CHAPTER 8

ON THE GROWTH ANALYSIS RELATED TO MAXIMUM TERM-BASED RELATIVE ORDER OF ENTIRE FUNCTIONS
8.1 Introduction, Definitions and Notations.

For an entire function \( f \) defined in the open complex plane \( \mathbb{C} \), the maximum term \( \mu_f (r) \) of \( f = \sum_{n=0}^{\infty} a_n z^n \) and the maximum modulus \( M_f (r) \) of \( f \) on \( |z| = r \) are respectively defined as

\[
\mu_f (r) = \max_{n \geq 0} (|a_n| r^n)
\]

and

\[
M_f (r) = \max_{|z|=r} |f(z)|.
\]

Though Definition 8.1.1 and Definition 8.1.2 have already been defined in Chapter 7 as Definition 7.1.1 and Definition 7.1.2 respectively, we state here again in order to keep a continuation of our discussion:

**Definition 8.1.1** The order \( \rho_f \) and lower order \( \lambda_f \) of an entire function \( f \) are defined as

\[
\rho_f = \limsup_{r \to \infty} \frac{\log^{[2]} M_f (r)}{\log r} \quad \text{and} \quad \lambda_f = \liminf_{r \to \infty} \frac{\log^{[2]} M_f (r)}{\log r},
\]

The results of this chapter have been communicated, see [21].
where $\log^{[k]}x = \log\left(\log^{[k-1]}x\right)$ for $k = 1, 2, 3, \ldots$ and $\log^{[0]}x = x$.

Using the inequalities $\mu_f(r) \leq M_f(r) \leq \frac{R}{R-r} \mu_f(R)$ \{cf. [48]\}, for $0 \leq r < R$ one may verify that

$$
\rho_f = \limsup_{r \to \infty} \frac{\log^{[2]} \mu_f(r)}{\log r} \quad \text{and} \quad \lambda_f = \liminf_{r \to \infty} \frac{\log^{[2]} \mu_f(r)}{\log r}.
$$

If an entire function $g$ is non-constant, then $M_g(r)$ is strictly increasing and its inverse $M_g^{-1} : ([g(0)], \infty) \to (0, \infty)$ exists and is such that $\lim_{s \to \infty} M_g^{-1}(s) = \infty$.

Bernal [2] introduced the definition of relative order of an entire function $f$ with respect to an entire function $g$, denoted by $\rho_g(f)$ as follows:

$$
\rho_g(f) = \inf \{ \mu > 0 : M_f(r) < M_g(r^\mu) \quad \text{for all} \quad r > r_0(\mu) > 0 \}
= \limsup_{r \to \infty} \frac{\log M_g^{-1}M_f(r)}{\log r}.
$$

The definition due to Bernal [2] coincides with the classical one \{cf.[50]\} if $g(z) = \exp z$.

Similarly, one can define the relative lower order of an entire function $f$ with respect to an entire function $g$ denoted by $\lambda_g(f)$ as follows:

$$
\lambda_g(f) = \liminf_{r \to \infty} \frac{\log M_g^{-1}M_f(r)}{\log r}.
$$

Datta and Maji [13] gave an alternative definition of relative order and relative lower order of an entire function with respect to another entire function in the following way:

**Definition 8.1.2** [13] The relative order $\rho_g(f)$ and relative lower order $\lambda_g(f)$ of an entire function $f$ with respect to another entire function $g$ are defined as follows:

$$
\rho_g(f) = \limsup_{r \to \infty} \frac{\log \mu_g^{-1}\mu_f(r)}{\log r} \quad \text{and} \quad \lambda_g(f) = \liminf_{r \to \infty} \frac{\log \mu_g^{-1}\mu_f(r)}{\log r}.
$$

In the chapter we wish to establish some results relating to the growth rates of composite entire function in terms of their maximum terms on the basis of relative order (relative lower order).

**8.2 Lemmas.**

In this section we present some lemmas which will be needed in the sequel.

