CHAPTER 5

GROWTH OF DIFFERENTIAL MONOMIALS AND DIFFERENTIAL POLYNOMIALS GENERATED BY ENTIRE OR EROMORPHIC FUNCTIONS IN THE LIGHT OF ZERO ORDER
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5.1 Introduction, Definitions and Notations.

For any two transcendental entire functions $f$ and $g$ defined in the open complex plane $\mathbb{C}$, Clunie [5] proved that

$$\lim_{r \to \infty} \frac{T(r, f \circ g)}{T(r, f)} = \infty \quad \text{and} \quad \lim_{r \to \infty} \frac{T(r, f \circ g)}{T(r, g)} = \infty.$$

Singh [45] proved some comparative growth properties of $\log T(r, f \circ g)$ and $T(r, f)$. He also raised the problem of investigating the comparative growth of $\log T(r, f \circ g)$ and $T(r, g)$ which he was unable to solve. However, some results on the comparative growth of $\log T(r, f \circ g)$ and $T(r, g)$ are proved in [33].

The results of this chapter have been published in the International Journal of Mathematical Archive(IJMA), see [17].
Let \( f \) be a non-constant meromorphic function defined in the open complex plane \( \mathbb{C} \). Also let \( n_{0j}, n_{1j}, \ldots, n_{kj} (k \geq 1) \) be non-negative integers such that for each \( j \), \( \sum_{i=0}^{k} n_{ij} \geq 1 \). We call \( M_j [f] = A_j (f)^{n_{0j}} (f^{(1)})^{n_{1j}} \ldots (f^{(k)})^{n_{kj}} \) where \( T(r, A_j) = S(r, f) \) to be a differential monomial generated by \( f \). The numbers \( \gamma_{M_j} = \sum_{i=0}^{k} n_{ij} \) and \( \Gamma_{M_j} = \sum_{i=0}^{k} (i+1)n_{ij} \) are called respectively the degree and weight of \( M_j [f] \) \([9],[43]\) . The expression \( P [f] = \sum_{j=1}^{s} M_j [f] \) is called a differential polynomial generated by \( f \). The numbers \( \gamma_P = \max_{1 \leq j \leq s} \gamma_{M_j} \) and \( \Gamma_P = \max_{1 \leq j \leq s} \Gamma_{M_j} \) are called respectively the degree and weight of \( P [f] \) \([9],[43]\) . Also we call the numbers \( \gamma_P = \min_{1 \leq j \leq s} \gamma_{M_j} \) and \( k \) (the order of the highest derivative of \( f \) ) the lower degree and the order of \( P [f] \) respectively. If \( \gamma_P = \gamma_P \), \( P [f] \) is called a homogeneous differential polynomial. In the chapter we further investigate the question of Singh \([45]\) mentioned earlier and prove some new results relating to the comparative growths of composite entire or meromorphic functions and differential monomials, differential polynomials generated by one of the factors. Throughout the chapter we consider only the non-constant differential polynomials and we denote by \( P_0 [f] \) a differential polynomial not containing \( f \) i.e., for which \( n_{0j} = 0 \) for \( j = 1, 2, \ldots, s \). We consider only those \( P [f] \), \( P_0 [f] \) singularities of whose individual terms do not cancel each other. We also denote by \( M [f] \) a differential monomial generated by a transcendental meromorphic function \( f \).

Though Definition 5.1.1 and Definition 5.1.2 have already been defined in Chapter 2 as Definition 2.1.1 and Definition 2.1.3 respectively, we state here again in order to keep a continuation of our discussion:

**Definition 5.1.1** The order \( \rho_f \) and lower order \( \lambda_f \) of a meromorphic function \( f \) are defined as

\[
\rho_f = \limsup_{r \to \infty} \frac{\log T(r, f)}{\log r} \quad \text{and} \quad \lambda_f = \liminf_{r \to \infty} \frac{\log T(r, f)}{\log r}.
\]

If \( f \) is entire, one can easily verify that

\[
\rho_f = \limsup_{r \to \infty} \frac{\log^{[2]} M(r, f)}{\log r} \quad \text{and} \quad \lambda_f = \liminf_{r \to \infty} \frac{\log^{[2]} M(r, f)}{\log r},
\]

where \( \log^{[k]} x = \log(\log^{[k-1]} x) \) for \( k = 1, 2, 3, \ldots \) and \( \log^{[0]} x = x \).

If \( \rho_f < \infty \) then \( f \) is of finite order. Also \( \rho_f = 0 \) means that \( f \) is of order zero. In this connection Datta and Biswas \([12]\) gave the following definition:

**Definition 5.1.2** \([12]\)Let \( f \) be a meromorphic function of order zero. Then the quantities \( \rho_f^{**} \) and \( \lambda_f^{**} \) of \( f \) are defined by

\[
\rho_f^{**} = \limsup_{r \to \infty} \frac{T(r, f)}{\log r} \quad \text{and} \quad \lambda_f^{**} = \liminf_{r \to \infty} \frac{T(r, f)}{\log r}.
\]
If \( f \) is an entire function then clearly
\[
\rho_f^* = \limsup_{r \to \infty} \frac{\log M(r, f)}{\log r} \quad \text{and} \quad \lambda_f^* = \liminf_{r \to \infty} \frac{\log M(r, f)}{\log r}.
\]

**Definition 5.1.3** Let \( a \) be a complex number, finite or infinite. The Nevanlinna deficiency and the Valiron deficiency of \( a \) with respect to a meromorphic function \( f \) are defined as
\[
\delta(a; f) = 1 - \limsup_{r \to \infty} \frac{N(r, a; f)}{T(r, f)} = \liminf_{r \to \infty} \frac{m(r, a; f)}{T(r, f)}
\]
and
\[
\Delta(a; f) = 1 - \liminf_{r \to \infty} \frac{N(r, a; f)}{T(r, f)} = \limsup_{r \to \infty} \frac{m(r, a; f)}{T(r, f)}.
\]

**Definition 5.1.4** The quantity \( \Theta(a; f) \) of a meromorphic function \( f \) is defined as follows
\[
\Theta(a; f) = 1 - \limsup_{r \to \infty} \frac{\tilde{N}(r, a; f)}{T(r, f)}.
\]

**Definition 5.1.5** [53] For \( a \in \mathbb{C} \cup \{\infty\} \), we denote by \( n(r, a; f | = 1) \) the number of simple zeros of \( f - a \) in \( |z| \leq r \). \( N(r, a; f | = 1) \) is defined in terms of \( n(r, a; f | = 1) \) in the usual way. We put
\[
\delta_1(a; f) = 1 - \limsup_{r \to \infty} \frac{N(r, a; f | = 1)}{T(r, f)},
\]
the deficiency of \( a \) corresponding to the simple \( a \)-points of \( f \) i.e., simple zeros of \( f - a \).

