CHAPTER 7

ON RELATIVE ORDER AND MAXIMUM TERM-RELATED COMPARATIVE GROWTH RATES OF ENTIRE FUNCTIONS
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7.1 Introduction, Definitions and Notations.

Let $\mathbb{C}$ the set of all finite complex numbers. Also let $f$ be an entire function defined on $\mathbb{C}$. The maximum term $\mu_f (r)$ of $f = \sum_{n=0}^{\infty} a_n z^n$ on $|z| = r$ is defined by $\mu_f (r) = \max_{n \geq 0} (|a_n| r^n)$ and the maximum modulus $M_f (r)$ of $f$ on $|z| = r$ is defined by $M_f (r) = \max_{|z|=r} |f(z)|$.

To start our chapter, we just recall the following definition:

**Definition 7.1.1** The order $\rho_f$ and lower order $\lambda_f$ of an entire function $f$ are defined as

$$\rho_f = \limsup_{r \to \infty} \frac{\log^2 M_f (r)}{\log r} \quad \text{and} \quad \lambda_f = \liminf_{r \to \infty} \frac{\log^2 M_f (r)}{\log r},$$

where $\log^k x = \log \left( \log^{k-1} x \right)$ for $k = 1, 2, 3, \ldots$ and $\log^0 x = x$.

The results of this chapter have been published in the *Journal of Tripura Mathematical Society* (*Journal Tri. Math. Soc.*), see [19].
Using the inequalities $\mu_f(r) \leq M_f(r) \leq \frac{R}{r} \mu_f(R)$ \{cf. [48]\}, for $0 \leq r < R$ one may verify that

$$
\rho_f = \limsup_{r \to \infty} \frac{\log \mu_f(r)}{\log r} \quad \text{and} \quad \lambda_f = \liminf_{r \to \infty} \frac{\log \mu_f(r)}{\log r}.
$$

If an entire function $g$ is non-constant then $M_g(r)$ is strictly increasing and continuous and its inverse $M_g^{-1} : ([g(0)], \infty) \to (0, \infty)$ exists and is such that $\lim_{s \to \infty} M_g^{-1}(s) = \infty$.

Bernal [2] introduced the definition of relative order of an entire function $f$ with respect to an entire function $g$, denoted by $\rho_g(f)$ as follows:

$$
\rho_g(f) = \inf \{ \mu > 0 : M_f(r) < M_g(r^\mu) \text{ for all } r > r_0(\mu) > 0 \}
$$

$$
= \limsup_{r \to \infty} \frac{\log M_g^{-1}M_f(r)}{\log r}.
$$

The definition coincides with the classical one \{cf.[50]\} if $g(z) = \exp z$.

Similarly, one can define the relative lower order of an entire function $f$ with respect to an entire function $g$ denoted by $\lambda_g(f)$ as follows:

$$
\lambda_g(f) = \liminf_{r \to \infty} \frac{\log M_g^{-1}M_f(r)}{\log r}.
$$

Datta and Maji [13] gave an alternative definition of relative order and relative lower order of an entire function with respect to another entire function in the following way:

**Definition 7.1.2** [13] The relative order $\rho_g(f)$ and relative lower order $\lambda_g(f)$ of an entire function $f$ with respect to an entire function $g$ are defined as follows:

$$
\rho_g(f) = \limsup_{r \to \infty} \frac{\log \mu_g^{-1}\mu_f(r)}{\log r} \quad \text{and} \quad \lambda_g(f) = \liminf_{r \to \infty} \frac{\log \mu_g^{-1}\mu_f(r)}{\log r}.
$$

In the chapter we wish to establish some results relating to the growth rates of composite entire functions in terms of their maximum terms on the basis of relative order (relative lower order).

### 7.2 Lemmas.

In this section we present some lemmas which will be needed in the sequel.

