CHAPTER 6

GROWTH OF WRONSKIANs GENERATED BY ENTIRE OR MEROMORPHIC FUNCTIONS ON THE BASIS OF ZERO ORDER AND WEAK TYPE
6.1 Introduction, Definitions and Notations.

A single valued function of one complex variable which has no singularities other than poles in the finite complex plane $\mathbb{C}$ is called a meromorphic function. On the other hand, an entire function is analytic in $\mathbb{C}$. Let $f$ be a meromorphic function and $g$ be an entire function defined on $\mathbb{C}$. In the sequel the following two notations are used:

\[ \log^{(k)} x = \log \left( \log^{(k-1)} x \right) \text{ for } k = 1, 2, 3, \ldots \text{ and } \]
\[ \log^{(0)} x = x. \]

and

\[ \exp^{(k)} y = \exp \left( \exp^{(k-1)} y \right) \text{ for } k = 1, 2, 3, \ldots \text{ and } \]
\[ \exp^{(0)} y = y. \]

The results of this chapter have been published in the Global Journal of Computational Science and Mathematics (GJCSM), see [18].
Though Definition 6.1.1, Definition 6.1.2, Definition 6.1.3, Definition 6.1.4 and Definition 6.1.5 have already been defined in Chapter 2 as Definition 2.1.1, Definition 2.1.2, Definition 2.1.3, Definition 2.1.4 and Definition 2.1.5 respectively, we state here again in order to keep a continuation of our discussion:

**Definition 6.1.1** The order $\rho_f$ and lower order $\lambda_f$ of an entire function $f$ are defined as

$$
\rho_f = \limsup_{r \to \infty} \frac{\log^2 M(r, f)}{\log r} \quad \text{and} \quad \lambda_f = \liminf_{r \to \infty} \frac{\log^2 M(r, f)}{\log r}.
$$

If $f$ is meromorphic then

$$
\rho_f = \limsup_{r \to \infty} \frac{\log T(r, f)}{\log r} \quad \text{and} \quad \lambda_f = \liminf_{r \to \infty} \frac{\log T(r, f)}{\log r}.
$$

If $\rho_f < \infty$ then $f$ is of finite order. Also $\rho_f = 0$ means that $f$ is of order zero. In this connection Liao and Yang [35] gave the following definition:

**Definition 6.1.2** [35]Let $f$ be a meromorphic function of order zero. Then the quantities $\rho_f^*$ and $\lambda_f^*$ of a meromorphic function $f$ are defined as :

$$
\rho_f^* = \limsup_{r \to \infty} \frac{\log^2 T(r, f)}{\log^2 r} \quad \text{and} \quad \lambda_f^* = \liminf_{r \to \infty} \frac{\log^2 T(r, f)}{\log^2 r}.
$$

If $f$ is entire, then

$$
\rho_f^* = \limsup_{r \to \infty} \frac{\log^2 M(r, f)}{\log^2 r} \quad \text{and} \quad \lambda_f^* = \liminf_{r \to \infty} \frac{\log^2 M(r, f)}{\log^2 r}.
$$

Datta and Biswas [12] gave an alternative definition of zero order and zero lower order of a meromorphic function in the following way:

**Definition 6.1.3** [12]Let $f$ be a meromorphic function of order zero. Then the quantities $\rho_f^{**}$ and $\lambda_f^{**}$ of $f$ are defined by

$$
\rho_f^{**} = \limsup_{r \to \infty} \frac{T(r, f)}{\log r} \quad \text{and} \quad \lambda_f^{**} = \liminf_{r \to \infty} \frac{T(r, f)}{\log r}.
$$

If $f$ is an entire function then clearly

$$
\rho_f^{**} = \limsup_{r \to \infty} \frac{\log M(r, f)}{\log r} \quad \text{and} \quad \lambda_f^{**} = \liminf_{r \to \infty} \frac{\log M(r, f)}{\log r}.
$$

**Definition 6.1.4** The type $\sigma_f$ and lower type $\overline{\sigma}_f$ of a meromorphic function $f$ are defined as

$$
\sigma_f = \limsup_{r \to \infty} \frac{T(r, f)}{r^{\rho_f}} \quad \text{and} \quad \overline{\sigma}_f = \liminf_{r \to \infty} \frac{T(r, f)}{r^{\rho_f}}, \quad 0 < \rho_f < \infty.
$$

When $f$ is entire, it can easily be verified that

$$
\sigma_f = \limsup_{r \to \infty} \frac{\log M(r, f)}{r^{\rho_f}} \quad \text{and} \quad \overline{\sigma}_f = \liminf_{r \to \infty} \frac{\log M(r, f)}{r^{\rho_f}}, \quad 0 < \rho_f < \infty.
$$
Datta and Jha [11] gave the definition of the weak type of a meromorphic function of finite positive lower order in the following way:

**Definition 6.1.5** [11] The weak type $\tau_f$ of a meromorphic function $f$ of finite positive lower order $\lambda_f$ is defined by

$$\tau_f = \liminf_{r \to \infty} \frac{T(r, f)}{r^{\lambda_f}}.$$  

For entire $f$,

$$\tau_f = \liminf_{r \to \infty} \frac{\log M(r, f)}{r^{\lambda_f}}, \ 0 < \lambda_f < \infty.$$  

Similarly, one can define the growth indicator $\bar{\tau}_f$ of a meromorphic function $f$ of finite positive lower order $\lambda_f$ as

$$\bar{\tau}_f = \limsup_{r \to \infty} \frac{T(r, f)}{r^{\lambda_f}}.$$  

When $f$ is entire, it can easily be verified that

$$\bar{\tau}_f = \limsup_{r \to \infty} \frac{\log M(r, f)}{r^{\lambda_f}}, \ 0 < \lambda_f < \infty.$$  

**Definition 6.1.6** A meromorphic function $a \equiv a(z)$ is called small with respect to $f$ if $T(r, a) = S(r, f)$.  

**Definition 6.1.7** Let $a_1, a_2, \ldots, a_k$ be linearly independent meromorphic functions and small with respect to $f$. We denote by $L(f) = W(a_1, a_2, \ldots, a_k, f)$ the Wronskian determinant of $a_1, a_2, \ldots, a_k, f$ i.e.,

$$L(f) = \begin{vmatrix} a_1 & a_2 & \ldots & a_k & f \\ a_1' & a_2' & \ldots & a_k' & f' \\ \vdots & \vdots & \ddots & \vdots \\ a_1^{(k)} & a_2^{(k)} & \ldots & a_k^{(k)} & f^{(k)} \end{vmatrix}.$$  

**Definition 6.1.8** If $a \in \mathbb{C} \cup \{\infty\}$, the quantity

$$\delta(a; f) = 1 - \limsup_{r \to \infty} \frac{N(r, a; f)}{T(r, f)} = \liminf_{r \to \infty} \frac{m(r, a; f)}{T(r, f)}$$

is called the Nevanlinna’s deficiency of the value ‘$a$’.