**Lemma 8.2.1** [47] Let $f$ and $g$ be any two entire functions. Then for every $\alpha > 1$ and $0 < r < R$,

$$
\mu_{fg}(r) \leq \frac{\alpha}{\alpha - 1} \mu_f\left(\frac{\alpha R}{R-r} \mu_g(R)\right).
$$
Lemma 8.2.2 [47] Let \( f \) and \( g \) be any two entire functions. Then for all sufficiently large values of \( r \),
\[
\mu_{fog}(r) \geq \frac{1}{2} \mu_f \left( \frac{1}{8} \mu_g \left( \frac{r}{4} \right) - |g(0)| \right).
\]

Lemma 8.2.3 [13] If \( f \) be entire and \( \alpha > 1 \), \( 0 < \beta < \alpha \), then for all sufficiently large \( r \),
\[
\mu_f(\alpha r) \geq \beta \mu_f(r).
\]

8.3 Theorems.

In this section we present the main results of the chapter.

Theorem 8.3.1 Let \( f \) and \( h \) be any two entire functions such that \( 0 < \lambda_h(f) \leq \rho_h(f) < \infty \) and \( g \) be an entire function with finite order. Then for every positive constant \( A \) and each \( \alpha \in (-\infty, \infty) \),
\[
\lim_{r \to \infty} \frac{\log \mu_h^{-1} \mu_{fog}(r)}{\exp(r)^A} = 0, \text{ if } A > (1 + \alpha) \rho_g.
\]

Proof. If \( 1 + \alpha \leq 0 \), then the theorem is obvious. We consider \( 1 + \alpha > 0 \).

Now taking \( R = \beta r \) in Lemma 8.2.1 and in view of Lemma 8.2.3, it follows for all sufficiently large values of \( r \) that
\[
\mu_{fog}(r) \leq \frac{\alpha}{\alpha - 1} \mu_f \left( \frac{\alpha \beta}{(\beta - 1)} \mu_g(\beta r) \right)
\]
\[
i.e., \quad \mu_{fog}(r) \leq \mu_f \left( \frac{2(\alpha - 1) \alpha \beta}{(\alpha - 1)(\beta - 1)} \mu_g(\beta r) \right).
\]
(8.3.1)

Since \( \mu_h^{-1}(r) \) is an increasing function of \( r \), from (8.3.1) it follows for all sufficiently large values of \( r \) that
\[
\mu_h^{-1} \mu_{fog}(r) \leq \mu_h^{-1} \mu_f \left( \frac{2(\alpha - 1) \alpha \beta}{(\alpha - 1)(\beta - 1)} \mu_g(\beta r) \right)
\]
\[
i.e., \quad \log \mu_h^{-1} \mu_{fog}(r) \leq \log \mu_h^{-1} \mu_f \left( \frac{2(\alpha - 1) \alpha \beta}{(\alpha - 1)(\beta - 1)} \mu_g(\beta r) \right)
\]
\[
i.e., \quad \log \mu_h^{-1} \mu_{fog}(r) \leq (\rho_h(f) + \varepsilon) \log \mu_g(\beta r) + O(1)
\]
(8.3.2)
\[
i.e., \quad \log \mu_h^{-1} \mu_{fog}(r) \leq (\rho_h(f) + \varepsilon) (\beta r)^{\rho_g + \varepsilon} + O(1).
\]
(8.3.3)

Again for all sufficiently large values of \( r \) we get that
\[
\log \mu_h^{-1} \mu_f \left\{ \exp(r)^A \right\} \geq (\lambda_h(f) - \varepsilon) r^A.
\]
(8.3.4)
Hence for all sufficiently large values of \( r \) we obtain from (8.3.3) and (8.3.4) that

\[
\frac{\log \mu_{h}^{-1} \mu_{fog}(r)}{\log \mu_{h}^{-1} \mu_{f} \left\{ \exp (r)^{A} \right\}} \leq \frac{(\rho_{h}(f) + \varepsilon)^{1+\alpha}.(\beta r)^{(\rho_{h}+\varepsilon)(1+\alpha)}.(1 + O(1))^{(\rho_{h}+\varepsilon)(1+\alpha)}}{((\lambda_{h}(f) - \varepsilon)r)^{A}},
\]

(8.3.5)

where we choose \( 0 < \varepsilon < \min \{ \lambda_{h}(f), \frac{A}{1+\alpha} - \rho_{g} \} \).

So from (8.3.5) we obtain that

\[
\lim_{r \to \infty} \frac{\log \mu_{h}^{-1} \mu_{fog}(r)}{\log \mu_{h}^{-1} \mu_{f} \left\{ \exp (r)^{A} \right\}} = 0.
\]

This proves the theorem. \( \blacksquare \)

**Remark 8.3.1** In Theorem 8.3.1 if we take the condition \( 0 < \rho_{h}(f) < \infty \) instead of \( 0 < \lambda_{h}(f) < \rho_{h}(f) < \infty \), the theorem remains true with “limit inferior” in place of “limit .”