**Lemma 7.2.1** [47] Let $f$ and $g$ be any two entire functions. Then for every $\alpha > 1$ and $0 < r < R$,

$$
\mu_{fog}(r) \leq \frac{\alpha}{\alpha - 1} \mu_f \left( \frac{\alpha R}{R - r} \mu_g(R) \right).
$$

**Lemma 7.2.2** [47] Let $f$ and $g$ be any two entire functions. Then for all sufficiently large values of $r$,

$$
\mu_{fog}(r) \geq \frac{1}{2} \mu_f \left( \frac{1}{8} \mu_g \left( \frac{r}{4} \right) - |g(0)| \right).
$$

**Lemma 7.2.3** [13] If $f$ be entire and $\alpha > 1$, $0 < \beta < \alpha$, then for all sufficiently large $r$,

$$
\mu_f(\alpha r) \geq \beta \mu_f(r).
$$
7.3 Theorems.

In this section we present the main results of the chapter.

**Theorem 7.3.1** Let $f$, $g$ and $h$ be any three entire functions where $g$ is of finite non zero lower order and $\lambda_{h}(f) > 0$, $\rho_{h}(g) < \infty$. Then for every positive constant $A$ and every real number $\alpha$,

\[
\lim_{r \to \infty} \frac{\log \mu_{h}^{-1} \mu_{fog}(r)}{\{\log \mu_{h}^{-1} \mu_{g}(r^{A})\}^{1+\alpha}} = \infty.
\]

**Proof.** If $\alpha$ be such that $1+\alpha \leq 0$, then the theorem is trivial. So we suppose that $1+\alpha > 0$. Now in view of Lemma 7.2.2 and Lemma 7.2.3, we have for all sufficiently large values of $r$ that

\[
\mu_{fog}(r) \geq \mu_{f} \left(1 + \frac{1}{24} \mu_{g} \left(\frac{r}{4}\right) - \frac{|g(0)|}{3}\right).
\]

Since $\mu_{h}^{-1}$ is an increasing function of $r$, it follows from above for all sufficiently large values of $r$ that

\[
\mu_{h}^{-1} \mu_{fog}(r) \geq \mu_{h}^{-1} \mu_{f} \left(1 + \frac{1}{24} \mu_{g} \left(\frac{r}{4}\right) - \frac{|g(0)|}{3}\right),
\]

i.e., $\log \mu_{h}^{-1} \mu_{fog}(r) \geq \log \mu_{h}^{-1} \mu_{f} \left(1 + \frac{1}{24} \mu_{g} \left(\frac{r}{4}\right) - \frac{|g(0)|}{3}\right)$

i.e., $\log \mu_{h}^{-1} \mu_{fog}(r) \geq (\lambda_{h}(f) - \varepsilon) \log \mu_{g} \left(\frac{r}{4}\right) + O(1)$

i.e., $\log \mu_{h}^{-1} \mu_{fog}(r) \geq (\lambda_{h}(f) - \varepsilon) \log \frac{r}{4}^{\lambda_{h} - \varepsilon} + O(1)$

(7.3.1)

where we choose $0 < \varepsilon < \min \left\{\lambda_{h}(f), \lambda_{g}\right\}$. Again from the definition of $\rho_{h}(f)$ in terms of maximum term it follows for all sufficiently large values of $r$ that

\[
\log \mu_{h}^{-1} \mu_{g}(r^{A}) \leq (\rho_{h}(g) + \varepsilon) \mu \log r
\]

i.e., $\{\log \mu_{h}^{-1} \mu_{g}(r^{A})\}^{1+\alpha} \leq (\rho_{h}(g) + \varepsilon)^{1+\alpha} A^{1+\alpha} (\log r)^{1+\alpha}$.