Yang [52] proved that there exists at most a denumerable number of complex numbers \( a \in \mathbb{C} \cup \{\infty\} \) for which \( \delta_1(a; f) > 0 \) and \( \sum_{a \in \mathbb{C} \cup \{\infty\}} \delta_1(a; f) \leq 4 \).

**Definition 5.1.6** [34] For \( a \in \mathbb{C} \cup \{\infty\} \), let \( n_p(r, a; f) \) denotes the number of zeros of \( f - a \) in \( |z| \leq r \), where a zero of multiplicity \( < p \) is counted according to its multiplicity and a zero of multiplicity \( \geq p \) is counted exactly \( p \) times; and \( N_p(r, a; f) \) is defined in terms of \( n_p(r, a; f) \) in the usual way. We define
\[
\delta_p(a; f) = 1 - \limsup_{r \to \infty} \frac{N_p(r, a; f)}{T(r, f)}.
\]

**Definition 5.1.7** [4] \( P[f] \) is said to be admissible if

(i) \( P[f] \) is homogeneous, or
(ii) \( P[f] \) is non homogeneous and \( m(r, f) = S(r, f) \).

In our subsequent discussion, we will use the following notation:
\[
\exp^k y = \exp \left( \exp^{k-1} y \right) \quad \text{for} \quad k = 1, 2, 3, \ldots \quad \text{and} \quad \exp^0 y = y.
\]
5.2 Lemmas.

In this section we present some lemmas which will be needed in the sequel.

Lemma 5.2.1 [1] If $f$ is meromorphic and $g$ is entire then for all sufficiently large values of $r$,
$$T(r, f \circ g) \leq \{1 + o(1)\} \frac{T(r, g)}{\log M(r, g)} T(M(r, g), f).$$

Lemma 5.2.2 [3] Let $f$ be meromorphic and $g$ be entire and suppose that $0 < \mu < \rho_g \leq \infty$. Then for a sequence of values of $r$ tending to infinity,
$$T(r, f \circ g) \geq T(\exp(r^\mu), f).$$

Lemma 5.2.3 [12] If $f$ be any meromorphic function of order zero. Then
$$\lim_{r \to \infty} \frac{\log T(r, f)}{\log^2 r} = 1.$$

Lemma 5.2.4 [4] Let $P_0[f]$ be admissible. If $f$ is of finite order or of non-zero lower order and
$$\sum_{a \neq \infty} \Theta(a; f) = 2.$$
Then
$$\lim_{r \to \infty} \frac{T(r, P_0[f])}{T(r, f)} = \Gamma_{P_0[f]}.$$

Lemma 5.2.5 [4] Let $f$ be either of finite order or of non-zero lower order such that
$$\Theta(\infty; f) = \sum_{a \neq \infty} \delta_p(a; f) = 1 \text{ or } \delta(\infty; f) = \sum_{a \neq \infty} \delta(a; f) = 1.$$ Then for homogeneous $P_0[f],$
$$\lim_{r \to \infty} \frac{T(r, P_0[f])}{T(r, f)} = \gamma_{P_0[f]}.$$

Lemma 5.2.6 Let $f$ be a meromorphic function of finite order or of non zero lower order. If
$$\sum_{a \neq \infty} \Theta(a; f) = 2,$$ then the order (lower order) of homogeneous $P_0[f]$ is same as that of $f$ if $f$ is of positive finite order.

Proof. By Lemma 5.2.4,
$$\lim_{r \to \infty} \frac{\log T(r, P_0[f])}{\log T(r, f)}$$
exists and is equal to 1.
$$\rho_{P_0[f]} = \limsup_{r \to \infty} \frac{\log T(r, P_0[f])}{\log r} = \limsup_{r \to \infty} \frac{\log T(r, f)}{\log r} \cdot \lim_{r \to \infty} \frac{\log T(r, P_0[f])}{\log T(r, f)} = \rho_f \cdot 1 = \rho_f.$$
Also

\[
\lambda_{P_0[f]} = \liminf_{r \to \infty} \frac{\log T(r, P_0[f])}{\log r} \\
= \liminf_{r \to \infty} \frac{\log T(r, f)}{\log r} \cdot \lim_{r \to \infty} \frac{\log T(r, P_0[f])}{\log T(r, f)} \\
= \lambda_f.1 = \lambda_f.
\]

This proves the lemma. ■

**Lemma 5.2.7** Let \( f \) be a meromorphic function of finite order or of non-zero lower order such that \( \Theta(\infty; f) = \sum_{a \neq \infty} \delta_p(a; f) = 1 \). Then the order (lower order) of homogeneous \( P_0[f] \) and \( f \) are same when \( f \) is of finite positive order.

We omit the proof of the lemma because it can be carried out in the line of Lemma 5.2.7 and with the help of Lemma 5.2.6.

In a similar manner we can state the following lemma without its proof:

**Lemma 5.2.8** Let \( f \) be a meromorphic function of finite order or of non-zero lower order such that \( \delta(\infty; f) = \sum_{a \neq \infty} \delta(a; f) = 1 \). Then for every homogeneous \( P_0[f] \), the order (lower order) of \( P_0[f] \) is same as that of \( f \) when \( f \) is of finite positive order.