From the second fundamental theorem it follows that the set of values of $a \in \mathbb{C} \cup \{\infty\}$ for which $\delta(a; f) > 0$ is countable and $\sum_{a \neq \infty} \delta(a; f) + \delta(\infty; f) \leq 2$ (cf [29], p.43). If in particular, $\sum_{a \neq \infty} \delta(a; f) + \delta(\infty; f) = 2$, we say that $f$ has the maximum deficiency sum.

Let $f$ be an entire function and $g$ be a transcendental entire function defined in the open complex plane $\mathbb{C}$. Polya [41] proved that if $\rho_{fg} < \infty$ then $\rho_f = 0$. Edrei and Fuchs [26]
proved that if \( f \) is a meromorphic function and \( g \) is a transcendental entire function then \( \lambda_{\text{fg}} < \infty \) implies that \( \lambda_f = 0 \). Under the same conditions, Gross [28] proved that if \( \rho_{\text{fg}} \) is finite then \( \rho_f = 0 \). Now a question may be investigated about the order and lower order of \( f \circ g \) where \( f \) is of finite order and \( g \) is of order zero. Datta and Biswas [12] worked on this question and proved that if \( \rho_f < \infty \) and \( \rho_g = 0 \) then \( \rho_{\text{fg}} \) is finite for meromorphic \( f \) and entire \( g \).

In this chapter we establish some newly developed results based on the comparative growth properties of composite entire or meromorphic functions considering the left factor or right factor to be of order zero and wronskians generated by one of the factors, which improve some earlier results.

### 6.2 Lemmas.

In this section we present some lemmas which will be needed in the sequel.

**Lemma 6.2.1** [1] Let \( f \) be meromorphic and \( g \) be entire then for all sufficiently large values of \( r \),

\[
T(r, f \circ g) \leq \{1 + o(1)\} \frac{T(r, g)}{\log M(r, g)} T(M(r, g), f).
\]

**Lemma 6.2.2** [3] Let \( f \) be meromorphic and \( g \) be entire and suppose that \( 0 < \mu < \rho_g \leq \infty \). Then for a sequence of values of \( r \) tending to infinity,

\[
T(r, f \circ g) \geq T(\exp(r^\mu), f).
\]

**Lemma 6.2.3** [5] If \( f \) and \( g \) are two entire functions then for all sufficiently large values of \( r \),

\[
M(r, f \circ g) \leq M(M(r, g), f).
\]

**Lemma 6.2.4** [37] Let \( f \) be a transcendental meromorphic function having the maximum deficiency sum. Then

\[
\lim_{r \to \infty} \frac{T(r, L(f))}{T(r, f)} = 1 + k - k\delta(\infty; f).
\]

**Lemma 6.2.5** If \( f \) be a transcendental meromorphic function with the maximum deficiency sum, then the order and lower order of \( L(f) \) are same as those of \( f \). Also \( \sigma_{L(f)}, \sigma_{\bar{L}(f)}, \tau_{L(f)} \) and \( \bar{\sigma}_{L(f)} \) are \( \{1 + k - k\delta(\infty; f)\} \) times that of \( f \), i.e., \( \rho_{L(f)} = \rho_f, \lambda_{L(f)} = \lambda_f, \sigma_{L(f)} = \{1 + k - k\delta(\infty; f)\} \sigma_f, \bar{\sigma}_{L(f)} = \{1 + k - k\delta(\infty; f)\} \bar{\sigma}_f, \tau_{L(f)} = \{1 + k - k\delta(\infty; f)\} \tau_f, \) and \( \bar{\tau}_{L(f)} = \{1 + k - k\delta(\infty; f)\} \bar{\tau}_f \) when \( f \) is of finite positive order.

**Proof.** By Lemma 6.2.4,

\[
\lim_{r \to \infty} \frac{\log T(r, L(f))}{\log T(r, f)}
\]
exists and is equal to 1. So

\[ \varrho_{L[f]} = \limsup_{r \to \infty} \frac{\log T(r, L(f))}{\log r} \]

\[ = \limsup_{r \to \infty} \frac{\log T(r, f)}{\log r} \cdot \lim_{r \to \infty} \frac{\log T(r, L(f))}{\log T(r, f)} \]

\[ = \varrho_f.1 = \varrho_f. \]

Also,

\[ \lambda_{L[f]} = \liminf_{r \to \infty} \frac{\log T(r, L(f))}{\log r} \]

\[ = \liminf_{r \to \infty} \frac{\log T(r, f)}{\log r} \cdot \lim_{r \to \infty} \frac{\log T(r, L(f))}{\log T(r, f)} \]

\[ = \lambda_f.1 = \lambda_f. \]

Again by Lemma 6.2.4,

\[ \sigma_{L[f]} = \limsup_{r \to \infty} \frac{T(r, L[f])}{r^{\varrho_{L[f]}}} \]

\[ = \lim \frac{T(r, L[f])}{T(r, f)} \cdot \limsup_{r \to \infty} \frac{T(r, f)}{r^{\varrho_f}} \]

\[ = \{1 + k - k\delta(\infty; f)\}\sigma_f. \]

