a_0| \left( \rho^L \right)^{N(r)} |z|^{p_m + 1} \left( \frac{1}{|z|} + \ldots + \frac{1}{|z|^{p_m+1-p_1}} + \frac{1}{|z|^{p_m+1}} \right). \] (10.3.27)

Using (10.3.26) and (10.3.27), we have for \( |z| \neq 0 \)
\[ |R(z)| \geq |a_{N(r)}| |z|^{N(r)} - |a_0| \left( \rho^L \right)^{N(r)} |z|^{p_m+1} \left( \frac{1}{|z|} + \ldots + \frac{1}{|z|^{p_m+1-p_1}} + \frac{1}{|z|^{p_m+1}} \right) \]
\[ > |a_{N(r)}| |z|^{N(r)} - |a_0| \left( \rho^L \right)^{N(r)} |z|^{p_m+1} \left( \frac{1}{|z|} + \ldots + \frac{1}{|z|^{p_m+1-p_1}} + \frac{1}{|z|^{p_m+1}} + \ldots \right) \]
\[ = |a_{N(r)}| |z|^{N(r)} - |a_0| \left( \rho^L \right)^{N(r)} |z|^{p_m+1} \sum_{k=1}^{\infty} \frac{1}{|z|^k}. \] (10.3.28)

The geometric series \( \sum_{k=1}^{\infty} \frac{1}{|z|^k} \) is convergent for
\[ \frac{1}{|z|} < 1 \]
i.e., for \( |z| > 1 \)
and converges to
\[ \frac{1}{|z|} \frac{1}{1 - \frac{1}{|z|}} = \frac{1}{|z| - 1}. \]

Therefore
\[ \sum_{k=1}^{\infty} \frac{1}{|z|^k} = \frac{1}{|z| - 1} \text{ for } |z| > 1. \]

So on \( |z| > 1 \),
\[ |R(z)| > 0 \text{ if } |a_{N(r)}| |z|^{N(r)} - \frac{|a_0| \left( \rho^L \right)^{N(r)} |z|^{p_m+1}}{|z| - 1} \geq 0 \]
i.e., if \[ |a_{N(r)}| |z|^{N(r)} \geq \frac{|a_0| \left( \rho^L \right)^{N(r)} |z|^{p_m+1}}{|z| - 1} \]
i.e., if \[ |a_{N(r)}| |z|^{N(r)+1} - |a_{N(r)}| |z|^{N(r)} \geq |a_0| \left( \rho^L \right)^{N(r)} |z|^{p_m+1} \]
i.e., if \[ |z|^{p_m+1} \left( |a_{N(r)}| |z|^{N(r)-p_m} - |a_{N(r)}| |z|^{N(r)-p_{m-1}} - |a_0| \left( \rho^L \right)^{N(r)} \right) \geq 0. \]
Let us consider

\[ g(t) \equiv |a_{N(r)}| |t|^{N(r) - p_m} - |a_{N(r)}| |t|^{N(r) - p_m - 1} - |a_0| (\rho^L)^{N(r)} = 0. \]

Clearly \( g(t) = 0 \) has one positive root because the maximum number of changes in sign in \( g(t) \) is one and \( g(0) = -|a_0| (\rho^L)^{N(r)} \) is \(-ve\), \( g(\infty) \) is \(+ve\).

Let \( t_0 \) be the positive root of \( g(t) = 0 \) and \( t_0 > 1 \). Clearly for \( t > t_0, g(t) \geq 0 \). If not for some \( t_1 > t_0, g(t_1) < 0 \).

Then \( g(t_1) < 0 \) and \( g(\infty) > 0 \). Therefore \( g(t) = 0 \) must have another positive root in \((t_1, \infty)\) which gives a contradiction.

Hence for \( t \geq t_0, g(t) \geq 0 \) and \( t_0 > 1 \). So \( |R(z)| > 0 \) for \( |z| \geq t_0 \).

Thus \( R(z) \) does not vanish in \( |z| \geq t_0 \).

Hence all the zeros of \( R(z) \) lie in \( |z| < t_0 \).

Let \( z = z_0 \) be any zero of \( P(z) \). So \( P(z_0) = 0 \). Clearly \( z_0 \neq 0 \) as \( a_0 \neq 0 \).

Putting \( z = \rho^L z_0 \) in \( R(z) \) we have

\[ R(\rho^L z_0) = (\rho^L)^{N(r)} P(z_0) = (\rho^L)^{N(r)} 0 = 0. \]

Therefore \( R(\rho^L z_0) = 0 \) and so \( z = \rho^L z_0 \) is a zero of \( R(z) \) and consequently \( |\rho^L z_0| < t_0 \) which implies \( |z_0| < \frac{t_0}{|\rho^L|} \). As \( z_0 \) is an arbitrary zero of \( P(z) \),

all the zeros of \( P(z) \) lie in \( |z| < \frac{t_0}{|\rho^L|} \).

Again let us consider

\[ F(z) = (\rho^L)^{N(r)} z^{N(r)} P \left( \frac{1}{\rho^L z} \right). \]

Now

\[
F(z) = (\rho^L)^{N(r)} z^{N(r)} \left\{ a_0 + a_{p_1} \left( \frac{1}{\rho^L} \right) z^{p_1} + \ldots + a_{p_m} \left( \frac{1}{\rho^L} \right)^m z^{p_m} + a_{N(r)} \left( \frac{1}{\rho^L} \right)^{N(r)} z^{N(r)} \right\}
\]

\[ = a_0 (\rho^L)^{N(r)} z^{N(r)} + a_{p_1} (\rho^L)^{N(r) - p_1} z^{N(r) - p_1} + \ldots + a_{p_m} (\rho^L)^{N(r) - p_m} z^{N(r) - p_m} + a_{N(r)}. \]