**Lemma 5.2.9** [36] Let \( f \) be a transcendental meromorphic function of finite order or of non-zero lower order and \( \sum_{a \in \mathbb{C}\setminus\{\infty\}} \delta_1(a; f) = 4 \). Then

\[
\lim_{r \to \infty} \frac{T(r, M[f])}{T(r, f)} = \Gamma_M - (\Gamma_M - \gamma_M)\Theta(\infty; f),
\]

where

\[
\Theta(\infty; f) = 1 - \limsup_{r \to \infty} \frac{N(r, f)}{T(r, f)}.
\]

**Lemma 5.2.10** If \( f \) be a transcendental meromorphic function of finite order or of non-zero lower order and \( \sum_{a \in \mathbb{C}\setminus\{\infty\}} \delta_1(a; f) = 4 \), then the order and lower order of \( M[f] \) are same as those of \( f \) when \( f \) is of finite positive order.

We omit the proof of the lemma because it can be carried out in the line of Lemma 5.2.6 and with the help of Lemma 5.2.9.
5.3 Theorems.

In this section we present the main results of the chapter.

Theorem 5.3.1 Let \( f \) be a meromorphic function and \( g \) be an entire function such that \( 0 < \lambda_f \leq \rho_f < \rho_g < \infty \). Also let \( \sum_{a \neq \infty} \Theta(a; f) = 2 \). Then

\[
\limsup_{r \to \infty} \frac{\log T(r, f \circ g)}{T(r, P_0[f])} = \infty.
\]

Proof. Since \( \rho_f < \rho_g \) we can choose \( \varepsilon (> 0) \) in such a way that

\[
\rho_f + \varepsilon < \rho_g - \varepsilon < \rho_g.
\] (5.3.1)

Now in view of Lemma 5.2.2 we obtain for a sequence of values of \( r \) tending to infinity that

\[
\log T(r, f \circ g) \geq \log T\{\exp r^{(\rho_g - \varepsilon)}, f\}
\]

i.e.,

\[
\log T(r, f \circ g) \geq (\lambda_f - \varepsilon) \log r^{(\rho_g - \varepsilon)}
\]

i.e.,

\[
\log T(r, f \circ g) \geq (\lambda_f - \varepsilon) r^{(\rho_g - \varepsilon)}.
\] (5.3.2)

Again by Lemma 5.2.6, we have for all sufficiently large values of \( r \),

\[
\log T(r, P_0[f]) \leq (\rho_{P_0[f]} + \varepsilon) \log r
\]

i.e.,

\[
\log T(r, P_0[f]) \leq (\rho_f + \varepsilon) \log r
\] (5.3.3)

i.e.,

\[
T(r, P_0[f]) \leq r^{(\rho_f + \varepsilon)}.
\] (5.3.4)

Therefore from (5.3.2) and (5.3.3), it follows for a sequence of values of \( r \) tending to infinity that

\[
\frac{\log T(r, f \circ g)}{T(r, P_0[f])} \geq \frac{(\lambda_f - \varepsilon) r^{(\rho_g - \varepsilon)}}{r^{(\rho_f + \varepsilon)}}.
\] (5.3.5)

Now in view of (5.3.1) it follows from (5.3.4) that

\[
\limsup_{r \to \infty} \frac{\log T(r, f \circ g)}{T(r, P_0[f])} = \infty.
\]

This proves the theorem. ■

Remark 5.3.1 The following example ensures the conclusion of Theorem 5.3.1.

Example 5.3.1 Let \( f = \exp z \) and \( g = \exp (z^2) \).

Further, taking \( s = 1, n_{1j} = 1 \) and \( n_{0j} = n_{2j} = ... = n_{kj} = 0 \) in the definitions of \( M[f] \) and \( P_0[f] \) (See Page 74), we obtain that \( P_0[f] = \exp z \).

Then

\[
\rho_f = \lambda_f = 1 \text{ and } \rho_g = 2.
\]
Now
\[ T(r, f \circ g) = T(r, \exp^2 (z^2)) \leq \log \left( \exp^2 (r^2) \right) = \exp (r^2) \]

so
\[ \log T(r, f \circ g) \leq r^2. \]

Again
\[ T(r, P_0 [f]) = T(r, \exp z) = \frac{r}{\pi} \]

and so
\[ \frac{\log T(r, f \circ g)}{T(r, P_0 [f])} \leq \frac{\pi}{r}. \]

Thus
\[ \limsup_{r \to \infty} \frac{\log T(r, f \circ g)}{T(r, P_0 [f])} = \infty. \]

In the line of Theorem 5.3.1, the following corollary may be deduced:

**Corollary 5.3.1** Let \( f \) be a meromorphic function and \( g \) be an entire function with \( 0 < \lambda_f \leq \rho_f < \lambda_g < \infty \). Also let \( \sum_{a \neq \infty} \Theta(a; f) = 2 \). Then
\[ \limsup_{r \to \infty} \frac{\log T(r, f \circ g)}{T(r, P_0 [f])} = \infty. \]

**Remark 5.3.2** The conclusion of Theorem 5.3.1 and Corollary 5.3.1 can also be drawn under the hypothesis \( \Theta(\infty; f) = \sum_{a \neq \infty} \delta_p (a; f) = 1 \) or \( \delta(\infty; f) = \sum_{a \neq \infty} \delta (a; f) = 1 \) instead of \( \sum_{a \neq \infty} \Theta(a; f) = 2 \).

