CHAPTER 4

ANALYSIS OF A BATCH ARRIVAL GENERAL BULK SERVICE QUEUEING SYSTEM WITH MULTIPLE VACATIONS AND RESTRICTED ADMISSION OF ARRIVING BATCHES

4.1 INTRODUCTION

In many instances, queueing systems are controlled by various control policies, namely, control on number of servers, control on service rates, control on admission of customers and control on queue discipline. One can refer Crabill, Gross and Magazine (1977), Rue and Rosenshine (1981), Stidham (1985), Neuts (1985) and Huang and Mc-Donald (1998) respectively for queueing models with control policies. Fikri et al (1997) analyzed duality relations for queues with arrival and service control. Perry et al (2000) studied a busy period analysis for M/G/1 and G/M/1 type queues with restricted accessibility. Madan and Abu Dayyeah (2002) studied some aspects of batch arrivals and Bernoulli vacation models with restricted admissibility, where all arriving batches are not allowed into the system at all-time. Madan and Choudhury (2004) analyzed a $\text{M}^X/G/1$ queue with Bernoulli vacation schedule under restricted admissibility policy.

mobile systems. Miao Miao Yu et al (2009) studied a computation of the GI\(^x\)/M\(^b\)/1/L queue with multiple working vacations and partial batch rejection in steady state. Madan (2010) analyzed a batch arrival queue, with two stages of heterogeneous service, restricted admissibility of arriving batches and modified Bernoulli single vacation policy. **Accepting all arrivals to join into the system is not realistic always.**

In this chapter, a bulk queueing system with multiple vacations under restricted admissibility policy of arriving batches is considered. Arrivals occur in bulk, according to Poisson process. But all the arrivals are not considered for service. During the busy period of the server, the arrivals are admitted with probability \(\alpha\), whereas, with probability \(\beta\), they are admitted when the server is idle. The service is done in bulk with minimum of \(a\) customers and maximum of \(b\) customers. The server is assigned for secondary jobs (vacations) repeatedly when the number of waiting jobs is inadequate to process. Analytical treatment of this model is obtained by the supplementary variable technique.

The motivation of the model comes from a real life situation observed in an industry involving Electroplating Process (EP). Electroplating is a process that is widely used in the automotive, aerospace, electronics, medical sciences and general engineering industries. Electroplating is used for corrosion prevention, aesthetic finishes, wear coatings to various components, etc. Some of the electroplating processes are hard chrome plating, nickel plating, copper plating, brass plating, etc. The electroplating process (Hard chrome) on the components is done in bulk. Once the process is started, the bulk operation has to continue successively for many batches of metals; otherwise, the operating cost will increase. Hence, the operator will start the electroplating process only when required numbers of pieces have been accumulated for processing. After completing
an electro plating process, if the number of pieces to be processed is less than
the batch quantity, say ‘a’, then, the operator stops the process and performs
the associated works, such as rinsing, unjigging the components, buffing,
inspection, etc., Further, in order to meet the customer satisfaction and to
deliver the processed electroplates in time, the management may reject new
order (arrivals) with some probability. However, the operator accepts only
certain percent of arriving batch when the server is busy and certain percent of
arriving batch when the server is on vacation. This situation can be modeled
as $M^X/G(a,b)/1$ queueing system with multiple vacations under a
restricted admissibility policy of arriving batches.

For the proposed model, the probability generating function (PGF)
of the steady state queue size distribution at an arbitrary time epoch is
obtained using supplementary variable technique. Particular cases and some
special cases are discussed. Various performance measures are derived. A
cost model for the queueing system is developed. Numerical solution for
particular values of parameters is presented.

4.2 MATHEMATICAL MODEL

Let $X$ be the group size random variable of the arrival, $\lambda$ be the
Poisson arrival rate. $g_k$ be the probability that ‘$k$’ customers arrive in a batch
and $X(z)$ be its probability generating function (PGF). An arriving batch is
allowed to join the queue during the busy period with probability ‘$\alpha$’ and with
probability ‘$\beta$’ during a vacation period. Let $S(x)$ ($s(x)$) $\{\tilde{S}(\theta)\}$ $[S^0(x)]$ be
the cumulative distribution function (probability density function) \{Laplace-
Stieltjes transform\} [remaining service time] of service. Let $V(x)$ ($v(x)$)
$\{\tilde{V}(\theta)\}$ $[V^0(x)]$ be the cumulative distribution function (probability density
function) \{Laplace - Stieltjes transform\} [remaining vacation time] of
vacation. \( N_q(t) \) denotes the number of customers waiting for service at time \( t \), \( N_s(t) \) denotes the number of customers under the service at time \( t \). The different states of the server at time \( t \)’ are defined as follows:

\[
C(t) = \begin{cases} 
0, & \text{if the server is busy with service} \\
1, & \text{if the server is on vacation} 
\end{cases}
\]

\( Z(t) = j, \) if the server is on \( j \)th vacation starting from the idle period