Similarly,

\[ \bar{\sigma}_{L[f]} = \liminf_{r \to \infty} \frac{T(r, L[f])}{r^{\varrho_{L[f]}}} \]

\[ = \lim \frac{T(r, L[f])}{T(r, f)} \cdot \liminf_{r \to \infty} \frac{T(r, f)}{r^{\varrho_f}} \]

\[ = \{1 + k - k\delta(\infty; f)\}\bar{\sigma}_f. \]

Also,

\[ \tau_{L(f)} = \liminf_{r \to \infty} \frac{T(r, L[f])}{r^{\lambda_{L[f]}}} \]

\[ = \lim \frac{T(r, L[f])}{T(r, f)} \cdot \liminf_{r \to \infty} \frac{T(r, f)}{r^{\lambda_f}} \]

\[ = \{1 + k - k\delta(\infty; f)\}\tau_f. \]

Analogously,

\[ \bar{\tau}_{L[f]} = \limsup_{r \to \infty} \frac{T(r, L[f])}{r^{\lambda_{L[f]}}} \]

\[ = \lim \frac{T(r, L[f])}{T(r, f)} \cdot \limsup_{r \to \infty} \frac{T(r, f)}{r^{\lambda_f}} \]

\[ = \{1 + k - k\delta(\infty; f)\}\bar{\tau}_f. \]

This proves the lemma.
6.3 Theorems.

In this section we present the main results of the chapter.

**Theorem 6.3.1** Let $f$ be a meromorphic function of order zero and $g$ be a transcendental entire function with the maximum deficiency sum such that $\rho_f^{**}$ is finite and $0 < \lambda_g \leq \rho_g < \infty$. Then for any $\mu > 0$ and all sufficiently large values of $r$

$$T(r, f \circ g) < T(\exp(\mu), L(g)).$$

**Proof.** In view of Lemma 6.2.1 and the inequality $T(r, g) \leq \log^+ M(r, g)$, we get for all sufficiently large values of $r$

$$T(r, f \circ g) \leq \{1 + o(1)\} \left(\rho_f^{**} + \varepsilon\right) \log M(r, g) + O(1)$$

*i.e.*, $\log T(r, f \circ g) \leq \log \left[2\right] M(r, g) + O(1)$

*i.e.*, $\log T(r, f \circ g) \leq (\rho_g + \varepsilon) \log r + O(1).$ \hspace{1cm} (6.3.1)

Again from the definition of $\lambda_{L(g)}$ and for any $\mu > 0$, we obtain for all sufficiently large values of $r$

$$\log T(\exp(\mu), L(g)) \geq (\lambda_{L(g)} - \varepsilon) \log \exp(\mu)$$

*i.e.*, $\log T(\exp(\mu), L(g)) \geq (\lambda_g - \varepsilon) r^\mu.$ \hspace{1cm} (6.3.2)

Now from (6.3.1) and (6.3.2), it follows for all sufficiently large values of $r$

$$\frac{\log T(\exp(\mu), L(g))}{\log T(r, f \circ g)} \geq \frac{(\lambda_g - \varepsilon) r^\mu}{(\rho_g + \varepsilon) \log r + O(1)}$$

*i.e.*, $\lim_{r \to \infty} \frac{\log T(\exp(\mu), L(g))}{\log T(r, f \circ g)} = \infty.$ \hspace{1cm} (6.3.3)

From (6.3.3) we obtain for all sufficiently large values of $r$ and for $K > 1$

$$\log T(\exp(\mu), L(g)) > K \log T(r, f \circ g)$$

*i.e.*, $\log T(\exp(\mu), L(g)) > \log \left\{T(r, f \circ g)\right\}^K$

*i.e.*, $\log T(\exp(\mu), L(g)) > \log T(r, f \circ g)$

*i.e.*, $T(\exp(\mu), L(g)) > T(r, f \circ g).$

This proves the theorem. $\blacksquare$

**Remark 6.3.1** The following examples verifies the conclusion of Theorem 6.3.1.

**Example 6.3.1** Let $f = z$, $g = \exp z$ and $\mu = 1$. Taking $a_1 = 1$ in Definition 6.1.7, we obtain that $L(g) = \exp z$. Then

$$\rho_f^{**} = 1 \text{ and } \rho_g = \lambda_g = 1.$$
Now
\[ T(r, f \circ g) = T(r, \exp z) = \frac{r}{\pi} \]
and
\[ T(\exp(r^\mu), L(g)) = T(\exp r, \exp z) = \frac{\exp r}{\pi}. \]
Therefore
\[ T(r, f \circ g) < T(\exp(r^\mu), L(g)). \]

**Example 6.3.2** Let \( f = z, g = \exp^{[2]} z \) and \( \mu = 1 \). Considering \( a_1 = 1 \) in Definition 6.1.7, we get that \( L(g) = \exp z \exp^{[2]} z \).

Then
\[ \rho_f^{**} = 1 \text{ and } \rho_g = \lambda_g = \infty. \]

Now
\[ T(r, f \circ g) = T(r, \exp^{[2]} z) \leq \log(\exp^{[2]} r) = \exp r. \]
and
\[
T(\exp(r^\mu), L(g)) = T(\exp r, \exp z \exp^{[2]} z) \\
\leq T(r, \exp z) + T(r, \exp^{[2]} z) + O(1) \\
\leq \frac{\exp r}{\pi} + \log(\exp^{[2]} z) + O(1) = \frac{\exp r}{\pi} + \exp r + O(1).
\]

Therefore
\[ T(r, f \circ g) < T(\exp(r^\mu), L(g)). \]

**Theorem 6.3.2** Let \( f \) be a transcendental meromorphic function with the maximum deficiency sum and \( g \) be an entire function such that \( \rho_f^{**} < \infty \) and \( 0 < \lambda_f \leq \rho_f < \infty \). Then for all sufficiently large values of \( r \),
\[ T(r, f \circ g) < T(\exp(r^\mu), L(f)), \]
where \( \mu > 0 \).