Also

\[
|a_{p_1} (\rho^L)^{N(r) - p_1} z^{N(r) - p_1} + \ldots + a_{p_m} (\rho^L)^{N(r) - p_m} z^{N(r) - p_m} + a_{N(r)}|
\]

\[ \leq |a_{p_1} (\rho^L)^{N(r) - p_1} |z|^{N(r) - p_1} + \ldots + |a_{p_m} (\rho^L)^{N(r) - p_m} |z|^{N(r) - p_m} + |a_{N(r)}|
\]

\[ \leq |a_{p_1} (\rho^L)^{N(r) - p_1} \left( |z|^{N(r) - p_1} + |z|^{N(r) - p_2} + \ldots + |z|^{N(r) - p_m} + 1 \right). \]
Therefore for $|z| \neq 0$,
\[
|F(z)| \\
\geq |a_0| (\rho^L)^{N(r)} |z|^{N(r)} \\
- |a_{p_1} (\rho^L)^{N(r)-p_1} z^{N(r)-p_1} + \ldots + a_{p_m} (\rho^L)^{N(r)-p_m} z^{N(r)-p_m} + a_{N(r)}| \\
\geq |a_0| (\rho^L)^{N(r)} |z|^{N(r)} - |a_{p_1} (\rho^L)^{N(r)-p_1} \left(|z|^{N(r)-p_1} + \ldots + |z|^{N(r)-p_m} + 1 \right) \\
= |a_0| (\rho^L)^{N(r)} |z|^{N(r)} \\
- |a_{p_1} (\rho^L)^{N(r)-p_1} |z|^{N(r)-p_1+1} \left(\frac{1}{|z|} + \frac{1}{|z|^{p_2-p_1+1}} + \ldots + \frac{1}{|z|^{N(r)-p_1+1}} \right)
\]
i.e., on $|z| \neq 0$,
\[
|F(z)| > |a_0| (\rho^L)^{N(r)} |z|^{N(r)} - |a_{p_1} (\rho^L)^{N(r)-p_1} |z|^{N(r)-p_1+1} \left(\sum_{k=1}^{\infty} \frac{1}{|z|^k} \right).
\]
The geometric series $\sum_{k=1}^{\infty} \frac{1}{|z|^k}$ is convergent for
\[
\frac{1}{|z|} < 1
\]
i.e., for $|z| > 1$
and converges to
\[
\frac{1}{|z|} \frac{1}{1 - \frac{1}{|z|}} = \frac{1}{|z| - 1}.
\]
Therefore
\[
\sum_{k=1}^{\infty} \frac{1}{|z|^k} = \frac{1}{|z| - 1} \quad \text{for} \quad |z| > 1.
\]
Therefore for $|z| > 1$,
\[
|F(z)| > |a_0| (\rho^L)^{N(r)} |z|^{N(r)} - |a_{p_1} (\rho^L)^{N(r)-p_1} |z|^{N(r)-p_1+1} \left(\frac{1}{|z| - 1} \right) \\
= (\rho^L)^{N(r)-p_1} \left(\frac{(\rho^L)^{p_1} |a_0| |z|^{N(r)} - |a_{p_1} | |z|^{N(r)-p_1+1}}{|z| - 1} \right) \\
= (\rho^L)^{N(r)-p_1} |z|^{N(r)-p_1+1} \left(|a_0| (\rho^L)^{p_1} |z|^{p_1-1} - \frac{|a_{p_1}|}{|z| - 1} \right)
\]
For $|z| > 1$,
\[
|F(z)| > 0 \quad \text{if} \quad |a_0| (\rho^L)^{p_1} |z|^{p_1-1} - \frac{|a_{p_1}|}{|z| - 1} \geq 0
\]
i.e., if
\[
|a_0| (\rho^L)^{p_1} |z|^{p_1-1} \geq \frac{|a_{p_1}|}{|z| - 1}
\]
i.e., if
\[
|a_0| (\rho^L)^{p_1} |z|^{p_1-1} - |a_0| (\rho^L)^{p_1} |z|^{p_1-1} - |a_{p_1}| \geq 0. \quad (10.3.30)
\]
Therefore on $|z| > 1, |F(z)| > 0$ if (10.3.30) holds.

Let us consider

$$f(t) = |a_0| (\rho^L)^{p_1} t^{p_1} - |a_0| (\rho^L)^{p_1 - 1} - |a_{p_1}| = 0.$$}

Clearly $f(t) = 0$ has exactly one positive root and is greater than one. Let $t'_0$ be the positive root of $f(t) = 0$. Therefore $t'_0 > 1$. Obviously if $t \geq t'_0$ then $f(t) \geq 0$. So for $|F(z)| > 0$, $|z| \geq t'_0$. Therefore $F(z)$ does not vanish in $|z| \geq t'_0$.

Hence all the zeros of $F(z)$ lie in $|z| < t'_0$.

Let $z = z_0$ be any zero of $P(z)$. Therefore $P(z_0) = 0$. Clearly $z_0 \neq 0$ as $a_0 \neq 0$.

Now putting $z = \frac{1}{\rho L z_0}$ in $F(z)$ we obtain that

$$F\left(\frac{1}{\rho L z_0}\right) = (\rho L)^{N(r)} \left(\frac{1}{\rho L z_0}\right)^{N(r)} P(z_0)$$

$$= \left(\frac{1}{z_0}\right)^{N(r)} P(z_0) = 0.$$}

Therefore $z = \frac{1}{\rho L z_0}$ is a zero of $F(z)$. Now

$$\frac{1}{\rho L z_0} < t'_0$$

i.e., $|1| < \rho L t'_0$

i.e., $|z_0| > \frac{1}{\rho L t'_0}$

As $z_0$ is an arbitrary zero of $P(z)$ therefore we obtain that

$$\text{all the zeros of } P(z) \text{ lie in } |z| > \frac{1}{\rho L t'_0}. \quad (10.3.31)$$

Using (10.3.29) and (10.3.31), we get that all the zeros of $P(z)$ lie in the ring shaped region

$$\frac{1}{\rho L t_0} < |z| < \frac{t_0}{\rho L}$$

where $t_0, t'_0$ are the unique positive roots of the equations $g(t) = 0$ and $f(t) = 0$ respectively whose form is given in the statement of Theorem 10.3.6.

This completes the proof of the theorem. ■

In the line of Theorem 10.3.11, the following theorem may be stated in view of Lemma 10.2.2:

**Theorem 10.3.12** Let $P(z)$ be an entire function with $L^*$-order $\rho^{L^*}$. For sufficiently large values of $r$ in the disc $|z| \leq [r^{L(r)}]$, the Taylor’s series expansion of $P(z)$

$$P(z) = a_0 + a_{p_1} z^{p_1} + a_{p_2} z^{p_2} + \ldots + a_{p_m} z^{p_m} + a_{N(r)} z^{N(r)}, a_0 \neq 0$$
be such that $1 \leq p_1 < p_2 \ldots < p_m \leq N(r) - 1$, $p_i$'s are integers and for $\rho^{L^*} > 0$,

$$|a_0| (\rho^{L^*})^{N(r)} \geq |a_{p_1}| (\rho^{L^*})^{N(r) - p_1} \geq \ldots \geq |a_{p_m}| (\rho^{L^*})^{N(r) - p_m}.$$ 

Then all the zeros of $P(z)$ lie in the proper ring shaped region

$$\frac{1}{\rho^{L^*} t_0} < |z| < \frac{t_0}{\rho^{L^*}}$$

where $t_0$ and $t_0'$ are the unique positive roots of the equations

$$g(t) \equiv |a_{N(r)}| \ t^{N(r) - p_m} - |a_{N(r)}| \ t^{N(r) - p_m - 1} - |a_0| \ (\rho^{L^*})^{N(r)} = 0$$

and

$$f(t) \equiv |a_0| \ (\rho^{L^*})^{p_1} t^{p_1} - |a_0| \ (\rho^{L^*})^{p_1 - 1} - |a_{p_1}| = 0$$

respectively.