In view of Theorem 8.3.1, the following theorem can be carried out:

**Theorem 8.3.2** Let \( f, g \) and \( h \) be any three entire functions where \( g \) is of finite order and \( \lambda_{h}(g) > 0, \rho_{h}(f) < \infty \). Then for every positive constant \( A \) and each \( \alpha \in (-\infty, \infty) \),

\[
\lim_{r \to \infty} \frac{\log \mu_{h}^{-1} \mu_{fog}(r)}{\log \mu_{h}^{-1} \mu_{g} \left\{ \exp (r)^{A} \right\}} = 0 \text{ if } A > (1 + \alpha)\rho_{g}.
\]

The proof is omitted.

**Remark 8.3.2** In Theorem 8.3.2 if we take the condition \( \rho_{h}(g) > 0 \) instead of \( \lambda_{h}(g) > 0 \), the theorem remains true with “limit replaced by limit inferior”.

**Theorem 8.3.3** Let \( f \) and \( h \) be any two entire functions such that \( 0 < \lambda_{h}(f) \leq \rho_{h}(f) < \infty \). Suppose \( g \) be an entire function with \( 0 < A < \rho_{g} \leq \infty \). Then for a sequence of values of \( r \) tending to infinity,

\[
\mu_{h}^{-1} \mu_{fog}(r) > \mu_{h}^{-1} \mu_{f} \left\{ \exp (r)^{A} \right\}.
\]

**Proof.** In view of Lemma 8.2.2 and Lemma 8.2.3, we obtain for all sufficiently large values of \( r \) that

\[
\mu_{fog}(r) \geq \mu_{f} \left( \frac{1}{24} \mu_{g} \left( r^{\frac{1}{4}} - \frac{|g(0)|}{3} \right) \right).
\]
Since $\mu_h^{-1}(r)$ is an increasing function of $r$, in view of Lemma 8.2.1, we get from (8.3.23) for a sequence of values of $r$ tending to infinity that

$$
\mu_h^{-1} \mu_f g(r) \geq \mu_h^{-1} \mu_f \left( \frac{1}{24} \mu_g \left( \frac{r}{4} \right) - \frac{|g(0)|}{3} \right)
$$

i.e., $\log \mu_h^{-1} \mu_f g(r) \geq \log \mu_h^{-1} \mu_f \left( \frac{1}{24} \mu_g \left( \frac{r}{4} \right) - \frac{|g(0)|}{3} \right)$

i.e., $\log \mu_h^{-1} \mu_f g(r) \geq (\lambda_h (f) - \varepsilon) \log \left( \frac{1}{24} \mu_g \left( \frac{r}{4} \right) - \frac{|g(0)|}{3} \right)$

i.e., $\log \mu_h^{-1} \mu_f g(r) \geq (\lambda_h (f) - \varepsilon) \log \mu_g \left( \frac{r}{4} \right) + O(1)$

i.e., $\log \mu_h^{-1} \mu_f g(r) \geq (\lambda_h (f) - \varepsilon) \left( \frac{r}{4} \right)^{\rho_g - \varepsilon} + O(1)$. (8.3.6)

Again from the definition of $\rho_h (f)$, we obtain for all sufficiently large values of $r$ that

$$
\log \mu_h^{-1} \mu_f \left\{ \exp (r)^A \right\} \leq (\rho_h (f) + \varepsilon) r^A.
$$

(8.3.7)

Now from (8.3.6) and (8.3.7), it follows for a sequence of values of $r$ tending to infinity that

$$
\frac{\log \mu_h^{-1} \mu_f g(r)}{\log \mu_h^{-1} \mu_f \left\{ \exp (r)^A \right\}} \geq \frac{(\lambda_h (f) - \varepsilon) \left( \frac{r}{4} \right)^{\rho_g - \varepsilon} + O(1)}{(\rho_h (f) + \varepsilon) r^A}.
$$

(8.3.8)

As $A < \rho_g$, we can choose $\varepsilon(>0)$ in such a way that

$$
A < \rho_g - \varepsilon.
$$

(8.3.9)

Thus from (8.3.8) and (8.3.9), we get that

$$
\limsup_{r \to \infty} \frac{\log \mu_h^{-1} \mu_f g(r)}{\log \mu_h^{-1} \mu_f \left\{ \exp (r)^A \right\}} = \infty.
$$