**Proof.** By Lemma 6.2.1 and the inequality \( T(r, g) \leq \log^+ M(r, g) \), we get for all sufficiently large values of \( r \) that
\[
T(r, f \circ g) \leq \{1 + o(1)\} T(M(r, g), f) \\
i.e., \quad \log T(r, f \circ g) \leq (\rho_f + \varepsilon) \log M(r, g) + O(1) \\
i.e., \quad \log T(r, f \circ g) \leq (\rho_f + \varepsilon) (\rho_g^{**} + \varepsilon) \log r + O(1). \tag{6.3.4}
\]
Again from the definition of $\lambda_{L(f)}$ and for any $\mu > 0$, we obtain for all sufficiently large values of $r$ that

$$\log T(\exp(r^\mu), L(f)) \geq (\lambda_f - \varepsilon) \log \exp(r^\mu)$$

i.e., \( \log T(\exp(r^\mu), L(f)) \geq (\lambda_f - \varepsilon) r^\mu \). \hspace{1cm} (6.3.5)

Now from (6.3.4) and (6.3.5), it follows for all sufficiently large values of $r$ that

$$\frac{\log T(\exp(r^\mu), L(f))}{\log T(r, f \circ g)} \geq \frac{(\lambda_f - \varepsilon) r^\mu}{(\rho_f + \varepsilon) (\rho_g^{**} + \varepsilon) \log r + O(1)}$$

i.e., \( \lim_{r \to \infty} \frac{\log T(\exp(r^\mu), L(f))}{\log T(r, f \circ g)} = \infty. \) \hspace{1cm} (6.3.6)

From (6.3.6) we obtain for all sufficiently large values of $r$ and for $K > 1$ that

$$\log T(\exp(r^\mu), L(f)) > K \log T(r, f \circ g)$$

i.e., \( \log T(\exp(r^\mu), L(f)) > \log \{T(r, f \circ g)\}^K \)

i.e., \( \log T(\exp(r^\mu), L(f)) > \log T(r, f \circ g) \)

i.e., \( T(\exp(r^\mu), L(f)) > T(r, f \circ g) \).

This proves the theorem. \[\blacksquare\]

Remark 6.3.2 The following examples check the conclusion of Theorem 6.3.2.

Example 6.3.3 Let $f = \exp z, g = z$ and $\mu = 1$. Considering $a_1 = 1$ in Definition 6.1.7, we obtain that $L(f) = \exp z$.

Then \( \rho_f = \lambda_f = 1 \) and $\rho_g^{**} = 1$.

Now

$$T(r, f \circ g) = T(r, \exp z) = \frac{r}{\pi}$$

and

$$T(\exp(r^\mu), L(f)) = T(\exp r, \exp z) = \frac{\exp r}{\pi}.$$

Thus

$$T(r, f \circ g) < T(\exp(r^\mu), L(f)).$$

Example 6.3.4 Let $f = \exp^{[2]} z, g = z$ and $\mu = 1$. Choosing $a_1 = 1$ in Definition 6.1.7, we obtain that $L(g) = \exp z \exp^{[2]} z$.
Then
\[ \rho_g^{**} = 1 \text{ and } \rho_f = \lambda_f = \infty. \]

Now
\[ T(r, f \circ g) = T(r, \exp^{[2]} z) \leq \log (\exp^{[2]} r) = \exp r \]
and
\[ T(\exp(r^\mu), L(f)) = T(\exp r, \exp z \exp^{[2]} z) \]
\[ \leq T(\exp r, \exp z) + T(\exp r, \exp^{[2]} z) + O(1) \]
\[ \leq \frac{\exp r}{\pi} + \log (\exp^{[2]} r) + O(1) = \frac{\exp r}{\pi} + \exp r + O(1). \]

Therefore
\[ T(r, f \circ g) < T(\exp(r^\mu), L(f)). \]

As an application of Theorem 6.3.1 and Theorem 6.3.2, we may prove the following four theorems:

**Theorem 6.3.3** Let \( f \) be a meromorphic function of order zero and \( g \) be a transcendental entire function with the maximum deficiency sum such that \( \rho_f^{**} \) is finite, \( \lambda_g > 0 \) and \( \bar{\sigma}_g > 0 \). Then
\[
\limsup_{r \to \infty} \frac{\log T(r, f \circ g)}{T(r, L(g))} \leq \frac{\rho_g}{\{1 + k - k\delta (\infty; f)\} \bar{\sigma}_g}.
\]

**Proof.** Since \( \rho_g > 0 \), taking \( \mu = \rho_g \) in Theorem 6.3.1 we obtain for all sufficiently large values of \( r \) that
\[
\log T(r, f \circ g) < \log T(\exp r^\rho_g, L(g))
\]
i.e., \( \log T(r, f \circ g) < (\rho_L(g) + \varepsilon) \log \exp r^\rho_g \)
i.e., \( \log T(r, f \circ g) < (\rho_g + \varepsilon) r^\rho_g. \quad (6.3.7) \]

Again we have for arbitrary positive \( \varepsilon \) and for all sufficiently large values of \( r \),
\[
T(r, L(g)) \geq \left( \sigma_L(g) - \varepsilon \right) r^\rho_L(g)
\]
i.e., \( T(r, L(g)) \geq \left(1 + k - k\delta (\infty; f)\right) \bar{\sigma}_g - \varepsilon \) \( r^\rho_g. \quad (6.3.8) \]

Therefore from (6.3.7) and (6.3.8) it follows for all sufficiently large values of \( r \) that
\[
\frac{\log T(r, f \circ g)}{T(r, L(g))} \leq \frac{(\rho_g + \varepsilon) r^\rho_g}{\left(1 + k - k\delta (\infty; f)\right) \bar{\sigma}_g - \varepsilon} r^\rho_g
\]
i.e., \( \limsup_{r \to \infty} \frac{\log T(r, f \circ g)}{T(r, L(g))} \leq \frac{\rho_g}{\{1 + k - k\delta (\infty; f)\} \bar{\sigma}_g}. \)

Thus the theorem is established. \( \blacksquare \)
Remark 6.3.3 The conclusion of Theorem 6.3.3 can be verified by the following example.

Example 6.3.5 Let $f = z$ and $g = \exp z$. Taking $a_1 = 1$ in Definition 6.1.7, we get that $L(g) = \exp z$.