To obtain the system equations, the following state probabilities are defined:

\[
P_{i,j}(x,t) \, dt = \Pr \left\{ N_s(t) = i, N_q(t) = j, x \leq \mathcal{S}^0(t) \leq x + dt, C(t) = 0 \right\}, \quad a \leq i \leq b, \quad j \geq 0
\]

\[
Q_{j,n}(x,t) \, dt = \Pr \left\{ N_q(t) = n, x \leq \mathcal{V}^0(t) \leq x + dt, C(t) = 1, Z(t) = j \right\}, \quad j \geq 1, \quad n \geq 0
\]

Now, the following system equations are obtained for the queueing system, using supplementary variable technique:

\[
P_{i,0}(x - \Delta t, t + \Delta t) = P_{i,0}(x,t)(1 - \lambda \Delta t) + \lambda(1 - \alpha) P_{i,0}(x,t) \Delta t + \sum_{m = a}^{b} \sum_{l = 1}^{\infty} P_{m,l}(0,t) s(x) \Delta t + \sum_{l = 1}^{\infty} Q_{l,0}(0,t) s(x) \Delta t ; \quad a \leq i \leq b
\]

\[
P_{i,j}(x - \Delta t, t + \Delta t) = P_{i,j}(x,t)(1 - \lambda \Delta t) + \lambda(1 - \alpha) P_{i,j}(x,t) \Delta t + \sum_{k = 1}^{j} P_{i,j-k}(x,t) \lambda g_k \Delta t ; \quad a \leq i \leq b - 1 \quad \& \quad j \geq 1
\]

\[
P_{b,j}(x - \Delta t, t + \Delta t) = P_{b,j}(x,t)(1 - \lambda \Delta t) + \lambda(1 - \alpha) P_{b,j}(x,t) \Delta t + \sum_{k = 1}^{j} P_{b,j-k}(x,t) \lambda g_k \Delta t + \sum_{m = a}^{b} P_{m,b+j}(0,t) s(x) \Delta t + \sum_{l = 1}^{\infty} Q_{l,b+j}(0,t) s(x) \Delta t ; \quad j \geq 1
\]
\[ Q_{1,0}(x - \Delta t, t + \Delta t) = Q_{1,0}(x, t) (1 - \lambda \Delta t) + \lambda (1 - \beta) Q_{1,0}(x, t) \Delta t + \sum_{m=a}^{b} P_{m,0}(0, t) v(x) \Delta t \]

\[ Q_{1,n}(x - \Delta t, t + \Delta t) = Q_{1,n}(x, t) (1 - \lambda \Delta t) + \lambda (1 - \beta) Q_{1,n}(x, t) \Delta t + \sum_{m=a}^{b} P_{m,n}(0, t) v(x) \Delta t + \beta \sum_{k=1}^{n} Q_{1,n-k}(x, t) \lambda g_k \Delta t ; \ 1 \leq n \leq a - 1 \]

\[ Q_{1,n}(x - \Delta t, t + \Delta t) = Q_{1,n}(x, t) (1 - \lambda \Delta t) + \lambda (1 - \beta) Q_{1,n}(x, t) \Delta t + \beta \sum_{k=1}^{n} Q_{1,n-k}(x, t) \lambda g_k \Delta t ; \quad n \geq a \]

\[ Q_{j,0}(x - \Delta t, t + \Delta t) = Q_{j,0}(x, t) (1 - \lambda \Delta t) + \lambda (1 - \beta) Q_{j,0}(x, t) \Delta t + Q_{j-1,0}(0, t) v(x) \Delta t \]

\[ j \geq 2 \]

\[ Q_{j,n}(x - \Delta t, t + \Delta t) = Q_{j,n}(x, t) (1 - \lambda \Delta t) + \lambda (1 - \beta) Q_{j,n}(x, t) \Delta t + \sum_{k=1}^{n} Q_{j,n-k}(x, t) \lambda g_k \Delta t + Q_{j-1,n}(0, t) v(x) \Delta t \]

\[ j \geq 2, \ 1 \leq n \leq a - 1 \]

\[ Q_{j,n}(x - \Delta t, t + \Delta t) = Q_{j,n}(x, t) (1 - \lambda \Delta t) + \lambda (1 - \beta) Q_{j,n}(x, t) \Delta t + \sum_{k=1}^{n} Q_{j,n-k}(x, t) \lambda g_k \Delta t ; \quad n \geq a, \ j \geq 2 \]