(8.3.10)

From (8.3.10) we obtain for a sequence of values of $r$ tending to infinity and for $K > 1$ that

$$
\log \mu_h^{-1} \mu_f g(r) > K \log \mu_h^{-1} \mu_f \left\{ \exp (r)^A \right\}
$$

i.e., $\log \mu_h^{-1} \mu_f g(r) > \left\{ \log \mu_h^{-1} \mu_f \left\{ \exp (r)^A \right\} \right\}^K$

i.e., $\log \mu_h^{-1} \mu_f g(r) > \log \mu_h^{-1} \mu_f \left\{ \exp (r)^A \right\}^K$

i.e., $\mu_h^{-1} \mu_f g(r) > \mu_h^{-1} \mu_f \left\{ \exp (r)^A \right\}^K$.

This proves the theorem. ■

**Theorem 8.3.4** Let $f$ and $h$ be any two entire functions such that $0 < \lambda_h (f) \leq \rho_h (f) < \infty$. Suppose $g$ be an entire function with finite relative order with respect to $h$ and $0 < A < \rho_g$. Then for a sequence of values of $r$ tending to infinity,

$$
\mu_h^{-1} \mu_f g(r) > \mu_h^{-1} \mu_g \left\{ \exp (r)^A \right\}.
$$
Proof. Let $0 < A < A_0 < \rho_g$.

Then in view of Theorem 8.3.3, we get for a sequence of values of $r$ tending to infinity that

$$
\mu_h^{-1}\mu_{fog}(r) > \mu_h^{-1}\mu_f\left\{\exp(r)^A\right\}
$$

i.e., $\log \mu_h^{-1}\mu_{fog}(r) > \log \mu_h^{-1}\mu_f\left\{\exp(r)^A\right\}$

i.e., $\log \mu_h^{-1}\mu_{fog}(r) > (\lambda_h(f) - \varepsilon) \log \left\{\exp(r)^A\right\}$

i.e., $\log \mu_h^{-1}\mu_{fog}(r) > (\lambda_h(f) - \varepsilon) r^A$. \hfill (8.3.11)

Again from the definition of $\rho_h(g)$, we obtain for all sufficiently large values of $r$ that

$$
\log \mu_h^{-1}\mu_g\left\{\exp(r)^A\right\} \leq (\rho_h(g) + \varepsilon) r^A.
$$

So combining (8.3.11) and (8.3.12), we obtain for a sequence of values of $r$ tending to infinity that

$$
\frac{\log \mu_h^{-1}\mu_{fog}(r)}{\log \mu_h^{-1}\mu_g\left\{\exp(r)^A\right\}} \geq \frac{(\lambda_h(f) - \varepsilon) r^A}{(\rho_h(g) + \varepsilon) r^A}.
$$

Since $A_0 > A$, from (8.3.13) it follows that

$$
\limsup_{r \to \infty} \frac{\log \mu_h^{-1}\mu_{fog}(r)}{\log \mu_h^{-1}\mu_g\left\{\exp(r)^A\right\}} = \infty.
$$

Thus the theorem follows from (8.3.14). \hfill ■

Theorem 8.3.5 Let $f$, $g$ and $h$ be any three entire functions with $0 < \lambda_h(f) \leq \rho_h(f) < \infty$ and $\lambda_g < A < \infty$. Then for a sequence of values of $r$ tending to infinity,

$$
\mu_h^{-1}\mu_{fog}(r) < \mu_h^{-1}\mu_f\left\{\exp(r)^A\right\}.
$$

Proof. Since $\mu_h^{-1}(r)$ is an increasing function of $r$, from (8.3.1) it follows for a sequence of values of $r$ tending to infinity that

$$
\log \mu_h^{-1}\mu_{fog}(r) \leq \log \mu_h^{-1}\mu_f\left(\frac{(2\alpha - 1) \alpha\beta}{(\alpha - 1) (\beta - 1)} \mu_g(\beta r)\right)
$$

i.e., $\log \mu_h^{-1}\mu_{fog}(r) \leq (\rho_h(f) + \varepsilon) \log \mu_g(\beta r) + O(1)$

i.e., $\log \mu_h^{-1}\mu_{fog}(r) \leq (\rho_h(f) + \varepsilon) (\beta r)^{\lambda_g + \varepsilon} + O(1)$. \hfill (8.3.15)