Then

$$
\rho_f^* = 1, \rho_g = \lambda_g = 1, \bar{\sigma}_g = 1 \text{ and } \delta(\infty, f) = 1.
$$

Now

$$
T(r, f \circ g) = T(r, \exp z) = \frac{r}{\pi}
$$

and so

$$
\log T(r, f \circ g) = \log r + O(1).
$$

Again

$$
T(r, L(g)) = T(r, \exp z) = \frac{r}{\pi}.
$$

Therefore

$$
\limsup_{r \to \infty} \frac{\log T(r, f \circ g)}{T(r, L(g))} = 0 \quad \text{and} \quad \frac{\rho_g}{\{1 + k - k\delta(\infty; f)\}\bar{\sigma}_g} = 1.
$$

Hence

$$
\limsup_{r \to \infty} \frac{\log T(r, f \circ g)}{T(r, L(g))} \leq \frac{\rho_g}{\{1 + k - k\delta(\infty; f)\}\bar{\sigma}_g}.
$$

Theorem 6.3.4 Let $f$ be a meromorphic function of order zero and $g$ be a transcendental entire function with the maximum deficiency sum such that $\rho_f^*$ is finite, $\lambda_g > 0$ and $\bar{\sigma}_g > 0$. Then

$$
\liminf_{r \to \infty} \frac{\log T(r, f \circ g)}{T(r, L(g))} \leq \min \left\{ \frac{1}{\{1 + k - k\delta(\infty; f)\}} \left\{ \frac{\rho_g}{\sigma_g}, \frac{\lambda_g}{\bar{\sigma}_g} \right\} \right\}.
$$

Theorem 6.3.4 can easily be proved in the line of Theorem 6.3.3. Hence its proof is omitted.

Remark 6.3.4 The following example ensures the conclusion of Theorem 6.3.4.

Example 6.3.6 Let $f = z$ and $g = \exp z$. Further, supposing $a_1 = 1$ in Definition 6.1.7, we obtain that $L(g) = \exp z$. 

\[ \rho_f^{**} = 1, \rho_g = \lambda_g = 1 \text{ and } \sigma_g = \sigma_g = 1. \]

Now
\[ T(r, f \circ g) = T(r, \exp z) = \frac{r}{\pi}. \]

So
\[ \log T(r, f \circ g) = \log r + O(1). \]

Again
\[ T(r, L(g)) = T(r, \exp z) = \frac{r}{\pi}. \]

Therefore
\[ \liminf_{r \to \infty} \frac{\log T(r, f \circ g)}{T(r, L(g))} = 0 \quad \text{and} \quad \frac{\rho_g}{\{1 + k - k \delta (\infty; f)\} \sigma_g} = 1. \]

Thus
\[ \liminf_{r \to \infty} \frac{\log T(r, f \circ g)}{T(r, L(g))} \leq \frac{\rho_f}{\{1 + k - k \delta (\infty; f)\} \sigma_f}. \]

**Theorem 6.3.5** Let \( f \) be a transcendental meromorphic function with the maximum deficiency sum and \( g \) be an entire function of order zero such that \( \rho_g^{**} < \infty, \lambda_f > 0 \) and \( \bar{\sigma}_f > 0 \). Then
\[ \limsup_{r \to \infty} \frac{\log T(r, f \circ g)}{T(r, L(f))} \leq \frac{\rho_f}{\{1 + k - k \delta (\infty; f)\} \sigma_f}. \]

**Proof.** Since \( \rho_f > 0 \), taking \( \mu = \rho_f \) in Theorem 6.3.2 we obtain for all sufficiently large values of \( r \) that
\[ \log T(r, f \circ g) < \log T\{\exp r^{\rho_f}, L(f)\} \]
\[ \quad \text{i.e.,} \quad \log T(r, f \circ g) < (\rho_{L(f)} + \varepsilon) \log \exp r^{\rho_{L(f)}} \]
\[ \quad \text{i.e.,} \quad \log T(r, f \circ g) < (\rho_f + \varepsilon) r^{\rho_f}. \quad (6.3.9) \]

Again we have for arbitrary positive \( \varepsilon \) and for all sufficiently large values of \( r \),
\[ T(r, L(f)) \geq \left(\bar{\sigma}_{L(f)} - \varepsilon\right) r^{\rho_f} \]
\[ \quad \text{i.e.,} \quad T(r, L(f)) \geq \{1 + k - k \delta (\infty; f)\} \bar{\sigma}_f - \varepsilon r^{\rho_f}. \quad (6.3.10) \]
Therefore from (6.3.9) and (6.3.10), it follows for all sufficiently large values of \( r \) that

\[
\frac{\log T(r, f \circ g)}{T(r, L(f))} \leq \frac{(\rho_f + \varepsilon) r^{\rho_f}}{\{1 + k - k\delta (\infty; f)\} \tilde{\sigma}_f - \varepsilon r^{\rho_f}}.
\]

i.e.,

\[
\limsup_{r \to \infty} \frac{\log T(r, f \circ g)}{T(r, L(f))} \leq \frac{\rho_f}{\{1 + k - k\delta (\infty; f)\} \tilde{\sigma}_f}.
\]

Thus the theorem is established.

\[\square\]

**Theorem 6.3.6** Let \( f \) be a transcendental meromorphic function with the maximum deficiency sum and \( g \) be an entire function of order zero such that \( \rho_g^{**} < \infty \), \( \lambda_f > 0 \) and \( \tilde{\sigma}_f > 0 \). Then

\[
\liminf_{r \to \infty} \frac{\log T(r, f \circ g)}{T(r, L(f))} \leq \min \left\{ \frac{\rho_g}{\{1 + k - k\delta (\infty; f)\} \tau_g}, \frac{\rho_f}{\{1 + k - k\delta (\infty; f)\} \tilde{\sigma}_f} \right\}.
\]

Theorem 6.3.6 can easily be proved in the line of Theorem 6.3.5. Hence its proof is omitted.