### 4.3 Steady State Queue Size Distribution

From the above equations, the steady state queue size equations are obtained as follows:

\[ -\frac{d}{dx} P_{1,0}(x) = -\lambda P_{1,0}(x) + \lambda (1 - \alpha) P_{1,0}(x) + \sum_{m=a}^{b} P_{m,i}(0) s(x) + \sum_{l=1}^{\infty} Q_{1,i}(0) s(x) \]

\[ a \leq i \leq b \] (4.1)
\[ \frac{d}{dx} P_{i,j}(x) = -\lambda P_{i,j}(x) + \sum_{k=1}^{j} \alpha \frac{d}{dx} P_{i,j-k}(x) \lambda g_k \]

\[ a \leq i \leq b - 1 \quad \& \quad j \geq 1 \tag{4.2} \]

\[ \frac{d}{dx} P_{b,j}(x) = -\lambda P_{b,j}(x) + \sum_{k=1}^{b} \alpha \lambda g_k \]

\[ + \sum_{m=a}^{b} P_{m,b+j}(0) s(x) + \sum_{l=1}^{\infty} Q_{1,b+j}(0) s(x) \quad j \geq 1 \tag{4.3} \]

\[ \frac{d}{dx} Q_{1,0}(x) = -\lambda Q_{1,0}(x) + \lambda(1-\beta)Q_{1,0}(x) + \sum_{m=a}^{b} P_{m,0}(0) v(x) \tag{4.4} \]

\[ \frac{d}{dx} Q_{1,n}(x) = -\lambda Q_{1,n}(x) + \lambda(1-\beta)Q_{1,n}(x) + \sum_{m=a}^{b} P_{m,n}(0) v(x) \]

\[ + \beta \sum_{k=1}^{n} Q_{1,n-k}(x) \lambda g_k \quad 1 \leq n \leq a - 1 \tag{4.5} \]

\[ \frac{d}{dx} Q_{1,n}(x) = -\lambda Q_{1,n}(x) + \lambda(1-\beta)Q_{1,n}(x) + \sum_{m=a}^{b} P_{m,n}(0) v(x) \]

\[ + \sum_{k=1}^{n} Q_{1,n-k}(x) \lambda g_k \quad n \geq a \tag{4.6} \]

\[ \frac{d}{dx} Q_{j,0}(x) = -\lambda Q_{j,0}(x) + \lambda(1-\beta)Q_{j,0}(x) + Q_{j-1,0}(0) v(x) \quad j \geq 2 \tag{4.7} \]

\[ \frac{d}{dx} Q_{j,n}(x) = -\lambda Q_{j,n}(x) + \lambda(1-\beta)Q_{j,n}(x) + \sum_{k=1}^{n} Q_{j,n-k}(x) \lambda g_k \]

\[ + Q_{j-1,n}(0) v(x) \quad j \geq 2, \quad 1 \leq n \leq a - 1 \tag{4.8} \]

\[ \frac{d}{dx} Q_{j,n}(x) = -\lambda Q_{j,n}(x) + \lambda(1-\beta)Q_{j,n}(x) + \sum_{k=1}^{n} Q_{j,n-k}(x) g_k \quad n \geq a, \quad j \geq 2 \tag{4.9} \]

The Laplace –Stieltjes transforms (LST) of \( P_{i,n}(x) \) and \( Q_{j,n}(x) \) are defined as

\[ \tilde{P}_{i,n}(\theta) = \int_{0}^{\infty} e^{-\theta x} P_{i,n}(x) dx \quad \text{and} \quad \tilde{Q}_{j,n}(\theta) = \int_{0}^{\infty} e^{-\theta x} Q_{j,n}(x) dx \]
Taking LST on both sides of the Equations (4.1) through (4.9), we have

\[
\tilde{\mathbf{P}}_{i,0}(\theta) - P_{i,0}(0) = \lambda \tilde{\mathbf{P}}_{i,0}(\theta) - (1-\alpha) \tilde{\mathbf{P}}_{i,0}(\theta) - \frac{b}{m=a} \sum P_{m,i}(0) \tilde{\xi}(\theta) - \sum_{l=1}^{\infty} Q_{i,j}(0) \tilde{\xi}(\theta) \quad a \leq i \leq b
\]

(4.10)

\[
\tilde{\mathbf{P}}_{i,j}(\theta) - P_{i,j}(0) = \lambda \tilde{\mathbf{P}}_{i,j}(\theta) - (1-\alpha) \tilde{\mathbf{P}}_{i,j}(\theta) - \lambda \alpha \sum_{k=1}^{j} \tilde{\mathbf{P}}_{i,j-k}(\theta) g_{k} ; \quad a \leq i \leq b-1, \quad j \geq 1
\]

(4.11)

\[
\tilde{\mathbf{P}}_{b,j}(\theta) - P_{b,j}(0) = \lambda \tilde{\mathbf{P}}_{b,j}(\theta) - (1-\alpha) \tilde{\mathbf{P}}_{b,j}(\theta) - \lambda \alpha \sum_{k=1}^{j} \tilde{\mathbf{P}}_{b,j-k}(\theta) g_{k}
\]

\[- \sum_{m=a}^{b} P_{m,b+j}(0) \tilde{\xi}(\theta) - \sum_{l=1}^{\infty} Q_{1,b+j}(0) \tilde{\xi}(\theta) ; \quad j \geq 1 \]

(4.12)

\[
\tilde{\mathbf{Q}}_{1,0}(\theta) - Q_{1,0}(0) = \lambda \tilde{\mathbf{Q}}_{1,0}(\theta) - (1-\beta) \tilde{\mathbf{Q}}_{1,0}(\theta) - \frac{b}{m=a} \sum P_{m,0}(0) \tilde{\nu}(\theta)
\]