The proof is omitted.

**Corollary 10.3.5** In view of Theorem 10.3.11 we may state that all the zeros of the polynomial $P(z) = a_0 + a_{p_1} z^{p_1} + \ldots + a_{p_m} z^{p_m} + a_n z^n$ of degree $n$ with $1 \leq p_1 < p_2 < \ldots < p_m \leq n - 1$, $p_i$'s are integers such that

$$|a_0| \geq |a_{p_1}| \geq \ldots \geq |a_n|$$

lie in ring shaped region

$$\frac{1}{t'_0} < |z| < t_0$$

where $t_0, t'_0$ are the unique positive roots of the equations

$$g(t) \equiv |a_{n}| \ t^{n - p_m} - |a_{n}| \ t^{n - p_m - 1} - |a_0| = 0$$

and

$$f(t) \equiv |a_0| \ t^{p_1} - |a_0| \ t^{p_1 - 1} - |a_{p_1}| = 0$$

respectively just substituting $\rho^{L^*} = 1$.

**Corollary 10.3.6** In view of Theorem 10.3.12 we may state that all the zeros of the polynomial $P(z) = a_0 + a_{p_1} z^{p_1} + \ldots + a_{p_m} z^{p_m} + a_n z^n$ of degree $n$ with $1 \leq p_1 < p_2 < \ldots < p_m \leq n - 1$, $p_i$'s are integers such that

$$|a_0| \geq |a_{p_1}| \geq \ldots \geq |a_n|$$

lie in ring shaped region

$$\frac{1}{t'_0} < |z| < t_0$$

where $t_0, t'_0$ are the unique positive roots of the equations

$$g(t) \equiv |a_{n}| \ t^{n - p_m} - |a_{n}| \ t^{n - p_m - 1} - |a_0| = 0$$

and

$$f(t) \equiv |a_0| \ t^{p_1} - |a_0| \ t^{p_1 - 1} - |a_{p_1}| = 0$$

respectively just substituting $\rho^{L^*} = 1$. 
Theorem 10.3.13 Let $P(z)$ be an entire function having $L$-order $\rho^L$ in the disc $|z| \leq [rL(r)]$ for sufficiently large $r$. Also let the Taylor’s series expansion of $P(z)$ be given by

$$P(z) = a_0 + a_{p_1}z^{p_1} + \ldots + a_{p_m}z^{p_m} + a_{N(r)}z^{N(r)}, a_0 \neq 0, a_{N(r)} \neq 0$$

with $1 \leq p_1 < p_2 < \ldots \ldots \ldots < p_m \leq N(r) - 1$, $p_i$’s are integers such that for $\rho^L > 0$,

$$|a_0| (\rho^L)^{N(r)} \geq |a_{p_1}| (\rho^L)^{N(r) - 1} \geq \ldots \ldots \ldots \geq |a_{p_m}| (\rho^L)^{N(r) - m} \geq |a_{N(r)}|.$$ 

Then all the zeros of $P(z)$ lie in the ring shaped region

$$\frac{1}{\rho^L} \left( 1 + \frac{|a_{p_1}|}{|a_0|\rho^L} \right) < |z| < \frac{1}{\rho^L} \left( 1 + \frac{|a_0|}{|a_{N(r)}|\rho^L} \right).$$

Proof. Given that

$$P(z) = a_0 + a_{p_1}z^{p_1} + \ldots + a_{p_m}z^{p_m} + a_{N(r)}z^{N(r)}$$

where $p_i$’s are integers and $1 \leq p_1 < p_2 < \ldots \ldots \ldots < p_m \leq N(r) - 1$. Then for $\rho^L > 0$ ,

$$|a_0| (\rho^L)^{N(r)} \geq |a_{p_1}| (\rho^L)^{N(r) - 1} \geq \ldots \ldots \ldots \geq |a_{p_m}| (\rho^L)^{N(r) - m} \geq |a_{N(r)}|.$$ 

Let us consider

$$Q(z) = (\rho^L)^{N(r)} P \left( \frac{z}{\rho^L} \right) = (\rho^L)^{N(r)} \left\{ a_0 + a_{p_1} \frac{z^{p_1}}{(\rho^L)^{p_1}} + \ldots + a_{p_m} \frac{z^{p_m}}{(\rho^L)^{p_m}} + a_{N(r)} \frac{z^{N(r)}}{(\rho^L)^{N(r)}} \right\} = a_0 (\rho^L)^{N(r)} + a_{p_1} (\rho^L)^{N(r) - 1} z^{p_1} + \ldots \ldots + a_{p_m} (\rho^L)^{N(r) - m} z^{p_m} + a_{N(r)} z^{N(r)}.$$ 

Therefore

$$|Q(z)| \geq |a_{N(r)} z^{N(r)}| - \left| a_0 (\rho^L)^{N(r)} + a_{p_1} (\rho^L)^{N(r) - 1} z^{p_1} + \ldots \ldots + a_{p_m} (\rho^L)^{N(r) - m} z^{p_m} \right|. \quad (10.3.32)$$

Now using the given condition of Theorem 10.3.7, we obtain that

$$|a_0 (\rho^L)^{N(r)} + a_{p_1} (\rho^L)^{N(r) - 1} z^{p_1} + \ldots \ldots + a_{p_m} (\rho^L)^{N(r) - m} z^{p_m}|$$

$$\leq |a_0| (\rho^L)^{N(r)} + |a_{p_1}| (\rho^L)^{N(r) - 1} |z|^{p_1} + \ldots \ldots + |a_{p_m}| (\rho^L)^{N(r) - m} |z|^{p_m}$$

$$= |a_0| (\rho^L)^{N(r)} |z|^{N(r)} \left( \frac{1}{|z|^{N(r) - p_1}} + \ldots \ldots + \frac{1}{|z|^{N(r)}} \right) \text{ for } |z| \neq 0.$$
Using (10.3.32) we get for \(|z| \neq 0\) that
\[
|Q(z)| \geq |a_{N(r)}| |z|^N(r) - |a_0| |z|^N(r) \left( \frac{1}{|z|^N(r) - \rho_m} + \cdots + \frac{1}{|z|^N(r)} \right)
\]
\[
> |a_{N(r)}| |z|^N(r) - |a_0| |z|^N(r) \left( \frac{1}{|z|^N(r) - \rho_m} + \cdots + \frac{1}{|z|^N(r)} + \cdots \right)
\]
\[
= |a_{N(r)}| |z|^N(r) - |a_0| |z|^N(r) \left( \sum_{k=1}^{\infty} \frac{1}{|z|^k} \right).
\] (10.3.33)