Using the notion of weak type, we may state the following four theorems without their proofs:

**Theorem 6.3.7** Let \( f \) be a meromorphic function of order zero and \( g \) be a transcendental entire function with the maximum deficiency sum such that \( \rho_f^{**} \) is finite, \( \rho_g < \infty \) and \( \tau_g > 0 \). Then

\[
\limsup_{r \to \infty} \frac{\log T(r, f \circ g)}{T(r, L(g))} \leq \frac{\rho_g}{\{1 + k - k\delta (\infty; f)\} \tau_g}.
\]

**Theorem 6.3.8** Let \( f \) be a meromorphic function of order zero and \( g \) be a transcendental entire function with the maximum deficiency sum such that \( \rho_f^{**} \) is finite, \( \rho_g < \infty \) and \( \tau_g > 0 \). Then

\[
\liminf_{r \to \infty} \frac{\log T(r, f \circ g)}{T(r, L(g))} \leq \min \left\{ \frac{\rho_g}{\{1 + k - k\delta (\infty; f)\} \tau_g}, \frac{\rho_f}{\{1 + k - k\delta (\infty; f)\} \tilde{\sigma}_f} \right\}.
\]

**Theorem 6.3.9** Let \( f \) be a transcendental meromorphic function with the maximum deficiency sum and \( g \) be an entire function of order zero such that \( \rho_g^{**} < \infty \), \( \rho_f < \infty \) and \( \tau_f > 0 \). Then

\[
\limsup_{r \to \infty} \frac{\log T(r, f \circ g)}{T(r, L(f))} \leq \frac{\rho_f}{\{1 + k - k\delta (\infty; f)\} \tau_f}.
\]

**Theorem 6.3.10** Let \( f \) be a transcendental meromorphic function with the maximum deficiency sum and \( g \) be an entire function of order zero such that \( \rho_g^{**} < \infty \), \( \rho_f < \infty \) and \( \tau_f > 0 \). Then

\[
\liminf_{r \to \infty} \frac{\log T(r, f \circ g)}{T(r, L(f))} \leq \min \left\{ \frac{\rho_f}{\{1 + k - k\delta (\infty; f)\} \tau_f}, \frac{\rho_f}{\{1 + k - k\delta (\infty; f)\} \tilde{\sigma}_f} \right\}.
\]
Theorem 6.3.11 Let \( f \) and \( g \) be any two entire functions such that \( 0 < \lambda_f^* < \infty \), \( 0 < \lambda_g \leq \rho_g < \infty \) and \( \sum_{a \neq \infty} \delta(a;g) + \delta(\infty;g) = 2 \). Then

\[
\lim_{r \to \infty} \frac{T(r, f \circ g)}{\log T(r^\lambda, L(g))} = \infty,
\]

where \( A \) is any positive real number.

Proof. We know that for \( r > 0 \) {cf.[40]}

\[
T(r, f \circ g) \geq \frac{1}{3} \log M \left\{ \frac{1}{8} M\left(\frac{r}{4}, g\right) + o(1), f \right\}. \tag{6.3.11}
\]

Let us choose \( \varepsilon \) in such a way that \( 0 < \varepsilon < \min \{ \lambda_f^{**}, \lambda_g \} \).

Now we get from (6.3.11) for all sufficiently large values of \( r \) that

\[
T(r, f \circ g) \geq \frac{1}{3} (\lambda_f^{**} - \varepsilon) \log M\left(\frac{r}{4}, g\right) + O(1)
\]

i.e., \( T(r, f \circ g) \geq \frac{1}{3} (\lambda_f^{**} - \varepsilon) \left(\frac{r}{4}\right)^{\lambda_g - \varepsilon} + O(1). \tag{6.3.12}
\]

Again from the definition of \( \rho_{L(g)} \) we have for arbitrary positive \( \varepsilon \) and for all sufficiently large values of \( r \),

\[
\log T(r^\lambda, L(g)) \leq A \left( \rho_{L(g)} + \varepsilon \right) \log r
\]

i.e., \( \log T(r^\lambda, L(g)) \leq A (\rho_g + \varepsilon) \log r. \tag{6.3.13}
\]

Now we obtain from (6.3.12) and (6.3.13) for all sufficiently large values of \( r \) that

\[
\frac{T(r, f \circ g)}{\log T(r^\lambda, L(g))} \geq \frac{\frac{1}{3} (\lambda_f^{**} - \varepsilon) \left(\frac{r}{4}\right)^{\lambda_g - \varepsilon} + O(1)}{A (\rho_g + \varepsilon) \log r}. \tag{6.3.14}
\]

As \( \lambda_g > 0 \), the theorem follows from (6.3.14). \( \blacksquare \)

Remark 6.3.5 The following example verifies the conclusion of Theorem 6.3.11.

Example 6.3.7 Let \( f = z, g = \exp z \) and \( A = 1 \). Taking \( a_1 = 1 \) in Definition 6.1.7, we have \( L(g) = \exp z \).

Then

\[
\lambda_f^{**} = 1 \text{ and } \lambda_g = \rho_g = 1.
\]

Now

\[
T(r, f \circ g) = T(r, \exp z) = \frac{r}{\pi} \text{ and } T(r^\lambda, L(g)) = T(r, \exp z) = \frac{r}{\pi}.
\]

Therefore
\[ \log T(r^A, L(g)) = \log r + O(1). \]

Hence
\[
\lim_{r \to \infty} \frac{T(r, f \circ g)}{\log T(r^A, L(g))} = \infty.
\]

**Remark 6.3.6** If we take \(0 < \rho_f^{**} < \infty\) instead of \(0 < \lambda_f^{**} < \infty\) in Theorem 6.3.11 and the other conditions remain the same, then in the line of Theorem 6.3.11 one can easily verify that
\[
\limsup_{r \to \infty} \frac{T(r, f \circ g)}{\log T(r^A, L(g))} = \infty.
\]

**Remark 6.3.7** Also if we consider \(0 < \lambda_g < \infty\) or \(0 < \rho_g < \infty\) instead of \(0 < \lambda_g \leq \rho_g < \infty\) in Theorem 6.3.11 and the other conditions remain the same, then in the line of Theorem 6.3.2 one can easily verify that
\[
\limsup_{r \to \infty} \frac{T(r, f \circ g)}{\log T(r^A, L(g))} = \infty.
\]

**Remark 6.3.8** In Theorem 6.3.11, if \(f\) be a meromorphic function with order zero and the other conditions remain the same, then with the help of Lemma 6.2.2 it can be shown that
\[
\limsup_{r \to \infty} \frac{T(r, f \circ g)}{\log T(r^A, L(g))} = \infty.
\]

**Theorem 6.3.12** Let \(f\) be a transcendental meromorphic function with the maximum deficiency sum and \(g\) be an entire function such that \(0 < \lambda_f \leq \rho_f < \infty\) and \(\lambda_f^{**} < \infty\). Then
\[
\lim_{r \to \infty} \frac{\log T(r, f \circ g)}{T(r^A, L(f))} = 0,
\]
where \(A\) is any positive real number.