(4.13)

\[
\tilde{\mathbf{Q}}_{1,n}(\theta) - Q_{1,n}(0) = \lambda \tilde{\mathbf{Q}}_{1,n}(\theta) - (1-\beta) \tilde{\mathbf{Q}}_{1,n}(\theta) - \lambda \beta \sum_{k=1}^{n} \tilde{\mathbf{Q}}_{1,n-k}(\theta) g_{k}
\]

\[- \sum_{m=a}^{b} P_{m,n}(0) \tilde{\nu}(\theta) ; \quad 1 \leq n \leq a-1 \]

(4.14)

\[
\tilde{\mathbf{Q}}_{1,n}(\theta) - Q_{1,n}(0) = \lambda \tilde{\mathbf{Q}}_{1,n}(\theta) - (1-\beta) \tilde{\mathbf{Q}}_{1,n}(\theta) - \lambda \beta \sum_{k=1}^{n} \tilde{\mathbf{Q}}_{1,n-k}(\theta) g_{k} ; \quad n \geq a
\]

(4.15)

\[
\tilde{\mathbf{Q}}_{j,0}(\theta) - Q_{j,0}(0) = \lambda \tilde{\mathbf{Q}}_{j,0}(\theta) - (1-\beta) \tilde{\mathbf{Q}}_{j,0}(\theta) - Q_{j-1,0}(0) \tilde{\nu}(\theta) ; \quad j \geq 2
\]

(4.16)

\[
\tilde{\mathbf{Q}}_{j,n}(\theta) - Q_{j,n}(0) = \lambda \tilde{\mathbf{Q}}_{j,n}(\theta) - (1-\beta) \tilde{\mathbf{Q}}_{j,n}(\theta) - \lambda \beta \sum_{k=1}^{n} \tilde{\mathbf{Q}}_{j,n-k}(\theta) g_{k}
\]

\[- Q_{j-1,n}(0) \tilde{\nu}(\theta) ; \quad 1 \leq n \leq a-1, \quad j \geq 2
\]

(4.17)
\[ \tilde{Q}_{j, n}(\theta) = Q_{j, n}(\theta) - \lambda \tilde{Q}_{j, n}(\theta) - \lambda(1-\beta)\tilde{Q}_{j, n}(\theta) - \lambda \beta \sum_{k=1}^{n} \tilde{Q}_{j, n-k}(\theta) g_k \] 
\[ n \geq a, \ j \geq 2 \]

### 4.3.1 Probability Generating Function

As discussed in section 2.3.1 of chapter II, to find the steady state probability generating function (PGF) of the number of customers in the queue at an arbitrary time epoch, the following probability generating functions are defined.

\[ \bar{P}_i(z, \theta) = \sum_{j=0}^{\infty} \bar{P}_{i,j}(\theta) z^j \quad \text{and} \quad P_i(z, 0) = \sum_{j=0}^{\infty} P_{i,j}(0) z^j, \quad a \leq i \leq b \]

\[ \tilde{Q}_j(z, \theta) = \sum_{n=0}^{\infty} \tilde{Q}_{j,n}(\theta) z^n \quad \text{and} \quad Q_j(z, 0) = \sum_{n=0}^{\infty} Q_{j,n}(0) z^n \]  
\[ (a, b) \]

The probability generating function \( P(z) \) of the number of customers in the queue at an arbitrary time epoch of the proposed model can be obtained using the following equation.

\[ P(z) = \sum_{i=a}^{b-1} \bar{P}_i(z, 0) + \tilde{P}_b(z, 0) + \sum_{j=1}^{\infty} \tilde{Q}_j(z, 0) \]  
\[ (4.20) \]

In order to find \( \bar{P}_i(z, 0), \tilde{P}_b(z, 0) \) and \( \tilde{Q}_j(z, 0) \) the following sequence of operations are done.

Multiplying (4.13) by \( z^0 \), (4.14) by \( z^n \) \( (1 \leq n \leq a-1) \), (4.15) by \( z^n \) \( (n \geq a) \), summing up from \( n = 0 \) to \( \infty \) and using (4.19), we get

\[ (\theta - \beta(\lambda - \lambda X(z)))\bar{Q}_1(z, 0) = Q_1(z, 0) - \tilde{V}(0) \sum_{n=0}^{a-1} \sum_{m=a}^{b} P_{m,n}(0) z^n \]  
\[ (4.21) \]
Multiplying (4.16) by \(z^0\), (4.17) by \(z^n\) \((1 \leq n \leq a-1)\), (4.18) by \(z^n\) \((n \geq a)\), summing up from \(n = 0\) to \(\infty\) and using (4.19), we get

\[
(\theta - \beta(\lambda - \lambda X(z))) \tilde{Q}_j(z,0) = Q_j(z,0) - \tilde{V}(\theta) \sum_{n=0}^{a-1} Q_{j-1,n}(0) z^n, \quad j \geq 2
\]

(4.22)