The geometric series \(\sum_{k=1}^{\infty} \frac{1}{|z|^k}\) is convergent for
\[
\frac{1}{|z|} < 1
\]
and converges to
\[
\frac{1}{|z| - 1}\]

Therefore
\[
\sum_{k=1}^{\infty} \frac{1}{|z|^k} = \frac{1}{|z| - 1} \quad \text{for } |z| > 1.
\]

Using (10.3.33) we get from above that for \(|z| > 1\)
\[
|Q(z)| \geq |a_{N(r)}| |z|^N(r) - |a_0| |z|^N(r) \left( \frac{1}{|z| - 1} \right)
\]
\[
= |z|^N(r) \left( |a_{N(r)}| - \frac{|a_0| (\rho^L)^N(r)}{|z| - 1} \right).
\]

Now for \(|z| > 1\),
\[
|Q(z)| \geq 0 \text{ if } |a_{N(r)}| > \frac{|a_0| (\rho^L)^N(r)}{|z| - 1} \geq 0
\]
i.e., if \(|a_{N(r)}| \geq \frac{|a_0| (\rho^L)^N(r)}{|z| - 1}\)
i.e., if \(|z| - 1 \geq \frac{|a_0| (\rho^L)^N(r)}{|a_{N(r)}|}\)
i.e., if \(|z| \geq 1 + \frac{|a_0| (\rho^L)^N(r)}{|a_{N(r)}|} > 1.
\]

Therefore \(|Q(z)| \geq 0\) if
\[
|z| \geq 1 + \frac{|a_0| (\rho^L)^N(r)}{|a_{N(r)}|}.
\]
Therefore $Q(z)$ does not vanish for

$$|z| \geq 1 + \left| \frac{a_0 \rho^L}{a_N(r)} \right|.$$ 

So all the zeros of $Q(z)$ lie in

$$|z| < 1 + \left| \frac{a_0 \rho^L}{a_N(r)} \right|.$$ 

Let $z = z_0$ be any zero of $P(z)$. Therefore $P(z_0) = 0$. Clearly $z_0 \neq 0$ as $a_0 \neq 0$. Putting $z = \rho^L z_0$ in $Q(z)$ we get that

$$Q(\rho^L z_0) = \left(\rho^L\right)^N(r) P(z_0) = \left(\rho^L\right)^N(r) \cdot 0 = 0.$$ 

So $z = \rho^L z_0$ is a zero of $Q(z)$. Hence

$$|\rho^L z_0| < 1 + \left| \frac{a_0 \rho^L}{a_N(r)} \right|$$ 

i.e.,

$$|z_0| < \frac{1}{\rho^L} \left(1 + \left| \frac{a_0 \rho^L}{a_N(r)} \right| \right).$$

Since $z_0$ is an arbitrary zero of $P(z)$, therefore all the zeros of $Q(z)$ lie in

$$|z| < \frac{1}{\rho^L} \left(1 + \left| \frac{a_0 \rho^L}{a_N(r)} \right| \right). \quad (10.3.34)$$

Again let us consider

$$R(z) = \left(\rho^L\right)^N(r) z^N(r) P\left(\frac{1}{\rho^L z}\right).$$

Therefore

$$R(z) = \left(\rho^L\right)^N(r) z^N(r)$$

$$= a_0 \rho^L z^N(r) + a_{p_1} \rho^L z^{-p_1} z^N(r) + \ldots + a_{p_m} \rho^L z^{-p_m} z^N(r) + a_N(r) \rho^L z^N(r)$$

$$= a_0 \rho^L z^N(r) + a_{p_1} \rho^L z^{-p_1} z^N(r) + \ldots + a_{p_m} \rho^L z^{-p_m} z^N(r) + a_N(r).$$

Now

$$|R(z)| \geq \left| a_0 \rho^L z^N(r) \right|$$

$$- \left| a_{p_1} \rho^L z^{-p_1} z^N(r) + \ldots + a_{p_m} \rho^L z^{-p_m} z^N(r) + a_N(r) \right|. \quad (10.3.35)$$
Also
\[
|a_{p_1} (\rho^L)^{N(r) - p_1} z^{N(r) - p_1} + \ldots + a_{p_m} (\rho^L)^{N(r) - p_m} z^{N(r) - p_m} + a_N(r)| \\
\leq |a_{p_1} (\rho^L)^{N(r) - p_1} z^{N(r) - p_1}| + \ldots + |a_{p_m} (\rho^L)^{N(r) - p_m} z^{N(r) - p_m}| + |a_N(r)| \\
\leq |a_{p_1}| (\rho^L)^{N(r) - p_1} |z|^{N(r) - p_1} + \ldots + |a_{p_m}| (\rho^L)^{N(r) - p_m} |z|^{N(r) - p_m} + |a_N(r)| \\
\leq |a_{p_1}| (\rho^L)^{N(r) - p_1} \left( |z|^{N(r) - p_1} + \ldots + |z|^{N(r) - p_m} + 1 \right). \quad (10.3.36)
\]

Using (10.3.36) we get from (10.3.35) that for $|z| \neq 0$
\[
|R(z)| \geq |a_0| (\rho^L)^{N(r)} |z|^{N(r)} - |a_{p_1}| (\rho^L)^{N(r) - p_1} \left( |z|^{N(r) - p_1} + \ldots + |z| + 1 \right) \\
= |a_0| (\rho^L)^{N(r)} |z|^{N(r)} - |a_{p_1}| (\rho^L)^{N(r) - p_1} |z|^{N(r)} \left( \frac{1}{|z|^{p_1}} + \ldots + \frac{1}{|z|^{p_m}} + \frac{1}{|z|^{N(r)}} \right) \\
> |a_0| (\rho^L)^{N(r)} |z|^{N(r)} - |a_{p_1}| (\rho^L)^{N(r) - p_1} |z|^{N(r)} \left( \frac{1}{|z|^{p_1}} + \ldots + \frac{1}{|z|^{p_m}} + \frac{1}{|z|^{N(r)}} + \ldots \right). \quad (10.3.37)
\]

Therefore for $|z| \neq 0$,
\[
|R(z)| > |a_0| (\rho^L)^{N(r)} |z|^{N(r)} - |a_{p_1}| (\rho^L)^{N(r) - p_1} |z|^{N(r)} \left( \sum_{k=1}^{\infty} \frac{1}{|z|^k} \right). \quad (10.3.37)
\]