(7.3.2)

Now from (7.3.1) and (7.3.2), it follows for all sufficiently large values of $r$ that

\[
\frac{\log \mu_{h}^{-1} \mu_{fog}(r)}{\{\log \mu_{h}^{-1} \mu_{g}(r^{A})\}^{1+\alpha}} \geq \frac{(\lambda_{h}(f) - \varepsilon) \left(\frac{r}{4}\right)^{\lambda_{h} - \varepsilon} + O(1)}{(\rho_{h}(g) + \varepsilon)^{1+\alpha} A^{1+\alpha} (\log r)^{1+\alpha}}.
\]

Since $\frac{r^{\lambda_{h} - \varepsilon}}{(\log r)^{1+\alpha}} \to \infty$ as $r \to \infty$, the theorem follows from above. $\blacksquare$

**Remark 7.3.1** In Theorem 7.3.1 if we take the condition $\lambda_{h}(g) < \infty$ instead of $\rho_{h}(g) < \infty$, then also Theorem 7.3.1 remains true with “limit superior” in place of “limit”.
In the line of Theorem 7.3.1, one may state the following theorem without its proof:

**Theorem 7.3.2** Let $f$ and $h$ be any two entire functions with $0 < \lambda_h(f) \leq \rho_h(f) < \infty$ and $g$ be an entire function with finite lower order. Then for every positive constant $A$ and every real number $\alpha$,

$$
\lim_{r \to \infty} \frac{\log \mu_h^{-1} \mu_{fog}(r)}{\log \mu_h^{-1} \mu_f(r^A)}^{1+\alpha} = \infty.
$$

**Remark 7.3.2** Theorem 7.3.2 is still valid with “limit superior” instead of “limit” if we replace the condition “$0 < \lambda_h(f) \leq \rho_h(f) < \infty$” by “$0 < \lambda_h(f) < \infty$”.

**Theorem 7.3.3** Let $f, g$ and $h$ be any three entire functions such that $\rho_h(f) < \infty$ and $\lambda_h(f \circ g) = \infty$. Then for every $A(0 > 0)$,

$$
\lim_{r \to \infty} \frac{\log \mu_h^{-1} \mu_{fog}(r)}{\log \mu_h^{-1} \mu_f(r^A)} = \infty.
$$

**Proof.** If possible, let there exist a constant $\beta$ such that for a sequence of values of $r$ tending to infinity,

$$
\log \mu_h^{-1} \mu_{fog}(r) \leq \beta \log \mu_h^{-1} \mu_f(r^A).
$$

Again from the definition of $\rho_h(f)$, it follows for all sufficiently large values of $r$ that

$$
\log \mu_h^{-1} \mu_f(r^A) \leq (\rho_h(f) + \varepsilon) A \log r.
$$

Now combining (7.3.3) and (7.3.4), we have for a sequence of values of $r$ tending to infinity,

$$
\log \mu_h^{-1} \mu_{fog}(r) \leq \beta (\rho_h(f) + \varepsilon) A \log r
$$

i.e.,

$$
\lambda_h(f \circ g) \leq \beta A (\rho_h(f) + \varepsilon),
$$

which contradicts the condition $\lambda_h(f \circ g) = \infty$.

So for all sufficiently large values of $r$ we get that

$$
\log \mu_h^{-1} \mu_{fog}(r) \geq \beta \log \mu_h^{-1} \mu_f(r^A),
$$

from which the theorem follows. □

**Remark 7.3.3** Theorem 7.3.3 is also valid with “limit superior” instead of “limit” if $\lambda_h(f \circ g) = \infty$ is replaced by $\rho_h(f \circ g) = \infty$ and the other conditions remain the same.

**Corollary 7.3.1** Under the assumptions of Theorem 7.3.3 and Remark 7.3.3,

$$
\lim_{r \to \infty} \frac{\log \mu_h^{-1} \mu_{fog}(r)}{\log \mu_h^{-1} \mu_f(r^A)} = \infty \quad \text{and} \quad \limsup_{r \to \infty} \frac{\log \mu_h^{-1} \mu_{fog}(r)}{\log \mu_h^{-1} \mu_f(r^A)} = \infty
$$

hold respectively.
Proof. By Theorem 7.3.3 we obtain for all sufficiently large values of \( r \) and for \( K > 1 \) that

\[
\log \mu_h^{-1} \mu_{fog} (r) \geq K \log \mu_h^{-1} \mu_f (r^A)
\]

i.e., \( \mu_h^{-1} \mu_{fog} (r) \geq \left\{ \mu_h^{-1} \mu_f (r^A) \right\}^K \),

from which the first part of the corollary follows.