In the line of Theorem 5.3.1 and with the help of Lemma 5.2.10, we may state the following theorem without its proof:

**Theorem 5.3.2** Let \( f \) be a meromorphic function and \( g \) be an entire function such that \( 0 < \lambda_f \leq \rho_f < \rho_g < \infty \). Also let \( \sum_{a \in \Omega \cup \infty} \delta_1 (a; f) = 4 \). Then
\[ \limsup_{r \to \infty} \frac{\log T(r, f \circ g)}{T(r, M[f])} = \infty. \]

In view of Theorem 5.3.2, the following corollary may also be deduced and hence its proof is omitted.
Corollary 5.3.2 Let $f$ be a meromorphic function and $g$ be an entire function such that $0 < \lambda_f \leq \rho_f < \lambda_g < \infty$. Also let $\sum_{a \in \Omega \cup \{\infty\}} \delta_1(a; f) = 4$. Then
\[
\limsup_{r \to \infty} \frac{\log T(r, f \circ g)}{T(r, M[f])} = \infty.
\]

Theorem 5.3.3 Let $f$ be a meromorphic function such that $0 < \lambda_f \leq \rho_f < \infty$, $\delta(\infty; f) = \sum_{a \neq \infty} \delta(a; f) = 1$ and $g$ be an entire function with finite order. Then for every positive constant $A$ and every real number $\alpha$,
\[
\limsup_{r \to \infty} \frac{\log T(r, f \circ g)}{\log T(r, P_0[f])^{1+\alpha}} = \infty.
\]

Proof. Let us suppose that
\[
0 < \varepsilon < \min \{\lambda_f, \lambda_g\}.
\]
If $\alpha$ is such that $1 + \alpha \leq 0$ then the theorem is trivial. So we suppose that $1 + \alpha > 0$. Now from Lemma 5.2.2, we get for a sequence values of $r$ tending to infinity that
\[
\log T(r, f \circ g) \geq \log T(\exp r^{(\rho_f - \varepsilon)}, f)
\]
i.e.,
\[
\log T(r, f \circ g) \geq (\lambda_f - \varepsilon) \log \exp r^{(\rho_f - \varepsilon)}
\]
i.e.,
\[
\log T(r, f \circ g) \geq (\lambda_f - \varepsilon) r^{(\rho_f - \varepsilon)}.
\]
(5.3.6)

Again from the definition of $\rho_{P_0[f]}$, it follows for all sufficiently large values of $r$ that
\[
\log T(r, P_0[f]) \leq \left(\rho_{P_0[f]} + \varepsilon\right) A \log r
\]
i.e.,
\[
\log T(r, P_0[f]) \leq (\rho_f + \varepsilon) A \log r
\]
i.e.,
\[
\left\{\log T(r, P_0[f])\right\}^{1+\alpha} \leq (\rho_f + \varepsilon)^{1+\alpha} A^{1+\alpha} (\log r)^{1+\alpha}.
\]
(5.3.7)

Now from (5.3.5) and (5.3.6), it follows for a sequence of values of $r$ tending to infinity that
\[
\frac{\log T(r, f \circ g)}{\left\{\log T(r, P_0[f])\right\}^{1+\alpha}} \geq \frac{(\lambda_f - \varepsilon) r^{(\rho_f - \varepsilon)}}{(\rho_f + \varepsilon)^{1+\alpha} A^{1+\alpha} (\log r)^{1+\alpha}}.
\]

Since $\frac{r^{\rho_f - \varepsilon}}{(\log r)^{1+\alpha}} \to \infty$ as $r \to \infty$, the theorem follows from above. \[\blacksquare\]

Remark 5.3.3 The following example verifies the conclusion of Theorem 5.3.3.

Example 5.3.2 Let $f = \exp z$, $g = \exp[z^2]$, $A = 1$ and $\alpha = 0$.

Taking $s = 1$, $n_{1j} = 1$ and $n_{0j} = n_{2j} = \ldots = n_{kj} = 0$ in the definitions of $M[f]$ and $P_0[f]$ (See Page 74), we obtain that $P_0[f] = \exp z$.

Then
\[
\rho_f = \lambda_f = 1 \text{ and } \rho_g = \infty.
\]
Now
\[ T(r, f \circ g) = T(r, \exp^3 z) \leq \log(\exp^3 r) = \exp^2 r. \]

So
\[ \log T(r, f \circ g) \leq \exp r. \]

Again
\[ T(r^A, P_0[f]) = T(r, \exp z) = \frac{r}{\pi} \]

and so
\[ \log T(r^A, P_0[f]) = \log r + O(1). \]

Therefore
\[ \frac{\log T(r, f \circ g)}{\{\log T(r^A, P_0[f])\}^{1+\alpha}} \leq \frac{\exp r}{\log r + O(1)} \]

i.e.,
\[ \limsup_{r \to \infty} \frac{\log T(r, f \circ g)}{\{\log T(r^A, P_0[f])\}^{1+\alpha}} = \infty. \]

**Remark 5.3.4** The conclusion of Theorem 5.3.3 can also deduced if we replace \( \delta(\infty; f) = \sum_{a \neq \infty} \delta(a; f) = 1 \) by \( \Theta(\infty; f) = \sum_{a \neq \infty} \delta_p(a; f) = 1 \) or \( \sum \Theta(a; f) = 2 \) respectively.

**Theorem 5.3.4** Let \( f \) be a meromorphic function such that \( 0 < \lambda_f \leq \rho_f < \infty \), \( \sum_{a \in \mathbb{C} \cup \{\infty\}} \delta_1(a; f) = 4 \) and \( g \) be an entire function with finite order. Then for every positive constant \( A \) and every real number \( \alpha \),
\[ \limsup_{r \to \infty} \frac{\log T(r, f \circ g)}{\{\log T(r^A, M[f])\}^{1+\alpha}} = \infty. \]

The proof of the theorem can be established in the line of Theorem 5.3.3 and with the help of Lemma 5.2.10 and therefore is omitted.

**Remark 5.3.5** The conclusion of Theorem 5.3.3, Theorem 5.3.4 and Remark 5.3.2 can also be deduced if we take \( g \) to be an entire function with non zero lower order instead of “finite order”.