Now from (8.3.4) and (8.3.15), it follows for a sequence of values of $r$ tending to infinity that

$$
\frac{\log \mu_h^{-1}\mu_{fog}(r)}{\log \mu_h^{-1}\mu_g\left\{\exp(r)^A\right\}} \geq \frac{(\lambda_h(f) - \varepsilon) r^A}{(\rho_h(f) + \varepsilon) (\beta r)^{\lambda_g + \varepsilon} + O(1)}.
$$

\hfill (8.3.16)
As \( \lambda_g < A \), we can choose \( \varepsilon (> 0) \) in such a way that
\[
\lambda_g + \varepsilon < A < \rho_g.
\] (8.3.17)

Thus from (8.3.16) and (8.3.17), we obtain that
\[
\limsup_{r \to \infty} \frac{\log \mu_h^{-1} \mu_f \left\{ \exp (r)^A \right\}}{\log \mu_h^{-1} \mu_f \left( g \right)} = \infty.
\] (8.3.18)

From (8.3.18) we obtain for a sequence of values of \( r \) tending to infinity and also for \( K > 1 \) that
\[
\log \mu_h^{-1} \mu_f (\exp r^A) > K \log \mu_h^{-1} \mu_f \left( g \right)
\]
i.e., \( \log \mu_h^{-1} \mu_f (\exp r^A) > \log \left[ \mu_h^{-1} \mu_f \left( g \right) \right]^K \)
i.e., \( \log \mu_h^{-1} \mu_f (\exp r^A) > \log \mu_h^{-1} \mu_f \left( g \right) \)
i.e., \( \mu_h^{-1} \mu_f (\exp r^A) > \mu_h^{-1} \mu_f \left( g \right) \).

Thus the theorem follows. \[ \blacksquare \]

In the line of Theorem 8.3.5, we may state the following theorem without its proof:

**Theorem 8.3.6** Let \( g \) and \( h \) be any two entire functions such that \( \lambda_h (g) > 0 \) and \( f \) be an entire function with finite relative order with respect to \( h \). Also suppose that \( \lambda_g < A < \infty \).

Then for a sequence of values of \( r \) tending to infinity,
\[
\mu_h^{-1} \mu_f \left( g \right) < \mu_h^{-1} \mu_f \left\{ \exp (r)^A \right\}.
\]

As an application of Theorem 8.3.3 and Theorem 8.3.5, we may prove the following theorem:

**Theorem 8.3.7** Let \( f, g \) and \( h \) be any three entire functions such that \( 0 < \lambda_h (f) \leq \rho_h (f) < \infty \) and \( \lambda_g < A < \rho_g \).

Then
\[
\liminf_{r \to \infty} \frac{\mu_h^{-1} \mu_f \left( g \right)}{\mu_h^{-1} \mu_f \left\{ \exp (r)^A \right\}} \leq 1 \leq \limsup_{r \to \infty} \frac{\mu_h^{-1} \mu_f \left( g \right)}{\mu_h^{-1} \mu_f \left\{ \exp (r)^A \right\}}.
\]

**Proof.** In view of Theorem 8.3.3, we get for a sequence of values of \( r \) tending to infinity that
\[
\frac{\mu_h^{-1} \mu_f \left( g \right)}{\mu_h^{-1} \mu_f \left\{ \exp (r)^A \right\}} > 1
\]
i.e., \( \limsup_{r \to \infty} \frac{\mu_h^{-1} \mu_f \left( g \right)}{\mu_h^{-1} \mu_f \left\{ \exp (r)^A \right\}} \geq 1. \) (8.3.19)
Again from Theorem 8.3.5, we obtain for a sequence of values of \( r \) tending to infinity that

\[
\frac{\mu_h^{-1} \mu_{fg} (r)}{\mu_h^{-1} \mu_f \left\{ \exp (r) \right\}^A} < 1
\]

i.e.,

\[
\liminf_{r \to \infty} \frac{\mu_h^{-1} \mu_{fg} (r)}{\mu_h^{-1} \mu_f \left\{ \exp (r) \right\}^A} \leq 1.
\] (8.3.20)

Thus the theorem follows from (8.3.19) and (8.3.20). ■

In view of Theorem 8.3.4 and Theorem 8.3.6 the following theorem can be carried out:

**Theorem 8.3.8** Let \( f, g \) and \( h \) be any three entire functions such that \( 0 < \lambda_h (f) \leq \rho_h (f) < \infty \), \( 0 < \lambda_h (g) \leq \rho_h (g) < \infty \) and \( 0 < \lambda_g < A < \rho_g < \infty \). Then

\[
\liminf_{r \to \infty} \frac{\mu_h^{-1} \mu_{fg} (r)}{\mu_h^{-1} \mu_f \left\{ \exp (r) \right\}^A} \leq 1 \leq \limsup_{r \to \infty} \frac{\mu_h^{-1} \mu_{fg} (r)}{\mu_h^{-1} \mu_g \left\{ \exp (r) \right\}^A}.
\]

The proof is omitted.

**Theorem 8.3.9** Let \( f, g \) and \( h \) be any three entire functions such that \( \rho_h (g) < \infty \) and \( \lambda_h (f \circ g) = \infty \). Then for every \( A (>0) \),

\[
\lim_{r \to \infty} \frac{\log \mu_h^{-1} \mu_{fg} (r)}{\log \mu_h^{-1} \mu_g (r^A)} = \infty.
\]

**Proof.** If possible, let there exists a constant \( \beta \) such that for a sequence of values of \( r \) tending to infinity,

\[
\log \mu_h^{-1} \mu_{fg} (r) \leq \beta \log \mu_h^{-1} \mu_g (r^A).
\] (8.3.21)

Again from the definition of \( \rho_h (g) \), it follows for all sufficiently large values of \( r \) that

\[
\log \mu_h^{-1} \mu_g (r^A) \leq (\rho_h (g) + \varepsilon) A \log r.
\] (8.3.22)

Now combining (8.3.21) and (8.3.22), we have for a sequence of values of \( r \) tending to infinity,

\[
\log \mu_h^{-1} \mu_{fg} (r) \leq \beta (\rho_h (g) + \varepsilon) A \log r
\]

i.e.,

\[
\lambda_h (f \circ g) \leq \beta A (\rho_h (g) + \varepsilon),
\]

which contradicts the condition \( \lambda_h (f \circ g) = \infty \).

So for all sufficiently large values of \( r \) we get that

\[
\log \mu_h^{-1} \mu_{fg} (r) \geq \beta \log \mu_h^{-1} \mu_g (r^A),
\]

from which the theorem follows. ■

**Remark 8.3.3** Theorem 8.3.9 is also valid with “limit superior” instead of “limit” if \( \lambda_h (f \circ g) = \infty \) is replaced by \( \rho_h (f \circ g) = \infty \) and the other conditions remain the same.
Corollary 8.3.1 Under the assumptions of Theorem 8.3.9 and Remark 8.3.3,
\[
\lim_{r \to \infty} \frac{\mu_h^{-1} \mu_{fog}(r)}{\mu_h^{-1} \mu_g(r^A)} = \infty \quad \text{and} \quad \limsup_{r \to \infty} \frac{\mu_h^{-1} \mu_{fog}(r)}{\mu_h^{-1} \mu_g(r^A)} = \infty
\]
respectively.

Proof. By Theorem 8.3.9, we obtain for all sufficiently large values of \( r \) and for \( K > 1 \) that
\[
\log \mu_h^{-1} \mu_{fog}(r) \geq K \log \mu_h^{-1} \mu_g(r^A)
\]
i.e., \( \mu_h^{-1} \mu_{fog}(r) \geq \left\{ \mu_h^{-1} \mu_g(r^A) \right\}^K \),
from which the first part of the corollary follows.
Similary using Remark 8.3.3, we obtain the second part of the corollary. \( \blacksquare \)

Theorem 8.3.10 Let \( l, f \) and \( h \) be any three entire functions such that \( \lambda_h(l) > 0 \) and \( \rho_h(f) < \infty \). Also for any two entire function \( g \) and \( k \) if \( \rho_g < \lambda_k \), then
\[
\lim_{r \to \infty} \frac{\mu_h^{-1} \mu_{lok}(r)}{\mu_h^{-1} \mu_{fog}(r) \cdot \mu_h^{-1} \mu_f(r)} = \infty.
\]