Similarly using Remark 7.3.3, we obtain the second part of the corollary. \( \blacksquare \)

**Theorem 7.3.4** If \( f, g \) and \( h \) be any three entire functions with \( \lambda_g < \lambda_h (f) \leq \rho_h (f) < \infty \) then

\[
\liminf_{r \to \infty} \frac{\log \mu_h^{-1} \mu_{fog} (r)}{\mu_h^{-1} \mu_f (r)} = 0 .
\]

**Proof.** Taking \( R = \beta r \) in Lemma 7.2.1 and in view of Lemma 7.2.3 it follows for all sufficiently large values of \( r \) that

\[
\mu_{fog} (r) \leq \frac{\alpha}{\alpha - 1} \mu_f \left( \frac{\alpha \beta}{(\beta - 1)} \mu_g (\beta r) \right)
\]

i.e., \( \mu_{fog} (r) \leq \mu_f \left( \frac{(2\alpha - 1) \alpha \beta}{(\alpha - 1)(\beta - 1)} \mu_g (\beta r) \right) . \) (7.3.5)

Since \( \mu_h^{-1} \) is an increasing function of \( r \), from (7.3.5) it follows for all sufficiently large values of \( r \) that

\[
\mu_h^{-1} \mu_{fog} (r) \leq \mu_h^{-1} \mu_f \left( \frac{(2\alpha - 1) \alpha \beta}{(\alpha - 1)(\beta - 1)} \mu_g (\beta r) \right)
\]

i.e., \( \log \mu_h^{-1} \mu_{fog} (r) \leq \log \mu_h^{-1} \mu_f \left( \frac{(2\alpha - 1) \alpha \beta}{(\alpha - 1)(\beta - 1)} \mu_g (\beta r) \right) . \)

From above we get for a sequence of values of \( r \) tending to infinity that

\[
\log \mu_h^{-1} \mu_{fog} (r) \leq (\rho_h (f) + \varepsilon) \log \mu_g (\beta r) + O(1)
\]

i.e., \( \log \mu_h^{-1} \mu_{fog} (r) \leq (\rho_h (f) + \varepsilon) (\beta r)^{\lambda_g + \varepsilon} + O(1) . \) (7.3.6)

Again from the definition of relative order we obtain for all sufficiently large values of \( r \) that

\[
\mu_h^{-1} \mu_f (r) \geq r^{(\lambda_h (f) - \varepsilon)} . \) (7.3.7)

In view of (7.3.6) and (7.3.7), we get for a sequence of values of \( r \) tending to infinity that

\[
\frac{\log \mu_h^{-1} \mu_{fog} (r)}{\mu_h^{-1} \mu_f (r)} \leq \frac{(\rho_h (f) + \varepsilon) (\beta r)^{\lambda_g + \varepsilon} + O(1)}{r^{(\lambda_h (f) - \varepsilon)}} . \) (7.3.8)

Now as \( \lambda_g < \lambda_h (f) \), we can choose \( \varepsilon (> 0) \) in such a way that \( \lambda_g + \varepsilon < \lambda_h (f) - \varepsilon \) and the theorem follows from (7.3.8). \( \blacksquare \)
Remark 7.3.4 If we take \( \rho_g < \lambda_h (f) \leq \rho_h (f) < \infty \) instead of \( \lambda_g < \lambda_h (f) \leq \rho_h (f) < \infty \) and the other conditions remain the same, the conclusion of Theorem 7.3.4 remains valid with “limit inferior” replaced by “limit” as we see in the following theorem:

Theorem 7.3.5 If \( f, g \) and \( h \) be any three entire functions with \( \rho_g < \lambda_h (f) \leq \rho_h (f) < \infty \) then
\[
\lim_{r \to \infty} \frac{\log \mu_h^{-1} \mu_{fog} (r)}{\mu_h^{-1} \mu_f (r)} = 0.
\]