In the line of Theorem 5.3.3 one may state the following theorem without proof:
Theorem 5.3.5  Let $f$ be a meromorphic function and $g$ be an entire function such that $0 < \lambda_g \leq \rho_g < \infty$ and $\Theta(\infty; g) = \sum_{a \neq \infty} \delta_p(a; g) = 1$ or $\delta(\infty; g) = \sum_{a \neq \infty} \delta(a; g) = 1$ or $\sum_{a \neq \infty} \Theta(a; g) = 2$. Then for every positive constant $A$ and every real number $\alpha$,

$$\limsup_{r \to \infty} \frac{\log T(r, f \circ g)}{(\log T(r^A, P_0[g]))^{1+\alpha}} = \infty.$$ 

Remark 5.3.6  The following example ensures the conclusion of Theorem 5.3.5.

Example 5.3.3  Let $f = \exp z$, $g = \exp z$, $A = 1$ and $\alpha = 0$.

Considering $s = 1$, $n_{1j} = 1$ and $n_{0j} = n_{2j} = \ldots = n_{kj} = 0$ in the definitions of $M[f]$ and $P_0[f]$ (See Page 74), we obtain that $P_0[f] = \exp z$.

Then

$$\rho_f = \lambda_f = 1, \lambda_g = \rho_g = 1 \text{ and } \delta(\infty, g) = 1.$$ 

Now

$$T(r, f \circ g) = T(r, \exp^{[2]} z) \leq \log (\exp^{[2]} r) = \exp r$$

i.e.,

$$\log T(r, f \circ g) \leq r.$$ 

Again

$$T(r^A, P_0[f]) = T(r, \exp z) = \frac{r}{\pi}$$

and so

$$\log T(r^A, P_0[f]) = \log r + O(1).$$

Therefore

$$\frac{\log T(r, f \circ g)}{(\log T(r^A, P_0[f]))^{1+\alpha}} \leq \frac{r}{\log r + O(1)}$$

i.e.,

$$\limsup_{r \to \infty} \frac{\log T(r, f \circ g)}{(\log T(r^A, P_0[f]))^{1+\alpha}} = \infty.$$ 

In the line of Theorem 5.3.5 and with the help of Lemma 5.2.10 we may state the following theorem without its proof:
Theorem 5.3.6 Let $f$ be a meromorphic function and $g$ be an entire function such that $0 < \lambda_g \leq \rho_g < \infty$ and \sum_{a \in \cup \{\infty\}} \delta_1(a; g) = 4$. Then for every positive constant $A$ and every real number $\alpha$,
\[ \limsup_{r \to \infty} \frac{\log T(r, f \circ g)}{\{\log T(r^A, M[g])\}^{1+\alpha}} = \infty. \]

Theorem 5.3.7 Let $f$ be a meromorphic function and $g$ be an entire function such that $0 < \lambda_f \leq \rho_f < \infty$, \sum_{a \neq \infty} \Theta(a; f) = 2$ and $0 < \rho_g < \infty$. Then
\[ \frac{\rho_g}{\rho_f} \leq \limsup_{r \to \infty} \frac{\log^{[2]} T(r, f \circ g)}{\log T(r, P_0[f])} \leq \frac{\rho_g}{\lambda_f}. \]

Proof. Since $T(r, g) \leq \log^+ M(r, g)$, we have from Lemma 5.2.1 for all sufficiently large values of $r$,
\[ T(r, f \circ g) \leq \{1 + o(1)\} T(M(r, g), f) \]
\[ i.e., \quad \log T(r, f \circ g) \leq (\rho_f + \varepsilon) \log M(r, g) + O(1) \]
\[ i.e., \quad \log^{[2]} T(r, f \circ g) \leq \log^{[2]} M(r, g) + O(1) \]
\[ i.e., \quad \log^{[2]} T(r, f \circ g) \leq (\rho_g + \varepsilon) \log r + O(1) \).

Again from (5.3.5) we obtain for a sequence of values of $r$ tending to infinity that
\[ \log^{[2]} T(r, f \circ g) \geq (\rho_g - \varepsilon) \log r + O(1) . \]

Also from the definition of $\lambda_{P_0[f]}$ we have for all sufficiently large values of $r$,
\[ \log T(r, P_0[f]) \geq (\lambda_{P_0[f]} - \varepsilon) \log r \]
\[ i.e., \quad \log T(r, P_0[f]) \geq (\lambda_f - \varepsilon) \log r. \]

Therefore from (5.3.8) and (5.3.10), we have for all sufficiently large values of $r$ that
\[ \frac{\log^{[2]} T(r, f \circ g)}{\log T(r, P_0[f])} \leq \frac{(\rho_g + \varepsilon) \log r + O(1)}{(\lambda_f - \varepsilon) \log r} \]
\[ i.e., \quad \limsup_{r \to \infty} \frac{\log^{[2]} T(r, f \circ g)}{\log T(r, P_0[f])} \leq \frac{\rho_g}{\lambda_f} . \]

Again from (5.3.3) and (5.3.9), it follows for a sequence of values of $r$ tending to infinity that
\[ \frac{\log^{[2]} T(r, f \circ g)}{\log T(r, P_0[f])} \geq \frac{(\rho_g - \varepsilon) \log r + O(1)}{(\rho_f + \varepsilon) \log r} \]
\[ i.e., \quad \limsup_{r \to \infty} \frac{\log^{[2]} T(r, f \circ g)}{\log T(r, P_0[f])} \geq \frac{\rho_g}{\rho_f} . \]

Thus the theorem follows from (5.3.11) and (5.3.12).
**Remark 5.3.7** The following example ensures the conclusion of Theorem 5.3.7.