Now the geometric series $\sum_{k=1}^{\infty} \frac{1}{|z|^k}$ is convergent for
\[
\frac{1}{|z|} < 1
\]
i.e., for $|z| > 1$
and converges to
\[
\frac{1}{|z| - 1} = \frac{1}{|z| - 1}.
\]
So
\[
\sum_{k=1}^{\infty} \frac{1}{|z|^k} = \frac{1}{|z| - 1} \quad \text{for } |z| > 1.
\]
Therefore for $|z| > 1$,
\[
|R(z)| > |a_0| (\rho^L)^{N(r)} |z|^{N(r)} - |a_{p_1}| (\rho^L)^{N(r) - p_1} |z|^{N(r)} \left( \frac{1}{|z| - 1} \right) \\
= |z|^{N(r)} (\rho^L)^{N(r) - p_1} \left( |a_0| (\rho^L)^{p_1} - \frac{|a_{p_1}|}{|z| - 1} \right).
i.e., for $|z| > 1$

$$|R(z)| > |z|^{N(r)} (\rho^L)^{N(r)-p_1} \left( |a_0| (\rho^L)^{p_1} - \frac{|a_{p_1}|}{|z| - 1} \right).$$

Now

$$|R(z)| > 0 \text{ if } |a_0| (\rho^L)^{p_1} - \frac{|a_{p_1}|}{|z| - 1} \geq 0$$

i.e., if $|a_0| (\rho^L)^{p_1} \geq \frac{|a_{p_1}|}{|z| - 1}$

i.e., if $|z| - 1 \geq \frac{|a_{p_1}|}{|a_0| (\rho^L)^{p_1}}$

i.e., if $|z| \geq 1 + \frac{|a_{p_1}|}{|a_0| (\rho^L)^{p_1}} > 1$.

Therefore

$$|R(z)| > 0 \text{ if } |z| \geq 1 + \frac{|a_{p_1}|}{|a_0| (\rho^L)^{p_1}}.$$ 

Since $R(z)$ does not vanish in

$$|z| \geq 1 + \frac{|a_{p_1}|}{|a_0| (\rho^L)^{p_1}},$$

all the zeros of $R(z)$ lie in

$$|z| < 1 + \frac{|a_{p_1}|}{|a_0| (\rho^L)^{p_1}}.$$

Let $z = z_0$ be any zero of $P(z)$. Therefore $P(z_0) = 0$. Clearly $z_0 \neq 0$ as $a_0 \neq 0$.

Putting $z = \frac{1}{\rho^L z_0}$ in $R(z)$ we obtain that

$$R \left( \frac{1}{\rho^L z_0} \right) = (\rho^L)^{N(r)} \left( \frac{1}{\rho^L z_0} \right)^{N(r)} P(z_0) = \left( \frac{1}{z_0} \right)^{N(r)} \cdot 0 = 0.$$

So

$$\left| \frac{1}{\rho^L z_0} \right| < 1 + \frac{|a_{p_1}|}{|a_0| (\rho^L)^{p_1}}$$

i.e.,

$$\frac{1}{z_0} < \rho^L \left( 1 + \frac{|a_{p_1}|}{|a_0| (\rho^L)^{p_1}} \right)$$

i.e. $z_0 > \frac{1}{\rho^L \left( 1 + \frac{|a_{p_1}|}{|a_0| (\rho^L)^{p_1}} \right)}$.

As $z_0$ is an arbitrary zero of $P(z)$, all the zeros of $P(z)$ lie in

$$|z| > \frac{1}{\rho^L \left( 1 + \frac{|a_{p_1}|}{|a_0| (\rho^L)^{p_1}} \right)}.$$  (10.3.38)
So from (10.3.34) and (10.3.38), we may conclude that all the zeros of $P(z)$ lie in the proper ring shaped region

$$
\frac{1}{\rho^L} \left( 1 + \frac{|a_{p_1}|}{|a_0| (\rho^L)^{|p_1|}} \right) < |z| < \frac{1}{\rho^L} \left( 1 + \frac{|a_0|}{|a_{N(r)}| (\rho^L)^{N(r)}} \right).
$$

This proves the theorem. ■

The next theorem is in fact an extension of Theorem 10.3.13 and can be proved by using Lemma 10.2.2.

**Theorem 10.3.14** Let $P(z)$ be an entire function having $L^*$-order $\rho L^*$ in the disc $|z| \leq [r e^{L(r)}]$ for sufficiently large $r$. Also let the Taylor’s series expansion of $P(z)$ be given by

$$
P(z) = a_0 + a_{p_1} z^{p_1} + \ldots + a_{p_m} z^{p_m} + a_{N(r)} z^{N(r)},
$$

with $1 \leq p_1 < p_2 < \ldots < p_m \leq N(r) - 1$, $p_i$’s are integers such that for $\rho L^* > 0$,

$$
|a_0| (\rho L^*)^{N(r)} \geq |a_{p_1}| (\rho L^*)^{N(r) - p_1} \geq \ldots \geq |a_{p_m}| (\rho L^*)^{N(r) - p_m} \geq |a_{N(r)}|.
$$

Then all the zeros of $P(z)$ lie in the ring shaped region

$$
\frac{1}{\rho L^*} \left( 1 + \frac{|a_{p_1}|}{|a_0| \rho L^{|p_1|}} \right) < |z| < \frac{1}{\rho L^*} \left( 1 + \frac{|a_0|}{|a_{N(r)}| \rho L^{|N(r)|}} \right).
$$

**Corollary 10.3.7** In view of Theorem 10.3.13 we may conclude that all the zeros of

$$
P(z) = a_0 + a_{p_1} z^{p_1} + \ldots + a_{p_m} z^{p_m} + a_n z^n
$$

of degree $n$ with $1 \leq p_1 < p_2 < \ldots < p_m \leq n - 1$, $p_i$’s are integers such that for $\rho L > 0$,

$$
|a_0| \geq |a_{p_1}| \geq \ldots \geq |a_n|
$$

lie in the ring shaped region

$$
\left( 1 + \frac{|a_{p_1}|}{|a_0|} \right) < |z| < \left( 1 + \frac{|a_0|}{|a_n|} \right)
$$

on putting $\rho L = 1$ in Theorem 10.3.13.

**Corollary 10.3.8** In view of Theorem 10.3.14, we may conclude that all the zeros of

$$
P(z) = a_0 + a_{p_1} z^{p_1} + \ldots + a_{p_m} z^{p_m} + a_n z^n
$$

of degree $n$ with $1 \leq p_1 < p_2 < \ldots < p_m \leq n - 1$, $p_i$’s are integers such that for $\rho L > 0$,

$$
|a_0| \geq |a_{p_1}| \geq \ldots \geq |a_n|
lie in the ring shaped region

\[
\frac{1}{1 + \left| \frac{a_{n+1}}{a_n} \right|} < |z| < \left( 1 + \frac{|a_0|}{|a_n|} \right)
\]
on putting \( \rho^L = 1 \) in Theorem 10.3.14.