**Proof.** In view of Lemma 6.2.1 and the inequality \(T(r, g) \leq \log^+ M(r, g)\), we get for all sufficiently large values of \(r\) that
\[
T(r, f \circ g) \leq \{1 + o(1)\} T(M(r, g), f)
\]

i.e., \(\log T(r, f \circ g) \leq (\rho_f + \varepsilon) \log M(r, g) + O(1)\)

i.e., \(\log T(r, f \circ g) \leq (\rho_f + \varepsilon) (\rho_g^{**} + \varepsilon) + O(1)\) \hspace{1cm} (6.3.15)

Again from the definition of \(\lambda_{L(f)}\), we have for arbitrary positive \(\varepsilon\) and for all sufficiently large values of \(r\),
\[
\log T(r^A, L(f)) \geq A (\lambda_{L(f)} - \varepsilon) \log r
\]

i.e., \(\log T(r^A, L(f)) \geq A (\lambda_f - \varepsilon) \log r
\]

i.e., \(T(r^A, L(f)) \geq r^A(\lambda_f - \varepsilon)\). \hspace{1cm} (6.3.16)
Therefore it follows from (6.3.15) and (6.3.16), for all sufficiently large values of \( r \) that
\[
\frac{\log T(r, f \circ g)}{T(r^A, L(f))} \leq \frac{(\rho_f + \varepsilon)(\rho_g^{**} + \varepsilon) + O(1)}{r^A(\lambda_f - \varepsilon)}.
\]
(6.3.17)

As \( \lambda_f > 0 \), the theorem follows from (6.3.17).

**Remark 6.3.9** The following example ensures the conclusion of Theorem 6.3.12.

**Example 6.3.8** Let \( f = \exp z, g = z \) and \( A = 1 \). Considering \( a_1 = 1 \) in Definition 6.1.7, we get that \( L(f) = \exp z \).

Then
\[
\lambda_f = \rho_f = 1 \text{ and } \lambda_g^{**} = 1.
\]

Since
\[
T(r, f \circ g) = T(r, \exp z) = \frac{r}{\pi} \text{ and so } \log T(r, f \circ g) = \log r + O(1).
\]

Again
\[
T(r^A, L(f)) = T(r, \exp z) = \frac{r}{\pi}.
\]

Therefore
\[
\log T(r^A, L(g)) = \log r + O(1).
\]

Hence
\[
\lim_{r \to \infty} \frac{\log T(r, f \circ g)}{T(r^A, L(f))} = 0.
\]

**Remark 6.3.10** If we take \( 0 < \rho_f < \infty \) or \( 0 < \lambda_f < \infty \) instead of \( 0 < \lambda_f \leq \rho_f < \infty \) in Theorem 6.3.12 and the other conditions remain the same, then in the line of Theorem 6.3.12 one can easily verify that
\[
\liminf_{r \to \infty} \frac{\log T(r, f \circ g)}{T(r^A, L(f))} = 0.
\]

**Remark 6.3.11** Also if we take \( \lambda_g^{**} < \infty \) instead of \( \rho_g^{**} < \infty \) in Theorem 6.3.12 and the other conditions remain the same, then in the line of Theorem 6.3.12 one can easily verify that
\[
\liminf_{r \to \infty} \frac{\log T(r, f \circ g)}{T(r^A, L(f))} = 0.
\]

**Theorem 6.3.13** Let \( f \) be a transcendental entire function with the maximum deficiency sum and \( g \) be an entire function such that \( 0 < \lambda_f \leq \rho_f < \infty \) and \( \rho_g^{**} > 0 \). Then for any positive real number \( A \),
\[
\limsup_{r \to \infty} \frac{\log T(r, f \circ g)}{\log T(r^A, L(f))} \geq \frac{\lambda_f \rho_g^{**}}{A \rho_f}.
\]
Proof. Suppose $0 < \varepsilon < \min \{ \lambda_f, \rho_g^{**} \}$.

Now from (6.3.11) we have for a sequence of values of $r$ tending to infinity that

$$\log T(r, f \circ g) \geq (\lambda_f - \varepsilon) \log M \left( \frac{r}{4}, g \right) + O(1)$$

i.e.,

$$\log T(r, f \circ g) \geq (\lambda_f - \varepsilon) \rho_g^{**} \log r + O(1).$$

(6.3.18)

Again from the definition of $\rho_{L(f)}$, we have for arbitrary positive $\varepsilon$ and for all sufficiently large values of $r$ that

$$\log T(r, f \circ g) \leq A \rho_{L(f)} \log r$$

i.e.,

$$\log T(r, f \circ g) \leq A (\rho_f + \varepsilon) \log r.$$  

(6.3.19)

So combining (6.3.18) and (6.3.19), we get for a sequence of values of $r$ tending to infinity that

$$\log T(r, f \circ g) \geq \lambda_f \rho_{L(f)} \log r$$

i.e.,

$$\limsup_{r \to \infty} \frac{\log T(r, f \circ g)}{\log T(r^A, L(f))} \geq \frac{\lambda_f \rho_{L(f)}^{**}}{A \rho_f^{**}}.$$  

This completes the proof of the theorem. □

Remark 6.3.12 The following example ensures the conclusion of Theorem 6.3.13.

Example 6.3.9 Let $f = \exp z, g = z$ and $A = 1$. Taking $a_1 = 1$ in Definition 6.1.7, we get that $L(f) = \exp z$.