**Example 5.3.4** Let $f = \exp z$ and $g = z$. Further let $A = 1$ and $\alpha = 0$.

Choosing $s = 1, n_{1j} = 1$ and $n_{0j} = n_{2j} = \ldots = n_{kj} = 0$ in the definitions of $M[f]$ and $P_0[f]$ (See Page 74), we obtain that $P_0[f] = \exp z$. Then

$$\rho_f = \lambda_f = 1, \lambda_g = \rho_g = 0.$$ 

Now

$$T(r, f \circ g) = T(r, \exp z) = \frac{r}{\pi}.$$ 

So

$$\log^2 T(r, f \circ g) = \log^2 r + O(1).$$

Again

$$T(r, P_0[f]) = T(r, \exp z) = \frac{r}{\pi}$$

and so

$$\log T(r, P_0[f]) = \log r + O(1).$$

Now

$$\frac{\log^2 T(r, f \circ g)}{\log T(r, P_0[f])} = \frac{\log^2 r + O(1)}{\log r + O(1)}$$

and therefore

$$\limsup_{r \to \infty} \frac{\log^2 T(r, f \circ g)}{\log T(r, P_0[f])} = 0.$$ 

Again

$$\frac{\rho_g}{\rho_f} = 0 \text{ and } \frac{\rho_g}{\lambda_f} = 0.$$ 

Hence

$$\frac{\rho_g}{\rho_f} = \limsup_{r \to \infty} \frac{\log^2 T(r, f \circ g)}{\log T(r, P_0[f])} = \frac{\rho_g}{\lambda_f}.$$ 

**Remark 5.3.8** In addition to the conditions of Theorem 5.3.7, if $f$ is of regular growth i.e., $\rho_f = \lambda_f$. Then

$$\limsup_{r \to \infty} \frac{\log^2 T(r, f \circ g)}{\log T(r, P_0[f])} = \frac{\rho_g}{\rho_f}.$$
Remark 5.3.9 The conclusion of Theorem 5.3.7 and Remark 5.3.4 can also be drawn under the hypothesis \( \Theta (\infty; f) = \sum_{a \neq \infty} \delta_p (a; f) = 1 \) or \( \delta (\infty; f) = \sum_{a \neq \infty} \delta (a; f) = 1 \) instead of \( \sum_{a \neq \infty} \Theta (a; f) = 2 \).

Theorem 5.3.8 Let \( f \) be a meromorphic function and \( g \) be an entire function such that \( 0 < \lambda_f \leq \rho_f < \infty, \sum_{a \in \text{CU}(\infty)} \delta_1 (a; f) = 4 \) and \( 0 < \rho_g < \infty. \) Then

\[
\frac{\rho_g}{\rho_f} \leq \limsup_{r \to \infty} \frac{\log^2 T (r, f \circ g)}{\log T (r, M [f])} \leq \frac{\rho_g}{\lambda_f}.
\]

The proof is omitted because it can be carried out in the line of Theorem 5.3.7 and with the help of Lemma 5.2.10.

Remark 5.3.10 In addition to the conditions of Theorem 5.3.8, let \( f \) be of regular growth i.e., \( \rho_f = \lambda_f. \) Then

\[
\limsup_{r \to \infty} \frac{\log^2 T (r, f \circ g)}{\log T (r, M [f])} = \frac{\rho_g}{\rho_f}.
\]

In the line of Theorem 5.3.7, the following two corollaries may be deduced:

Corollary 5.3.3 Let \( f \) be a meromorphic function and \( g \) be an entire function such that \( 0 < \lambda_f \leq \rho_f < \infty, 0 < \lambda_g \leq \rho_g < \infty \) and \( \Theta (\infty; g) = \sum_{a \neq \infty} \delta_p (a; g) = 1 \) or \( \delta (\infty; g) = \sum_{a \neq \infty} \delta (a; g) = 1 \) or \( \sum_{a \neq \infty} \Theta (a; g) = 2. \) Then

\[
\frac{\rho_g}{\rho_f} \leq \limsup_{r \to \infty} \frac{\log^2 T (r, f \circ g)}{\log T (r, M [g])} \leq \frac{\rho_g}{\lambda_f}.
\]

In addition, if \( g \) is of regular growth then

\[
\limsup_{r \to \infty} \frac{\log^2 T (r, f \circ g)}{\log T (r, M [g])} = 1.
\]

Corollary 5.3.4 Let \( f \) be a meromorphic function and \( g \) be an entire function such that \( 0 < \lambda_f \leq \rho_f < \infty, 0 < \lambda_g \leq \rho_g < \infty \) and \( \sum_{a \in \text{CU}(\infty)} \delta_1 (a; g) = 4. \) Then

\[
\frac{\rho_g}{\rho_f} \leq \limsup_{r \to \infty} \frac{\log^2 T (r, f \circ g)}{\log T (r, M [g])} \leq \frac{\rho_g}{\lambda_f}.
\]

In addition, if \( g \) is of regular growth then

\[
\limsup_{r \to \infty} \frac{\log^2 T (r, f \circ g)}{\log T (r, M [g])} = 1.
\]
Theorem 5.3.9 Let \( f \) be a meromorphic function of order zero and \( g \) be an entire function of non zero finite order. Also let \( \Theta (\infty; g) = \sum_{a \neq \infty} \delta_p (a; g) = 1 \). Then

\[
\limsup_{r \to \infty} \frac{\log T (r, f \circ g)}{\log T (r^A, P_0 [g])} \geq \frac{1}{\lambda},
\]

where \( \lambda > 0 \).

Proof. In view of Lemma 5.2.2 and Lemma 5.3.3, we obtain for a sequence of values of \( r \) tending to infinity that

\[
\log T (r, f \circ g) \geq \log T \{ \exp (\rho_g - \varepsilon), f \}
\]

\( i.e., \log T (r, f \circ g) \geq (1 - \varepsilon) \log T (r^A, P_0 [g]) \).