**Theorem 10.3.15** Let \( P(z) \) be an entire function having \( L \)-order \( \rho^L \). For sufficiently large \( r \) in the disc \( |z| \leq [r \mu(r)] \), the Taylor’s series expansion of \( P(z) \) be given by \( P(z) = a_0 + a_1 z + \ldots + a_{N(r)} z^{N(r)} \), \( a_0 \neq 0 \). Further for \( \rho^L > 0 \),

\[
|a_0| (\rho^L)^{N(r)} \geq |a_1| (\rho^L)^{N(r)-1} \geq \ldots \geq |a_{N(r)}|.
\]

Then all the zeros of \( P(z) \) lie in the ring shaped region.

\[
\frac{1}{\rho^L t_0^L} < |z| < \frac{1}{\rho^L t_0}
\]

where \( t_0 \) and \( t_0' \) are the greatest roots of

\[
g(t) \equiv |a_{N(r)}| t^{N(r)+1} - (|a_{N(r)}| + (\rho^L)^{N(r)} |a_0|) t^{N(r)} + (\rho^L)^{N(r)} |a_0| = 0
\]

and

\[
f(t) \equiv |a_0| \rho^L t^{N(r)+1} - (|a_0| \rho^L + |a_1|) t^{N(r)} + |a_1| = 0.
\]

**Proof.** Let

\[
P(z) = a_0 + a_1 z + \ldots + a_{N(r)} z^{N(r)},
\]

by applying Lemma 10.2.1 and in view of Taylor’s series expansion of \( P(z) \). Also

\[
|a_0| (\rho^L)^{N(r)} \geq |a_1| (\rho^L)^{N(r)-1} \geq \ldots \geq |a_{N(r)}|.
\]

Let us consider

\[
Q(z) = (\rho^L)^{N(r)} P \left( \frac{z}{\rho^L} \right)
\]

\[
= (\rho^L)^{N(r)} \left\{ a_0 + a_1 \frac{z}{\rho^L} + a_2 \left( \frac{z^2}{(\rho^L)^2} \right) + \ldots + a_{N(r)} \frac{z^{N(r)}}{(\rho^L)^{N(r)}} \right\}
\]

\[
= a_0 (\rho^L)^{N(r)} + a_1 (\rho^L)^{N(r)-1} z + \ldots + a_{N(r)} z^{N(r)}.
\]

Now

\[
|Q(z)| \geq |a_{N(r)}| |z|^{N(r)} - |a_0 (\rho^L)^{N(r)} + a_1 (\rho^L)^{N(r)-1} z + \ldots + a_{N(r)-1} z^{N(r)-1} |.
\]
Also applying the condition \( |a_0| (\rho^L)^{N(r)} \geq |a_1| (\rho^L)^{N(r)-1} \geq \ldots \geq |a_{N(r)}| \), we get from above that
\[
|a_0 (\rho^L)^{N(r)} + a_1 (\rho^L)^{N(r)-1} z + \ldots + a_{N(r)} z^{N(r)-1}| \\
\leq |a_0| (\rho^L)^{N(r)} + |a_1| (\rho^L)^{N(r)-1} |z| + \ldots + |a_{N(r)}| |z|^{N(r)-1} \\
\leq |a_0| (\rho^L)^{N(r)} \left( 1 + |z| + \ldots + |z|^{N(r)-1} \right) \\
= |a_0| (\rho^L)^{N(r)} \frac{|z|^{N(r)} - 1}{|z| - 1} \quad \text{for} \quad |z| \neq 1.
\]

Therefore it follows from above that
\[
|Q(z)| \geq |a_{N(r)}| |z|^{N(r)} - |a_0| (\rho^L)^{N(r)} \frac{|z|^{N(r)} - 1}{|z| - 1}.
\]

Now
\[
|Q(z)| > 0 \quad \text{if} \quad |a_{N(r)}| |z|^{N(r)} - |a_0| (\rho^L)^{N(r)} \frac{|z|^{N(r)} - 1}{|z| - 1} > 0
\]
i.e., if
\[
|a_{N(r)}| |z|^{N(r)} > |a_0| (\rho^L)^{N(r)} \frac{|z|^{N(r)} - 1}{|z| - 1}
\]
i.e., if
\[
|a_{N(r)}| |z|^{N(r)} (|z| - 1) > |a_0| (\rho^L)^{N(r)} \left( |z|^{N(r)} - 1 \right)
\]
i.e., if \( |a_{N(r)}| |z|^{N(r)+1} - (|a_{N(r)}| + |a_0| (\rho^L)^{N(r)}) |z|^{N(r)} + |a_0| (\rho^L)^{N(r)} > 0. \)

Let us consider
\[
g(t) \equiv |a_{N(r)}| t^{N(r)+1} - (|a_{N(r)}| + |a_0| (\rho^L)^{N(r)}) t^{N(r)} + |a_0| (\rho^L)^{N(r)} = 0. \quad (10.3.39)
\]

The maximum number of positive roots of (10.3.39) is two because maximum number of changes of sign in \( g(t) = 0 \) is two and if it is less, less by two. Clearly \( t = 1 \) is a positive root of \( g(t) = 0 \). Therefore \( g(t) = 0 \) must have exactly one positive root other than 1. Let the positive root of \( g(t) \) be \( t_1 \). Let us take \( t_0 = \max \{1, t_1\} \). Clearly for \( t > t_0, g(t) > 0 \). If not for some \( t_2 > t_0, g(t_2) < 0 \). Also \( g(\infty) > 0 \). Therefore \( g(t) = 0 \) has another positive root in \((t_2, \infty)\) which gives a contradiction.

So for \( t > t_0, g(t) > 0 \). Also \( t_0 \geq 1 \). Therefore \( |Q(z)| > 0 \) if \( |z| > t_0 \). So \( Q(z) \) does not vanish in \( |z| > t_0 \). Hence all the zeros of \( Q(z) \) lie in \( |z| \leq t_0 \).

Let \( z = z_0 \) be a zero of \( P(z) \). So \( P(z_0) = 0 \). Clearly \( z_0 \neq 0 \) as \( a_0 \neq 0 \).