Proof. In view of Lemma 8.2.2 and Lemma 8.2.3, we obtain for all sufficiently large values of \( r \) that
\[
\mu_{lok}(r) \geq \mu_l \left( \frac{1}{24} \mu_k \left( \frac{r}{4} \right) - \frac{|g(0)|}{3} \right).
\] (8.3.23)
Since \( \mu_h^{-1}(r) \) is an increasing function of \( r \), we get from (8.3.23) for all sufficiently large values of \( r \) that
\[
\mu_h^{-1} \mu_{lok}(r) \geq \mu_h^{-1} \mu_l \left( \frac{1}{24} \mu_k \left( \frac{r}{4} \right) - \frac{|k(0)|}{3} \right)
\]
i.e., \( \log \mu_h^{-1} \mu_{lok}(r) \geq \log \mu_h^{-1} \mu_l \left( \frac{1}{24} \mu_k \left( \frac{r}{4} \right) - \frac{|k(0)|}{3} \right) \)
i.e., \( \log \mu_h^{-1} \mu_{lok}(r) \geq (\lambda_h(l) - \varepsilon) \log \left( \frac{1}{24} \mu_k \left( \frac{r}{4} \right) - \frac{|k(0)|}{3} \right) \)
i.e., \( \log \mu_h^{-1} \mu_{lok}(r) \geq (\lambda_h(l) - \varepsilon) \log \mu_k \left( \frac{r}{4} \right) + O(1) \)
i.e., \( \log \mu_h^{-1} \mu_{lok}(r) \geq (\lambda_h(l) - \varepsilon) \left( \frac{r}{4} \right) \lambda_k^{-\varepsilon} + O(1) \)
i.e., \( \mu_h^{-1} \mu_{lok}(r) \geq \exp \left[ (\lambda_h(l) - \varepsilon) \left( \frac{r}{4} \right) \lambda_k^{-\varepsilon} + O(1) \right] \). (8.3.24)
Also from (8.3.3), it follows for all sufficiently large values of \( r \) that
\[
\mu_h^{-1} \mu_{fog}(r) \leq \exp \left[ (\rho_h(f) + \varepsilon) (\beta r)^{\rho_g + \varepsilon} + O(1) \right].
\] (8.3.25)
Again from the definition of relative order in terms of maximum term we have for all sufficiently large values of $r$ that
\[
\log \mu_{h}^{-1} \mu_f (r) \leq (\rho_h (f) + \varepsilon) \log r
\]
i.e.,
\[
\mu_{h}^{-1} \mu_f (r) \leq r^{(\rho_h (f) + \varepsilon)}.
\] (8.3.26)
From (8.3.25) and (8.3.26), it follows for all sufficiently large values of $r$ that
\[
\mu_{h}^{-1} \mu_{f + g} (r) \cdot \mu_{h}^{-1} \mu_f (r) \leq r^{(\rho_h (f) + \varepsilon)} \exp \left[ (\rho_h (f) + \varepsilon) r^{\rho_g + \varepsilon} + O (1) \right].
\] (8.3.27)
Combining (8.3.24) and (8.3.27), we get for all sufficiently large values of $r$ that
\[
\frac{\mu_{h}^{-1} \mu_{lok} (r)}{\mu_{h}^{-1} \mu_{f + g} (r) \cdot \mu_{h}^{-1} \mu_f (r)} \geq \exp \left[ (\lambda_h (l) - \varepsilon) \left( \frac{r}{4} \right)^{\lambda_k - \varepsilon} + O (1) \right] \exp \left[ (\rho_h (f) + \varepsilon) r^{\rho_g + \varepsilon} + O (1) \right].
\] (8.3.28)
Since $\rho_g < \lambda_k$, we can choose $\varepsilon (> 0)$ in such a manner that
\[
\rho_g + \varepsilon < \lambda_k - \varepsilon.
\] (8.3.29)
Thus the theorem follows from (8.3.28) and (8.3.29).

**Remark 8.3.4** If we consider $\rho_g < \rho_k$ instead of $\rho_g < \lambda_k$ and the other conditions remain the same, the conclusion of Theorem 8.3.10 remains valid with “limit” replaced by “limit superior” as we see in the following theorem:

**Theorem 8.3.11** Let $l$, $f$ and $h$ be any three entire functions such that $\lambda_h (l) > 0$ and $\rho_h (f) < \infty$. Also for any two entire functions $g$ and $k$ if $\rho_g < \rho_k$, then
\[
\limsup_{r \to \infty} \frac{\mu_{h}^{-1} \mu_{lok} (r)}{\mu_{h}^{-1} \mu_{f + g} (r) \cdot \mu_{h}^{-1} \mu_f (r)} = \infty.
\]