\( i.e., \log T (r, f \circ g) \geq (1 - \varepsilon) (\rho_g - \varepsilon) \log r. \) (5.3.13)

Again by Lemma 5.2.7, we get for all sufficiently large values of \( r \) that

\[
\log T (r^A, P_0 |g|) \leq (\rho_g |g| + \varepsilon) \log r
\]

\( i.e., \log T (r^A, P_0 [g]) \leq A (\rho_g + \varepsilon) \log r. \) (5.3.14)

Therefore from (5.3.13) and (5.3.14), we obtain for a sequence of values of \( r \) tending to infinity that

\[
\frac{\log T (r, f \circ g)}{\log T (r^A, P_0 [g])} \geq \frac{(1 - \varepsilon) (\rho_g - \varepsilon) \log r}{A (\rho_g + \varepsilon) \log r}.
\]

(5.3.15)

Since \( \varepsilon (> 0) \) is arbitrary, the theorem follows from (5.3.15). \( \square \)

Remark 5.3.11 The conclusion of Theorem 5.3.9 can also be deduced if we replace \( \Theta (\infty; g) = \sum_{a \neq \infty} \delta_p (a; g) = 1 \) by \( \sum_{a \neq \infty} \Theta (a; g) = 2 \) or \( \delta (\infty; g) = \sum_{a \neq \infty} \delta (a; g) = 1 \) respectively.

Theorem 5.3.10 Let \( f \) be a meromorphic function of order zero and \( g \) be an entire function of non zero finite order. Also let \( \sum_{a \in \mathbb{C} \cup \{\infty\}} \delta_1 (a; g) = 4 \). Then for any positive real number \( A \),

\[
\limsup_{r \to \infty} \frac{\log T (r, f \circ g)}{\log T (r^A, M [g])} \geq \frac{1}{\lambda}.
\]

The proof is omitted because it can be carried out in the line of Theorem 5.3.9 and with the help of Lemma 5.2.10.

Theorem 5.3.11 Let \( f \) and \( g \) be any two entire functions such that \( 0 < \lambda_f^* < \infty \), \( 0 < \lambda_g \leq \lambda_g < \infty \) and \( \Theta (\infty; g) = \sum_{a \neq \infty} \delta_p (a; g) = 1 \) . Then

\[
\lim_{r \to \infty} \frac{T (r, f \circ g)}{\log T (r^A, P_0 [g])} = \infty,
\]

where \( A \) is any positive real number.
Proof. We know that for \( r > 0 \) \{cf.[40]\}

\[
T(r,f \circ g) \geq \frac{1}{3} \log M \left\{ \frac{1}{8} M\left(\frac{r}{4},g\right) + o(1), f \right\}.
\]

Let us choose \( \varepsilon \) in such a way that \( 0 < \varepsilon < \min \{ \lambda_f^*, \lambda_g \} \).

Now we get from (5.3.16) for all sufficiently large values of \( r \) that

\[
T(r,f \circ g) \geq \frac{1}{3} (\lambda_f^* - \varepsilon) \log M\left(\frac{r}{4},g\right) + O(1)
\]

i.e.,

\[
T(r,f \circ g) \geq \frac{1}{3} (\lambda_f^* - \varepsilon) \left(\frac{r}{4}\right)^{\lambda_g - \varepsilon} + O(1).
\]

Therefore we obtain from (5.3.14) and (5.3.17) for all sufficiently large values of \( r \) that

\[
\frac{T(r,f \circ g)}{\log T(r^A, P_0[g])} \geq \frac{1}{3} \left(\frac{\lambda_f^* - \varepsilon}{A} \right) \left(\frac{r}{4}\right)^{\lambda_g - \varepsilon} + O(1).
\]

As \( \lambda_g > 0 \), the theorem follows from (5.3.19).

Remark 5.3.12 If we take \( 0 < \rho_f^* < \infty \) instead of \( 0 < \lambda_f^* < \infty \) in Theorem 5.3.11 and the other conditions remain the same, then in the line of Theorem 5.3.11 one can easily verify that

\[
\limsup_{r \to \infty} \frac{T(r,f \circ g)}{\log T(r^A, P_0[g])} = \infty.
\]

Remark 5.3.13 Also if we consider \( 0 < \lambda_g < \infty \) or \( 0 < \rho_g < \infty \) instead of \( 0 < \lambda_g \leq \rho_g < \infty \) in Theorem 5.3.11 and the other conditions remain the same, then in the line of Theorem 5.3.11 one can easily verify that

\[
\limsup_{r \to \infty} \frac{T(r,f \circ g)}{\log T(r^A, P_0[g])} = \infty.
\]

Remark 5.3.14 The conclusion of Theorem 5.3.11, Remark 5.3.8 and Remark 5.3.9 can also be deduced if we replace \( \Theta(\infty; g) = \sum_{a \neq \infty} \delta_p(a; g) = 1 \) by \( \sum_{a \neq \infty} \Theta(a; g) = 2 \) or \( \delta(\infty; g) = \sum_{a \neq \infty} \delta(a; g) = 1 \) respectively.

In the line of Theorem 5.3.11 and with the help of Lemma 5.2.10, we may state the following theorem without its proof:

Theorem 5.3.12 Let \( f \) and \( g \) be any two entire functions such that \( 0 < \lambda_f^* < \infty \), \( 0 < \lambda_g \leq \rho_g < \infty \) and \( \sum_{a \in \mathbb{C} \cup \{\infty\}} \delta_1(a; g) = 4 \). Then

\[
\lim_{r \to \infty} \frac{T(r,f \circ g)}{\log T(r^A, M[g])} = \infty
\]

where \( A \) is any real number.
Remark 5.3.15 If we take $0 < \rho_f^* < \infty$ instead of $0 < \lambda_f^* < \infty$ in Theorem 5.3.12 and the other conditions remain the same, then in the line of Theorem 5.3.12 one can easily verify that
\[
\limsup_{r \to \infty} \frac{T(r, f \circ g)}{\log T(r^A, M \| g)} = \infty.
\]

Remark 5.3.16 Also if we consider $0 < \lambda_g < \infty$ or $0 < \rho_g < \infty$ instead of $0 < \lambda_g < \rho_g < \infty$ in Theorem 5.3.12 and the other conditions remain the same, then in the line of Theorem 5.3.12 it can be shown that
\[
\limsup_{r \to \infty} \frac{T(r, f \circ g)}{\log T(r^A, M \| g)} = \infty.
\]

Remark 5.3.17 If we take $f$ to be a meromorphic function with order zero in Theorem 5.3.11 and Theorem 5.3.12 and the other conditions remain the same then Theorem 5.3.11 and Theorem 5.3.12 remain valid with “limsup” instead of “lim”.