Putting \( z = \rho^L z_0 \) in \( Q(z) \) we get that
\[
Q(\rho^L z_0) = (\rho^L)^{N(r)} P(z_0) = (\rho^L)^{N(r)} . 0 = 0.
\]

Therefore \( z = \rho^L z_0 \) is a zero of \( Q(z) \). So \( |\rho^L z_0| \leq t_0 \) or \( |z_0| \leq \frac{1}{\rho^L} t_0 \). As \( z_0 \) is an arbitrary zero of \( P(z) \),
\[
\text{all the zeros of } P(z) \text{ lie in the region } |z| \leq \frac{1}{\rho^L} t_0. \quad (10.3.40)
\]
In order to prove the lower bound of Theorem 10.3.8, let us consider

$$R(z) = (\rho^L)^N z^N P \left( \frac{1}{\rho^N z} \right).$$

Then

$$R(z) = (\rho^L)^N z^N \left( a_0 + \frac{a_1}{\rho^L z} + \ldots + a_N (\rho^L)^N z^N \right)$$

$$= a_0 (\rho^L)^N z^N + a_1 (\rho^L)^{N-1} z^{N-1} + \ldots + a_N (\rho^L)^N z^N.$$

Now

$$|R(z)| \geq |a_0| (\rho^L)^N |z|^N - |a_1| (\rho^L)^{N-1} |z|^{N-1} + \ldots + |a_N|.$$

Also

$$|a_1 (\rho^L)^{N-1} |z|^{N-1} + \ldots + |a_N| \leq |a_1| (\rho^L)^{N-1} |z|^{N-1} + \ldots + |a_N|.$$  

So applying the condition $|a_0| (\rho^L)^N \geq |a_1| (\rho^L)^{N-1} \geq \ldots \geq |a_N|$, we get from above that

$$- |a_1 (\rho^L)^{N-1} |z|^{N-1} + \ldots + |a_N|$$

$$\geq - |a_1| (\rho^L)^{N-1} |z|^{N-1} - \ldots - |a_N|$$

$$\geq - |a_1| (\rho^L)^{N-1} \left( |z|^{N-1} + \ldots + 1 \right)$$

$$= - |a_1| (\rho^L)^{N-1} \frac{|z|^{N-1} - 1}{|z| - 1} \text{ for } |z| \neq 1. \tag{10.3.41}$$

Using (10.3.41) we get for $|z| \neq 1$ that

$$|R(z)| \geq (\rho^L)^{N-1} \left( |a_0| \rho^L |z|^{N-1} - |a_1| \frac{|z|^{N-1} - 1}{|z| - 1} \right). \tag{10.3.42}$$

Now

$$|R(z)| > 0 \text{ if } (\rho^L)^{N-1} \left( |a_0| \rho^L |z|^{N-1} - |a_1| \frac{|z|^{N-1} - 1}{|z| - 1} \right) > 0$$

i.e., if $|a_0| \rho^L |z|^{N-1} - |a_1| \frac{|z|^{N-1} - 1}{|z| - 1} > 0$

i.e., if $|a_0| \rho^L |z|^{N-1} > |a_1| \frac{|z|^{N-1} - 1}{|z| - 1}$

i.e., if $|a_0| \rho^L |z|^{N-1} (|z| - 1) > |a_1| \left( |z|^{N-1} - 1 \right)$

i.e., if $|a_0| \rho^L |z|^{N+1} - (|a_0| \rho^L + |a_1|) |z|^{N+1} + |a_1| > 0.$
Let us consider
\[ f(t) \equiv |a_0| \rho^L t^{N(r)} + 1 - (|a_0| \rho^L + |a_1|) t^{N(r)} + |a_1| = 0. \]

Clearly \( f(t) = 0 \) has two positive roots, because the number of changes of sign of \( f(t) \) is two. If it is less, less by two.

Also \( t = 1 \) is the one of the positive roots of \( f(t) = 0 \). Let us suppose that \( t = t_2 \) be the other positive root. Also let \( t_0' = \max \{1, t_2\} \) and so \( t_0' \geq 1 \). Now \( t > t_0' \) implies \( f(t) > 0 \).

If not then there exists some \( t_3 > t_0' \) such that \( f(t_3) < 0 \). Also \( f(\infty) > 0 \). Therefore there exists another positive root in \( (t_3, \infty) \) which is a contradiction.

So \( |R(z)| > 0 \) if \( |z| > t_0' \). Thus \( R(z) \) does not vanish in \( |z| > t_0' \). In other words all the zeros of \( R(z) \) lie in \( |z| \leq t_0' \).

Let \( z = 0 \) be any zero of \( P(z) \). Then \( P(z_0) = 0 \). Clearly \( z_0 \neq 0 \) as \( a_0 \neq 0 \).

Putting \( z = \frac{1}{\rho^L z_0} \) in \( R(z) \) we get that
\[ R \left( \frac{1}{\rho^L z_0} \right) = (\rho^L) \left( \frac{1}{\rho^L z_0} \right)^{N(r)} P(z_0) = \left( \frac{1}{z_0} \right)^{N(r)} = 0 \]

Therefore \( \frac{1}{\rho^L z_0} \) is a root of \( R(z) \). So \( \left| \frac{1}{\rho^L z_0} \right| \leq t_0' \) implies \( |z_0| \geq \frac{1}{\rho^L t_0'} \). As \( z_0 \) is an arbitrary zero of \( P(z) = 0 \),

\[ \text{all the zeros of } P(z) \text{ lie in } |z| \geq \frac{1}{\rho^L t_0'}. \quad (10.3.43) \]

From (10.3.40) and (10.3.43), we have all the zeros of \( P(z) \) lying in the ring shaped region given by
\[ \frac{1}{\rho^L t_0} \leq |z| \leq \frac{1}{\rho^L t_0} \]

where \( t_0 \) and \( t_0' \) are the greatest positive roots of \( g(t) = 0 \) and \( f(t) = 0 \) respectively.

This completes the proof of the theorem. \( \blacksquare \)

In the line of Theorem 10.3.15, we may state the following theorem in view of Lemma 10.2.2:

**Theorem 10.3.16** Let \( P(z) \) be an entire function having \( L^* \)-order \( \rho^{L^*} \). For sufficiently large \( r \) in the disc \( |z| \leq |re^{L(r)}| \), the Taylor’s series expansion of \( P(z) \) be given by \( P(z) = a_0 + a_1 z + \ldots + a_{N(r)} z^{N(r)}, a_0 \neq 0 \). Further for \( \rho^{L^*} > 0 \),
\[ |a_0| (\rho^{L^*})^{N(r)} \geq |a_1| (\rho^{L^*})^{N(r)-1} \geq \ldots \geq |a_{N(r)}|. \]

Then all the zeros of \( P(z) \) lie in the ring shaped region.
\[ \frac{1}{\rho^{L^*} t_0} < |z| < \frac{1}{\rho^{L^*} t_0} \]

where \( t_0 \) and \( t_0' \) are the greatest roots of
\[ g(t) \equiv |a_{N(r)}| t^{N(r)+1} - (|a_{N(r)}| + (\rho^{L^*})^{N(r)} |a_0|) t^{N(r)} + (\rho^{L^*})^{N(r)} |a_0| = 0 \]

and
\[ f(t) \equiv |a_0| \rho^{L^*} t^{N(r)+1} - (|a_0| \rho^{L^*} + |a_1|) t^{N(r)} + |a_1| = 0. \]
The proof is omitted.