Then

$$\lambda_f = \rho_f = 1 \text{ and } \rho_g^{**} = 1.$$  

Now

$$T(r, f \circ g) = T(r, \exp z) = \frac{r}{\pi} \text{ and so } \log T(r, f \circ g) = \log r + O(1).$$

Since

$$T(r^A, L(f)) = T(r, \exp z) = \frac{r}{\pi},$$

therefore

$$\log T(r^A, L(f)) = \log r + O(1).$$

Now

$$\limsup_{r \to \infty} \frac{\log T(r, f \circ g)}{\log T(r^A, L(f))} = 1 \text{ and } \frac{\lambda_f \rho_{g}^{**}}{A \rho_f} = 1.$$  

Therefore

$$\limsup_{r \to \infty} \frac{\log T(r, f \circ g)}{\log T(r^A, L(f))} = \frac{\lambda_f \rho_{g}^{**}}{A \rho_f}.$$
Remark 6.3.13 Under the same conditions of Theorem 6.3.13, if $f$ is of regular growth then
\[ \limsup_{r \to \infty} \frac{\log T(r, f \circ g)}{\log T(r^A, L(f))} \geq \rho^{**}_g. \]

Remark 6.3.14 In Theorem 6.3.13 if we take $\lambda_g^{**} > 0$ instead of $\rho_g^{**} > 0$ and the other conditions remain the same, then it can be shown that
\[ \liminf_{r \to \infty} \frac{\log T(r, f \circ g)}{\log T(r^A, L(f))} \geq \frac{\lambda_f\lambda_g^{**}}{A\rho_f}. \]

In addition, if $f$ is of regular growth then
\[ \liminf_{r \to \infty} \frac{T(r, f \circ g)}{T(r^A, L(f))} \geq \frac{\lambda_g^{**}}{A}. \]

Remark 6.3.15 Also if we consider $0 < \lambda_g < \infty$ or $0 < \rho_g < \infty$ instead of $0 < \lambda_g \leq \rho_g < \infty$ in Theorem 6.3.13 and the other conditions remain the same, then one can easily verify that
\[ \limsup_{r \to \infty} \frac{\log T(r, f \circ g)}{\log T(r^A, L(f))} \geq \frac{\lambda_g^{**}}{A}. \]

Theorem 6.3.14 Let $f$ be a transcendental meromorphic function with the maximum deficiency sum and $g$ be an entire function such that $0 < \lambda_f < \infty$ and $\rho_g < \infty$. Then for any positive real number $A$,
\[ \frac{\log T(r, f \circ g)}{\log T(r^A, L(f))} \leq \frac{\rho_f\rho^{**}_g}{A\lambda_f}. \]

Proof. From the definition of $\lambda_{L(f)}$, we have for arbitrary positive $\varepsilon$ and for all sufficiently large values of $r$ that
\[ \log T(r^A, L(f)) \geq A(\lambda_{L(f)} - \varepsilon) \log r \]
\[ \text{i.e., } \log T(r^A, L(f)) \geq A(\lambda_f - \varepsilon) \log r. \] (6.3.20)

Now combining (6.3.15) and (6.3.20), we get for all sufficiently large values of $r$ that
\[ \frac{\log T(r, f \circ g)}{\log T(r^A, L(f))} \leq \frac{(\rho_f + \varepsilon)(\rho^{**}_g + \varepsilon) + O(1)}{A(\lambda_f - \varepsilon) \log r} \]
\[ \text{i.e., } \limsup_{r \to \infty} \frac{\log T(r, f \circ g)}{\log T(r^A, L(f))} \leq \frac{\rho_f\rho^{**}_g}{A\lambda_f}. \]

This completes the proof of the theorem. ■

Remark 6.3.16 The following example checks the conclusion of Theorem 6.3.14.

Example 6.3.10 Let $f = \exp z$, $g = z$ and $A = 1$. Further, letting $a_1 = 1$ in Definition 6.1.7, we obtain that $L(f) = \exp z$. 
Then

\[ \lambda_f = \rho_f = 1 \text{ and } \rho^{**}_g = 1. \]

Now

\[ T(r, f \circ g) = T(r, \exp z) = \frac{r}{\pi} \text{ and so } \log T(r, f \circ g) = \log r + O(1). \]

Again

\[ T(r^A, L(f)) = T(r, \exp z) = \frac{r}{\pi}. \]

So

\[ \log T(r^A, L(f)) = \log r + O(1). \]

Therefore

\[ \limsup_{r \to \infty} \frac{\log T(r, f \circ g)}{\log T(r^A, L(f))} = 1 \text{ and } \frac{\rho_f \rho^{**}_g}{A \lambda_f} = 1. \]

Hence

\[ \limsup_{r \to \infty} \frac{\log T(r, f \circ g)}{\log T(r^A, L(f))} = \frac{\lambda_f \rho^{**}_g}{A \rho_f}. \]

**Remark 6.3.17** Under the same conditions of Theorem 6.3.14, if \( f \) is of regular growth then

\[ \limsup_{r \to \infty} \frac{\log T(r, f \circ g)}{\log T(r^A, L(f))} \leq \frac{\rho^{**}_g}{A}. \]

**Remark 6.3.18** In Theorem 6.3.14 if we take \( \lambda^{**}_g < \infty \) instead of \( \rho^{**}_g < \infty \) and the other conditions remain the same then it can be shown that

\[ \liminf_{r \to \infty} \frac{\log T(r, f \circ g)}{\log T(r^A, L(f))} \leq \frac{\rho_f \lambda^{**}_g}{A \lambda_f}. \]

In addition, if \( f \) is of regular growth then

\[ \liminf_{r \to \infty} \frac{\log T(r, f \circ g)}{\log T(r^A, L(f))} \leq \frac{\lambda^{**}_g}{A}. \]

**Remark 6.3.19** If we take \( 0 < \rho_f < \infty \) or \( 0 < \lambda_f < \infty \) instead of \( 0 < \lambda_f \leq \rho_f < \infty \) in Theorem 6.3.14 and the other conditions remain the same, then one can easily verify that

\[ \liminf_{r \to \infty} \frac{\log T(r, f \circ g)}{\log T(r^A, L(f))} \leq \frac{\rho^{**}_g}{A}. \]