Proof. Since \( \mu_h^{-1} \) is an increasing function of \( r \), from (7.3.5) it follows for all sufficiently large values of \( r \) that
\[
\log \mu_h^{-1} \mu_{fog} (r) \leq \log \mu_h^{-1} \mu_f \left( \frac{(2 \alpha - 1) \alpha \beta}{(\alpha - 1) (\beta - 1)} \mu_g (\beta r) \right)
\]
i.e.,
\[
\log \mu_h^{-1} \mu_{fog} (r) \leq (\rho_h (f) + \varepsilon) \log \mu_g (\beta r) + O(1)
\]
i.e.,
\[
\log \mu_h^{-1} \mu_{fog} (r) \leq (\rho_h (f) + \varepsilon) (\beta r)^{\rho_g + \varepsilon} + O(1).
\]

Now combining (7.3.7) and (7.3.9), it follows for all sufficiently large values of \( r \) that
\[
\frac{\log \mu_h^{-1} \mu_{fog} (r)}{\mu_h^{-1} \mu_f (r)} \leq \frac{(\rho_h (f) + \varepsilon) (\beta r)^{\rho_g + \varepsilon} + O(1)}{r^{(\lambda_h (f) - \varepsilon)}}.
\]

As \( \rho_g < \lambda_h (f) \) we can choose \( \varepsilon (> 0) \) in such a manner that \( \rho_g + \varepsilon < \lambda_h (f) - \varepsilon \) and thus the theorem follows from (7.3.10).

Theorem 7.3.6 Let \( f \) and \( h \) be any two entire functions such that \( 0 < \lambda_h (f) \leq \rho_h (f) < \infty \). Then for any entire function \( g \),
\[
\lim_{r \to \infty} \frac{\log \mu_h^{-1} \mu_{fog} (r)}{\mu_h^{-1} \mu_f \{\exp (r^A)\}} = \infty,
\]
where \( 0 < A < \lambda_g \).

Proof. From the definition of relative order we obtain for all sufficiently large values of \( r \) that
\[
\log \mu_h^{-1} \mu_f \{\exp (r^A)\} \leq (\rho_h (f) + \varepsilon) \log \{\exp (r^A)\}
\]
i.e.,
\[
\log \mu_h^{-1} \mu_f \{\exp (r^A)\} \leq (\rho_h (f) + \varepsilon) r^A.
\]

So combining (7.3.1) and (7.3.11), we obtain for all sufficiently large values of \( r \) that
\[
\frac{\log \mu_h^{-1} \mu_{fog} (r)}{\log \mu_h^{-1} \mu_f \{\exp (r^A)\}} \geq \frac{(\lambda_h (f) - \varepsilon) (\frac{\varepsilon}{4})^{\lambda_g - \varepsilon} + O(1)}{(\rho_h (f) + \varepsilon) r^A}.
\]

Since \( 0 < A < \lambda_g \), we can choose \( \varepsilon (\varepsilon > 0) \) in such a way that
\[
A < \lambda_g - \varepsilon.
\]

Thus the theorem follows from (7.3.12) and (7.3.13).
Corollary 7.3.2 Under the assumptions of Theorem 7.3.6,
\[
\lim_{r \to \infty} \frac{\mu_h^{-1}\mu_{fog}(r)}{\mu_h^{-1}\mu_f \{\exp(r^A)\}} = \infty \quad \text{where} \quad 0 < A < \lambda_y.
\]

The proof of Corollary 7.3.2 is omitted because it can be carried out in the line of Corollary 7.3.1 and from Theorem 7.3.6.

Analogously one may state the following theorem and corollary without its proof as those can be carried out in the line of Theorem 7.3.6 and Corollary 7.3.2 respectively.