**Corollary 10.3.9** From Theorem 10.3.15 we can easily conclude that all the zeros of
\[ P(z) = a_0 + a_1 z + \ldots + a_n z^n \]
of degree \( n \) with property \( |a_0| \geq |a_1| \geq \ldots \geq |a_n| \) lie in the ring shaped region
\[ \frac{1}{t'_0} \leq |z| \leq t_0 \]
where \( t_0 \) and \( t'_0 \) are the greatest positive roots of
\[ g(t) \equiv |a_n| t^{n+1} - (|a_n| + |a_0|) t^n + |a_0| = 0 \]
and
\[ f(t) \equiv |a_0| t^{n+1} - (|a_0| + |a_1|) t^n + |a_1| = 0 \]
respectively by putting \( \rho^L = 1 \).

**Corollary 10.3.10** From Theorem 10.3.16 we can easily conclude that all the zeros of
\[ P(z) = a_0 + a_1 z + \ldots + a_n z^n \]
of degree \( n \) with property \( |a_0| \geq |a_1| \geq \ldots \geq |a_n| \) lie in the ring shaped region
\[ \frac{1}{t'_0} \leq |z| \leq t_0 \]
where \( t_0 \) and \( t'_0 \) are the greatest positive roots of
\[ g(t) \equiv |a_n| t^{n+1} - (|a_n| + |a_0|) t^n + |a_0| = 0 \]
and
\[ f(t) \equiv |a_0| t^{n+1} - (|a_0| + |a_1|) t^n + |a_1| = 0 \]
respectively by putting \( \rho^L = 1 \).

**Corollary 10.3.11** Under the conditions of Theorem 10.3.15 and
\[ P(z) = a_0 + a_{p_1} z^{p_1} + \ldots + a_{p_m} z^{p_m} + a_{N(r)} z^{N(r)} \]
with
\[ 1 \leq p_1 \leq p_2 \leq \ldots \leq p_m \leq p_{N(r)-1}, \]
where \( p_i \)'s are integers and \( a_0, a_{p_1}, \ldots, a_{N(r)} \) are non vanishing coefficients with
\[ |a_0| (\rho^L)^{N(r)} \geq |a_{p_1}| (\rho^L)^{N(r)-p_1} \geq \ldots \geq |a_{p_m}| (\rho^L)^{N(r)-p_m} \geq |a_{N(r)}|, \]
all the zeros of \( P(z) \) lie in
\[ \frac{1}{\rho^L t'_0} \leq |z| \leq \frac{1}{\rho^L t_0} \]
where \( t_0 \) and \( t'_0 \) are the greatest positive roots of
\[ g(t) \equiv |a_{N(r)}| t^{N(r)+1} - (|a_{N(r)}| + |a_0| (\rho^L)^{N(r)}) t^{N(r)} + |a_0| (\rho^L)^{N(r)} = 0 \]
and
\[ f(t) \equiv |a_0| (\rho^L)^{p_1} t^{N(r)+1} - (|a_0| (\rho^L)^{p_1} + |a_{p_1}|) t^{N(r)} - |a_{p_1}| = 0 \]
respectively.
Corollary 10.3.12 Under the conditions of Theorem 10.3.16 and
\[ P(z) = a_0 + a_{p_1}z^{p_1} + \ldots + a_{p_m}z^{p_m} + a_{N(r)}z^{N(r)} \]

with
\[ 1 \leq p_1 \leq p_2 \leq \ldots \leq p_m \leq p_{N(r)-1}, \]

where \( p_i \)'s are integers and \( a_0, a_{p_1}, \ldots, a_{N(r)} \) are non vanishing coefficients with
\[ |a_0| (\rho^L)^{N(r)} \geq |a_{p_1}| (\rho^L)^{N(r)-p_1} \geq \ldots \geq |a_{p_m}| (\rho^L)^{N(r)-p_m} \geq |a_{N(r)}|, \]

all the zeros of \( P(z) \) lie in
\[ \frac{1}{\rho^L t_0'} \leq |z| \leq \frac{1}{\rho^L t_0} \]

where \( t_0 \) and \( t_0' \) are the greatest positive roots of
\[ g(t) \equiv |a_{N(r)}| t^{N(r)+1} - (|a_{N(r)}| + |a_0| (\rho^L)^{N(r)}) t^{N(r)} + |a_0| (\rho^L)^{N(r)} = 0 \]

and
\[ f(t) \equiv |a_0| (\rho^L)^{p_1} t^{N(r)+1} - (|a_0| (\rho^L)^{p_1} + |a_{p_1}|) t^{N(r)} - |a_{p_1}| = 0 \]

respectively.

Corollary 10.3.13 If we put \( \rho^L = 1 \) in Corollary 10.3.11 then all the zeros of
\[ P(z) = a_0 + a_{p_1}z^{p_1} + \ldots + a_{p_m}z^{p_m} + a_nz^n \]

lie in the ring shaped region
\[ \frac{1}{t_0'} \leq |z| \leq t_0 \]

where \( t_0 \) and \( t_0' \) are the greatest positive roots of
\[ g(t) \equiv |a_n| t^{n+1} - (|a_n| + |a_0|) t^n + |a_0| = 0 \]

and
\[ f(t) \equiv |a_0| t^{n+1} - (|a_0| + |a_{p_1}|) t^n - |a_{p_1}| = 0 \]

respectively

provided
\[ |a_0| \geq |a_{p_1}| \geq \ldots \geq |a_{p_m}| \geq |a_n|. \]

Corollary 10.3.14 If we put \( \rho^L = 1 \) in Corollary 10.3.12 then all the zeros of
\[ P(z) = a_0 + a_{p_1}z^{p_1} + \ldots + a_{p_m}z^{p_m} + a_nz^n \]

lie in the ring shaped region
\[ \frac{1}{t_0'} \leq |z| \leq t_0 \]

where \( t_0 \) and \( t_0' \) are the greatest positive roots of
\[ g(t) \equiv |a_n| t^{n+1} - (|a_n| + |a_0|) t^n + |a_0| = 0 \]
and

\[ f(t) \equiv |a_0| t^{n+1} - (|a_0| + |a_{p_1}|) t^n - |a_{p_1}| = 0 \quad \text{respectively} \]

provided

\[ |a_0| \geq |a_{p_1}| \geq \ldots \ldots \geq |a_{p_m}| \geq |a_n|. \]

* * * * * * *

The figures related to Remark 10.3.1 and Remark 10.3.3 are given in the APPENDIX, see Page a and Page b respectively.