**Proof.** As $\rho_g < \rho_k$, we can choose $\varepsilon (> 0)$ in such a manner that
\[
\rho_g + \varepsilon < \lambda_k - \varepsilon.
\] (8.3.30)
Since $\mu_{h}^{-1} (r)$ is an increasing function of $r$, we get from (8.3.23) for a sequence of values of $r$ tending to infinity that
\[
\mu_{h}^{-1} \mu_{lok} (r) \geq \mu_{h}^{-1} \mu_l \left( \frac{1}{24} \mu_k \left( \frac{r}{4} \right) - \frac{|k(0)|}{3} \right)
\]
i.e.,
\[
\log \mu_{h}^{-1} \mu_{lok} (r) \geq \log \mu_{h}^{-1} \mu_l \left( \frac{1}{24} \mu_k \left( \frac{r}{4} \right) - \frac{|k(0)|}{3} \right)
\]
i.e.,
\[
\log \mu_{h}^{-1} \mu_{lok} (r) \geq (\lambda_h (l) - \varepsilon) \log \mu_k \left( \frac{r}{4} \right) + O (1)
\]
i.e.,
\[
\log \mu_{h}^{-1} \mu_{lok} (r) \geq (\lambda_h (l) - \varepsilon) (\frac{r}{4})^{\rho_k - \varepsilon} + O (1)
\]
i.e.,
\[
\mu_{h}^{-1} \mu_{lok} (r) \geq \exp \left[ (\lambda_h (l) - \varepsilon) \left( \frac{r}{4} \right)^{\rho_k - \varepsilon} + O (1) \right].
\] (8.3.31)
Now combining (8.3.31) and (8.3.27), we obtain for a sequence of values of \( r \) tending to infinity that

\[
\frac{\mu_h^{-1} \mu_{fog}(r)}{\mu_h^{-1} \mu_{fog}(r) \cdot \mu_h^{-1} \mu_f(r)} \geq \exp \left( (\lambda_h(l) - \varepsilon) \left( \frac{\rho_h - \varepsilon}{4} \right) + O(1) \right)
\]

Thus in view of (8.3.30) the theorem follows from (8.3.32).  

**Theorem 8.3.12** Let \( h \) and \( f \) be any two entire functions such that \( 0 < \lambda_h(f) \leq \rho_h(f) < \infty \). Then for any entire function \( g \) with finite order,

\[
\limsup_{r \to \infty} \frac{\log \mu_h^{-1} \mu_{fog}(r)}{\log \mu_h^{-1} \mu_f(r)} \leq \frac{\rho_g}{\lambda_h(f)}.
\]

**Proof.** From (8.3.2) it follows for all sufficiently large values of \( r \) that

\[
\log \mu_h^{-1} \mu_{fog}(r) \leq \log \mu_g(\beta r) + O(1)
\]

\[
\frac{\log \mu_h^{-1} \mu_{fog}(r)}{\log \mu_h^{-1} \mu_f(r)} \leq \frac{\log \mu_g(\beta r) + O(1)}{\log \mu_h^{-1} \mu_f(r)}
\]

\[
i.e., \limsup_{r \to \infty} \frac{\log \mu_h^{-1} \mu_{fog}(r)}{\log \mu_h^{-1} \mu_f(r)} \leq \limsup_{r \to \infty} \frac{\log \mu_g(\beta r) + O(1)}{\log \mu_h^{-1} \mu_f(r)}
\]

\[
i.e., \limsup_{r \to \infty} \frac{\log \mu_h^{-1} \mu_{fog}(r)}{\log \mu_h^{-1} \mu_f(r)} \leq \rho_g \cdot \frac{1}{\lambda_h(f)} = \frac{\rho_g}{\lambda_h(f)}.
\]

This proves the theorem.  

**Theorem 8.3.13** Let \( f, g \) and \( h \) be any three entire functions satisfying the conditions \( i) \rho_h(f) < \infty, (ii) \lambda_h(g) > 0 \) and \( iii) \rho_g < \infty \). Then

\[
\limsup_{r \to \infty} \frac{\log \mu_h^{-1} \mu_{fog}(r)}{\log \mu_h^{-1} \mu_g(r)} \leq \frac{\rho_g}{\lambda_h(g)}.
\]

The proof of Theorem 8.3.13 is omitted as it can be carried out in the line of Theorem 8.3.12.

**********