Theorem 5.3.13 Let $f$ be meromorphic and $g$ be an entire function such that $0 < \lambda_f \leq \rho_f < \infty$, $\sum_{a \neq \infty} \Theta(a; f) = 2$ and $\rho_g^* < \infty$. Then
\[
\lim_{r \to \infty} \frac{\log T(r, f \circ g)}{T(r^A, P_0 \| f)} = 0,
\]
where $A$ is any positive real number.

Proof. In view of Lemma 5.2.1 and the inequality $T(r, g) \leq \log^+ M(r, g)$, we get for all sufficiently large values of $r$
\[
T(r, f \circ g) \leq \{1 + o(1)\} T(M(r, g), f)
\]
\[
i.e., \quad \log T(r, f \circ g) \leq (\rho_f + \varepsilon) \log M(r, g) + O(1)
\]
\[
i.e., \quad \log T(r, f \circ g) \leq (\rho_f + \varepsilon) (\rho_g^* + \varepsilon) \log r + O(1).
\]
Again from the definition of $\lambda_{P_0 \| f}$ we have for arbitrary positive $\varepsilon$ and for all sufficiently large values of $r$,
\[
\log T(r^A, P_0 \| f) \geq A \left( \lambda_{P_0 \| f} - \varepsilon \right) \log r
\]
\[
i.e., \quad T(r^A, P_0 \| f) \geq r^A(\lambda_f - \varepsilon).
\]
Therefore it follows from (5.3.20) and (5.3.21) for all sufficiently large values of $r$ that
\[
\frac{\log T(r, f \circ g)}{T(r^A, P_0 \| f)} \leq \frac{(\rho_f + \varepsilon)(\rho_g^* + \varepsilon) \log r + O(1)}{r^A(\lambda_f - \varepsilon)}.
\]
As $\lambda_f > 0$, the theorem follows from (5.3.22).

Remark 5.3.18 The following example verifies the conclusion of Theorem 5.3.13.
Example 5.3.5 Let $f = \exp z$, $g = z$ and $A = 1$.

Considering $s = 1$, $n_{1j} = 1$ and $n_{0j} = n_{2j} = \ldots = n_{kj} = 0$ in the definitions of $M[f]$ and $P_0[f]$ (See Page 74), we obtain that $P_0[f] = \exp z$.

Then

$$\rho_f = \lambda_f = 1 \text{ and } \rho_g^{**} = 1.$$ 

Now

$$T(r, f \circ g) = T(r, \exp z) = \frac{r}{\pi}.$$ 

So

$$\log T(r, f \circ g) = \log r + O(1).$$

Again

$$T(r^A, P_0[f]) = T(r, \exp z) = \frac{r}{\pi}.$$ 

Therefore

$$\frac{\log T(r, f \circ g)}{T(r^A, P_0[f])} = \frac{\log r + O(1)}{r/\pi}$$

and so

$$\lim_{r \to \infty} \frac{\log T(r, f \circ g)}{T(r^A, P_0[f])} = 0.$$ 

Remark 5.3.19 If we take $0 < \rho_f < \infty$ or $0 < \lambda_f < \infty$ instead of $0 < \lambda_f \leq \rho_f < \infty$ in Theorem 5.3.13 and the other conditions remain the same, then in the line of Theorem 5.3.13 one can easily verify that

$$\liminf_{r \to \infty} \frac{\log T(r, f \circ g)}{T(r^A, P_0[f])} = 0.$$ 

Remark 5.3.20 Also if we take $\lambda_g^{**} < \infty$ instead of $\rho_g^{**} < \infty$ in Theorem 5.3.13 and the other conditions remain the same, then in the line of Theorem 5.3.13 one can easily verify that

$$\liminf_{r \to \infty} \frac{\log T(r, f \circ g)}{T(r^A, P_0[f])} = 0.$$ 

Remark 5.3.21 The conclusion of Theorem 5.3.13, Remark 5.3.14 and Remark 5.3.15 can also be deduced if we replace $\sum_{a \neq \infty} \Theta(a; f) = 2$ by $\Theta(\infty; f) = \sum_{a \neq \infty} \delta_p(a; f) = 1$ or $\delta(\infty; f) = \sum_{a \neq \infty} \delta(a; f) = 1$ respectively.

In the line of Theorem 5.3.13 and with the help of Lemma 5.2.10 we may state the following theorem without its proof:
**Theorem 5.3.14** Let $f$ be meromorphic and $g$ be an entire function such that $0 < \lambda_f \leq \rho_f < \infty$, $\sum_{a \in C \cup \{\infty\}} \delta_1(a; f) = 4$ and $\rho_g^{**} < \infty$. Then for any positive real number $A$,

$$\lim_{r \to \infty} \frac{\log T(r, f \circ g)}{T(r^A, M[f])} = 0.$$ 

**Remark 5.3.22** If we take $0 < \rho_f < \infty$ or $0 < \lambda_f < \infty$ instead of $0 < \lambda_f \leq \rho_f < \infty$ in Theorem 5.3.14 and the other conditions remain the same, then in the line of Theorem 5.3.14 one can easily verify that

$$\liminf_{r \to \infty} \frac{\log T(r, f \circ g)}{T(r^A, M[f])} = 0.$$ 

**Remark 5.3.23** Also if we take $\lambda_g^{**} < \infty$ instead of $\rho_g^{**} < \infty$ in Theorem 5.3.14 and the other conditions remain the same, then in the line of Theorem 5.3.14 one can easily verify that

$$\liminf_{r \to \infty} \frac{\log T(r, f \circ g)}{T(r^A, M[f])} = 0,$$

where $A$ is any real number.

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