Multiplying (4.10) by \(z^0\), (4.11) by \(z^j\) \((j \geq 1)\), summing up from \(j = 0\) to \(\infty\) and using (4.19), we get

\[
(\theta - \alpha(\lambda - \lambda X(z))) \tilde{P}_i(z,0) = P_i(z,0) - \tilde{S}(\theta) \left( \sum_{m=a}^{b} P_{m,i}(0) + \sum_{l=1}^{\infty} Q_{l,i}(0) \right); \quad a \leq i \leq b-1
\]

(4.23)

Multiplying (4.10) by \(z^0\) with \(i = b\), (4.12) by \(z^j\) \((j \geq 1)\), summing up from \(j = 0\) to \(\infty\) and using (4.19), we get

\[
z^b (\theta - \alpha(\lambda - \lambda X(z))) \tilde{P}_b(z,0) = z^b P_b(z,0)
\]

\[
- \tilde{S}(\theta) \left( \sum_{m=a}^{b} \left( P_{m,a}(z,0) - \sum_{j=0}^{b-1} P_{m,j}(0) z^j \right) \right)
\]

\[
- \tilde{S}(\theta) \left( \sum_{j=1}^{\infty} Q_j(z,0) - \sum_{j=0}^{b-1} Q_{j,0} z^j \right)
\]

(4.24)

By substituting \(\theta = \beta(\lambda - \lambda X(z))\) in the Equations (4.21) and (4.22), we get

\[
Q_1(z,0) = \tilde{V}(\beta(\lambda - \lambda X(z))) \sum_{n=0}^{a-1} \sum_{m=a}^{b} P_{m,n}(0) z^n
\]

(4.25)

\[
Q_j(z,0) = \tilde{V}(\beta(\lambda - \lambda X(z))) \sum_{n=0}^{a-1} Q_{j-1,n}(0) z^n, \quad j \geq 2
\]

(4.26)
By substituting $\theta = \alpha(\lambda - \lambda X(z))$ in the Equations (4.23) and (4.24), we get

\[
P_i(z,0) = \tilde{S}(\alpha(\lambda - \lambda X(z))) \left\{ \sum_{m=a}^{b} P_{m,i}(0) + \sum_{l=1}^{\infty} Q_{l,i}(0) \right\}; a \leq i \leq b - 1 \tag{4.27}
\]

\[
z^{b} P_{b}(z,0) = \tilde{S}(\alpha(\lambda - \lambda X(z))) \left\{ \sum_{m=a}^{b} \left( P_{m}(z,0) - \sum_{j=0}^{b-1} P_{m,j}(0) z^{j} \right) \right\} + \sum_{l=1}^{\infty} \left( Q_{l}(z,0) - \sum_{j=0}^{b-1} Q_{l,j}(0) z^{j} \right) \tag{4.28}
\]

From the Equation (4.28), we get

\[
P_{b}(z,0) = \frac{\tilde{S}(\alpha(\lambda - \lambda X(z))) f(z)}{z^{b} - \tilde{S}(\alpha(\lambda - \lambda X(z)))} \tag{4.29}
\]

where

\[
f(z) = \tilde{S}(\alpha(\lambda - \lambda X(z))) \sum_{i=a}^{b-1} \left( \sum_{m=a}^{b} P_{m,i}(0) + \sum_{l=1}^{\infty} Q_{l,i}(0) \right) - \sum_{j=0}^{b-1} \sum_{l=1}^{\infty} Q_{l,j}(0) z^{j} + \tilde{V}(\beta(\lambda - \lambda X(z))) \left( \sum_{n=0}^{a-1} \sum_{m=a}^{b} P_{m,n}(0) z^{n} + \sum_{n=0}^{a-1} \sum_{l=1}^{\infty} Q_{l,n}(0) z^{n} \right) \tag{4.30}
\]

From the Equations (4.21) and (4.25), we get

\[
\tilde{Q}_{1}(z,\theta) = \frac{1}{(\theta - \beta(\lambda - \lambda X(z)))} \left( \tilde{V}(\beta(\lambda - \lambda X(z))) - \tilde{V}(\theta) \right) \sum_{n=0}^{a-1} \sum_{m=a}^{b} P_{m,n}(0) z^{n} \tag{4.31}
\]

From the Equations (4.22) and (4.26), we get

\[
\tilde{Q}_{j}(z,\theta) = \frac{1}{(\theta - \beta(\lambda - \lambda X(z)))} \left( \tilde{V}(\beta(\lambda - \lambda X(z))) - \tilde{V}(\theta) \right) \sum_{n=0}^{a-1} \sum_{j=1}^{\infty} Q_{j-1,n}(0) z^{n}, j \geq 2 \tag{4.32}
\]
From the Equations (4.23) and (4.27), we get

\[
\tilde{P}_1(z,0) = \frac{1}{(\theta - \alpha(\lambda - \lambda X(z)))} \left[ \left( \tilde{S}(\alpha(\lambda - \lambda X(z))) - \tilde{S}(\theta) \right) \sum_{m=a}^{b} P_{m,i}(0) + \sum_{l=1}^{\infty} Q_{i,l}(0) \right]_{a \leq i \leq b - 1}
\]

(4.33)

From the Equations (4.24) and (4.29), we get

\[
\tilde{P}_b(z,0) = \frac{\left( \tilde{S}(\alpha(\lambda - \lambda X(z))) - \tilde{S}(\theta) \right) f(z)}{(\theta - \alpha(\lambda - \lambda X(z))) \left( z^b - \tilde{S}(\alpha(\lambda - \lambda X(z))) \right)}
\]

(4.34)

where \( f(z) \) is given in (4.30).