Theorem 7.3.7 Let \( f \) and \( h \) be any two entire functions such that \( 0 < \lambda_h(f) \leq \rho_h(f) < \infty \).
Suppose \( g \) be an entire function of finite relative order with respect to entire function \( h \).
Then for every \( A \) with \( 0 < A < \lambda_y \),
\[
\lim_{r \to \infty} \frac{\log \mu_h^{-1}\mu_{fog}(r)}{\log \mu_h^{-1}\mu_g \{\exp(r^A)\}} = \infty.
\]

Corollary 7.3.3 Under the assumptions of Theorem 7.3.7,
\[
\lim_{r \to \infty} \frac{\mu_h^{-1}\mu_{fog}(r)}{\mu_h^{-1}\mu_g \{\exp(r^A)\}} = \infty \quad \text{where} \quad 0 < A < \lambda_y.
\]

Theorem 7.3.8 Let \( f \), \( g \) and \( h \) be any three entire functions with \( 0 < \lambda_h(f) \leq \rho_h(f) < \infty \).
Then for any entire \( g \) with finite order,
\[
\lim_{r \to \infty} \frac{\log \mu_h^{-1}\mu_{fog}(r)}{\log \mu_h^{-1}\mu_{fog}(r)} = \infty,
\]
where \( \rho_g < A < \infty \) holds.

Proof. From the definition of relative lower order we obtain for all sufficiently large values of \( r \) that
\[
\log \mu_h^{-1}\mu_f \{\exp(r^A)\} \geq (\lambda_h(f) - \varepsilon) \log \{\exp(r^A)\}
\]
\[\text{i.e.,} \quad \log \mu_h^{-1}\mu_f \{\exp(r^A)\} \geq (\lambda_h(f) - \varepsilon) r^A. \quad (7.3.14)\]

Now from (7.3.9) and (7.3.14), it follows for a sequence of values of \( r \) tending to infinity that
\[
\frac{\log \mu_h^{-1}\mu_f \{\exp(r^A)\}}{\log \mu_h^{-1}\mu_{fog}(r)} \geq \frac{(\lambda_h(f) - \varepsilon) r^A}{(\rho_h(f) + \varepsilon) (\beta r)^{\rho_g + \varepsilon} + O(1)}. \quad (7.3.15)
\]

As \( \rho_g < \mu \), we can choose \( \varepsilon (> 0) \) in such a way that
\[
\rho_g + \varepsilon < A. \quad (7.3.16)
\]

Thus from (7.3.15) and (7.3.16), we obtain that
\[
\lim_{r \to \infty} \frac{\log \mu_h^{-1}\mu_f \{\exp(r^A)\}}{\log \mu_h^{-1}\mu_{fog}(r)} = \infty.
\]
Thus the theorem follows. ■

In the line of Theorem 7.3.8, we may state the following theorem without its proof:
**Theorem 7.3.9** Let \( g \) and \( h \) be any two entire functions with \( \lambda_h(g) > 0 \) and \( f \) be an entire function with finite relative order with respect to \( h \). Then for every \( A \) with \( \rho_g < A < \infty \),

\[
\lim_{r \to \infty} \frac{\log \mu_h^{-1} \mu_g \{ \exp (r^A) \}}{\log \mu_h^{-1} \mu_{fog} (r)} = \infty.
\]

**Corollary 7.3.4** Under the assumptions of Theorem 7.3.8,

\[
\lim_{r \to \infty} \frac{\mu_h^{-1} \mu_f \{ \exp (r^A) \}}{\mu_h^{-1} \mu_{fog} (r)} = \infty \quad \text{where} \quad \rho_g < A < \infty.
\]

**Corollary 7.3.5** Under the hypothesis of Theorem 7.3.9,

\[
\lim_{r \to \infty} \frac{\mu_h^{-1} \mu_g \{ \exp (r^A) \}}{\mu_h^{-1} \mu_{fog} (r)} = \infty \quad \text{where} \quad \rho_g < A < \infty.
\]

The proof of the above two corollaries are omitted as those may be carried out from Theorem 7.3.8 and Theorem 7.3.9 respectively in the line of Corollary 7.3.1.