Let \( \sum_{m=a}^{b} P_{m,i}(0) = p_i \), \( \sum_{l=1}^{\infty} Q_{i,l}(0) = q_i \) and \( c_i = p_i + q_i \).

Substituting \( \tilde{P}_1(z,0) \), \( \tilde{P}_b(z,0) \) and \( \tilde{Q}_j(z,0) \) from the Equations (4.31) – (4.34) in the Equation (4.20), the probability generating function of the queue size \( P(z) \) at an arbitrary time epoch is obtained as

\[
P(z) = \frac{\beta^{b-1} \sum_{l=a}^{b} \left( \tilde{S}(\alpha(\lambda - \lambda X(z))) - 1 \right) \left( z^b - z^l \right) c_i}{\alpha^\beta \left( z^b - \tilde{S}(\alpha(\lambda - \lambda X(z))) \right) (\lambda X(z) - \lambda)} + \left( \tilde{V}(\beta(\lambda - \lambda X(z))) - 1 \right) \sum_{n=0}^{a-1} \beta^{a-1} \left( \tilde{S}(\alpha(\lambda - \lambda X(z))) - 1 \right) + \alpha \left( z^b - \tilde{S}(\alpha(\lambda - \lambda X(z))) \right) c_n z^n}
\]

(4.35)
The probability generating function \( P(z) \) has to satisfy \( P(1) = 1 \). In order to satisfy the condition, applying L’Hospital’s rule and evaluating \( \lim_{z \to 1} P(z) \) and equating the expression to 1, \( b - \alpha \lambda E(X)E(S) > 0 \) is obtained.

Define ‘\( \rho \)’ as \( \frac{\alpha \lambda E(X)E(S)}{b} \). Thus \( \rho < 1 \) is the condition to be satisfied for the existence of steady state for the model under consideration.

### 4.3.2 Computational Aspects of Unknown Probabilities

Equation (4.35) gives the probability generating function \( P(z) \) of the number of customers in the queue at an arbitrary time epoch, which involves ‘\( b \)’ unknown probabilities namely, \( c_0, c_1, c_2, \ldots, c_{b-1} \). By Rouche’s theorem, the expression \( z^b - \tilde{S}(\alpha(\lambda - \lambda X(z))) \) has \( b-1 \) zeros inside and one on the unit circle \( |z|=1 \). Since \( P(z) \) is analytic within and on the unit circle, the numerator of (4.35) must vanish at these points, which gives ‘\( b \)’ equations and ‘\( b \)’ unknowns. These equations can be solved by suitable numerical techniques.

The unknown probabilities \( q_0, q_1, q_2, \ldots, q_{a-1} \) are expressed in terms of \( p_0, p_1, p_2, \ldots, p_{a-1} \) in theorem (4.1).

### Theorem 4.1

The constants \( q_n \) involved in \( P(z) \) are expressed in terms of \( p_n \) as,

\[
q_n = \sum_{i=0}^{n} b_{n-i} p_i ; \quad n = 0, 1, 2, \ldots, a-1, \text{ where } b_0 = \frac{\omega_0}{1 - \omega_0}, \quad b_n = \frac{\omega_n + \sum_{j=0}^{n-1} b_j \cdot \omega_j}{1 - \omega_0},
\]

\( n = 1, 2, \ldots, a-1 \) and \( \omega_i \) is the probability that ‘\( i \)’ customers arrive during a vacation period.
Proof:

From the Equations (4.19), (4.25) and (4.26), we have

$$\sum_{j=1}^{\infty} Q_j(z,0) = \sum_{n=0}^{\infty} q_n z^n = \tilde{V}(\beta(\alpha - \lambda X(z))) \sum_{n=0}^{a-1} \left( p_n + q_n \right) z^n$$

which, using Lemma (2.1), gives

$$\sum_{n=0}^{\infty} q_n z^n = \left( \sum_{n=0}^{\infty} \omega_n z^n \right) \sum_{n=0}^{a-1} \left( p_n + q_n \right) z^n$$

$$= \sum_{n=0}^{a-1} \left( \sum_{i=0}^{n} \omega_{n-i} \left( p_i + q_i \right) \right) z^n + \sum_{n=a}^{\infty} \left( \sum_{i=0}^{a-1} \omega_{n-i} \left( p_i + q_i \right) \right) z^n$$

Equating the coefficients of $z^n$, $n=\ 0,1,2,3,\ldots a-1$, on both sides of the equation, we have

$q_n = \sum_{i=0}^{n} \omega_{n-i} \left( p_i + q_i \right)$,

$$q_n = \frac{\sum_{i=0}^{n} \omega_{n-i} p_i + \sum_{i=0}^{n-1} \omega_{n-i} q_i}{1 - \omega_0}$$

Coefficient of $p_n$ in $q_n$ is $\frac{\omega_0}{1 - \omega_0} = b_0$

Coefficient of $p_{n-1}$ in $q_n$ is $[\omega_1 + \omega_1 \text{ Coefficient of } p_{n-1} \text{ in } q_{n-1}] / (1 - \omega_0) = \frac{\omega_1 + \omega_1 b_0}{1 - \omega_0} = b_1$
Coefficient of \( p_{n-2} \) in \( q_n \) is \([ \omega_2 + \omega_1 \) Coefficient of \( p_{n-2} \) in \( q_{n-1} \) \\
+ \omega_2 \) Coefficient of \( p_{n-2} \) in \( q_{n-2} \)]/1 - \( \omega_0 \)

\[
= \frac{\omega_2 + (\omega_1 b_1 + \omega_2 b_0)}{1 - \omega_0} = b_2
\]

Proceeding like this, we get

Coefficient of \( p_0 \) in \( q_n \) is \( \frac{\omega_n + \sum_{i=1}^{n} \omega_i b_{n-i}}{1 - \omega_0} = b_n \)

Therefore, \( q_n = \sum_{i=0}^{n} b_{n-i} p_i \); \( n = 0,1,2\ldots,a-1 \) \hspace{1cm} (4.36)

Hence the theorem \( \Box \)

4.4 PERFORMANCE MEASURES

In this section, some useful performance measures of the proposed model like, expected number of customers in the queue \( E(Q) \), expected length of idle period \( E(I) \), expected length of busy period \( E(B) \) are derived which are useful to find the total average cost of the system. Also, probability that the server is on vacation \( P(V) \) and probability that the server is busy \( P(B) \) are derived.

4.4.1 Expected Queue Length

The expected queue length \( E(Q) \) (i.e. mean number of customers waiting in the queue) at an arbitrary time epoch, is obtained by differentiating \( P(z) \) at \( z = 1 \) and is given by

\[
\lim_{z \to 1} P(z) = E(Q)
\]
\[
E(Q) = \left( \sum_{i=a}^{b-1} \beta c_i (b(b-1)-i(i-1)) f_1(X,S) + \sum_{i=a}^{b-1} \beta c_i (b-i) f_2(X,S) \right) \times \left( \sum_{i=0}^{a-1} \beta c_i \left( f_3(X,S,V) - f_4(X,S,V) \right) + \sum_{i=0}^{a-1} \alpha c_i \left( f_5(X,S,V) - f_6(X,S,V) \right) \right) \times \frac{2\alpha \beta E(X)}{(T1)^2}
\]

where \( S_1 = a\lambda E(X)E(S) \); \( T_1 = b-S_1 \); \( V_1 = \beta \lambda E(X)E(V) \);
\( V_2 = \beta \lambda X^*(1)E(V) + \beta^2 \lambda^2 E^2(X)E(V^2) \);
\( S_2 = a\alpha X^*(1)E(S) + a^2 \lambda^2 E^2(X)E(S^2) \);
\( S_3 = \lambda X^*(1)(T1) + \lambda E(X)b(b-1) - \lambda E(X)(S2) \);
\( f_1(X,S) = (T1)(S1) \);
\( f_2(X,S) = (T1)(S2) - \alpha E(S)S3 \);
\( f_3(X,S,V) = (T1)(2i(V1)(S1)+(S2)(V1)+(S1)(V2)) \);
\( f_4(X,S,V) = (S3)(S1)\beta E(V) \);
\( f_5(X,S,V) = (T1)(2i(V1)(T1)+b(b-1)(V1)-(V1)(S2)+(T1)(V2)) \) and
\( f_6(X,S,V) = (S3)(T1)\beta E(V) \)

### 4.4.2 Expected Length of Idle Period

Let \( I \) be the idle period random variable. Another random variable \( U_1 \) is defined as,

\[
U_1 = \begin{cases} 
0, & \text{if the server finds at least ‘a’ customers after the first vacation} \\
1, & \text{if the server finds less than ‘a’ customers after the first vacation} 
\end{cases}
\]

Now, the expected length of idle period due to multiple vacations \( E(I) \) is given by

\[
E(I) = E(I / U_1 = 0)P(U_1 = 0) + E(I / U_1 = 1)P(U_1 = 1)
\]

\[
= E(V)P(U_1 = 0) + (E(V) + E(I))P(U_1 = 1)
\]

where \( E(V) \) is the expected vacation time.
Solving for $E(I)$, we have

$$E(I) = \frac{E(V)}{P(U_1 = 0)} \quad (4.38)$$

To find $P(U_1 = 0)$, we do some algebra using the Equation (4.25), then

$$Q_1(z, 0) = \sum_{n=0}^{\infty} Q_{1,n}(0) = \tilde{V}(\beta(\lambda - \lambda X(z))) \sum_{n=0}^{a-1} p_n z^n$$

$$= \left( \sum_{n=0}^{\infty} \omega_n z^n \right) \left( \sum_{n=0}^{a-1} p_n z^n \right)$$

$$= \left( \sum_{n=0}^{a-1} \sum_{i=0}^{n} \omega_i p_{n-i} z^n \right) \left( \sum_{n=a_i=0}^{\infty} \omega_i z^n \right)$$

Equating the coefficients of $z^n$ ($n = 0, 1, 2, 3, \ldots a-1$) on both sides, we get

we get $Q_{1,n}(0) = \sum_{i=0}^{n} \omega_i p_{n-i}$

Therefore $P(U_1 = 0) = 1 - \sum_{n=0}^{a-1} \sum_{i=0}^{n} \omega_i p_{n-i} \quad (4.39)$

Using (4.39) in (4.38), the expected length of idle period $E(I)$ is obtained as

$$E(I) = \frac{E(V)}{1 - \sum_{n=0}^{a-1} \sum_{i=0}^{n} \omega_i p_{n-i}} \quad (4.40)$$
4.4.3 Expected Length of Busy Period

Let $B$ be the busy period random variable. Another random variable $J$ is defined as,

$J = \begin{cases} 
0, & \text{if the server finds less than 'a' customers after the first service} \\
1, & \text{if the server finds at least 'a' customers after the first service} 
\end{cases}$

Now, the expected length of busy period $E(B)$ is given by

$$E(B) = E(B \mid J = 0)P(J = 0) + E(B \mid J = 1)P(J = 1) = E(S)P(J = 0) + (E(S) + E(B))P(J = 1)$$

Solving for $E(B)$, we get

$$E(B) = \frac{E(S)}{P(J = 0)}$$

Thus, the expected length of busy period is obtained as

$$E(B) = \frac{E(S)}{\sum_{i=0}^{a-1} p_i}, \text{ where } E(S) \text{ is the expected service time.} \quad (4.41)$$

4.4.4 Probability that the Server is on Vacation

Let $V$ be the random variable for multiple vacations and $P(V)$ be the probability that the server is on multiple vacations at time $t$.

From the Equations (4.31) and (4.32), we have

$$\sum_{j=1}^{\infty} \tilde{Q}_j(z,0) = \frac{\tilde{V}(\beta(\lambda, \lambda X(z)) - 1)}{(-\lambda + \lambda X(z))} \left( \sum_{n=0}^{a-1} c_n z^n \right)$$
Now, the probability that the server is on vacation is given by

\[
P(V) = \lim_{z \to 1} \sum_{j=1}^{\infty} \tilde{Q}_j(z,0)
\]

\[
= \lim_{z \to 1} \left( \frac{\hat{V}(\beta(\lambda - \lambda X(z))) - 1}{(-\lambda + \lambda X(z))} \right) \left( \sum_{n=0}^{a-1} c_n z^n \right)
\]

\[
P(V) = E(V) \left( \sum_{n=0}^{a-1} c_n \right) \quad (4.42)
\]

### 4.4.5 Probability that the Server is Busy

Let \( B \) be the busy period random variable and \( P(B) \) be the probability that the server is busy at time \( t \).

From the Equations (4.33) and (4.34), we have

\[
P(B) = \lim_{z \to 1} \sum_{i=a}^{b} \tilde{P}_i(z,0)
\]

\[
= \lim_{z \to 1} \left( \sum_{i=a}^{b-1} \tilde{P}_i(z,0) + \tilde{P}_b(z,0) \right)
\]

\[
P(B) = E(S) \sum_{i=a}^{b-1} c_i + \frac{E(S)f'(1)}{b(1-\rho)} \quad (4.43)
\]

where \( f'(1) = \alpha \lambda E(X)E(S) \sum_{i=a}^{b-1} c_i + \beta \lambda E(X)E(V) \sum_{i=0}^{a-1} c_i - \sum_{i=a}^{b-1} ic_i \) and \( \rho = \frac{\alpha \lambda E(X)E(S)}{b} \)

### 4.5 PARTICULAR CASES

In this section, some of the existing models are deduced as a particular case of the proposed model.
Case (i): If all arrivals are allowed to join the system, (i.e. $\alpha = 1$ and $\beta = 1$), then the Equation (4.35) reduces to

$$P(z) = \frac{\left[ \sum_{i=a}^{b-1} \left( S(\lambda - \lambda X(z)) - 1 \right) z^i \right] c_1 + \left( \tilde{V}(\lambda - \lambda X(z)) - 1 \right) \sum_{n=0}^{a-1} \left( z^n - 1 \right) c_n z^n}{\left( z^b - S(\lambda - \lambda X(z)) \right) \left( \lambda X(z) - \lambda \right)}$$

which exactly coincides with the result $M^X / G(a,b)/1$ and multiple vacations without setup time and N – Policy of Krishna Reddy et al (1998).

Case (ii): If all arrivals are allowed to join the system, (i.e. $\alpha = 1$, $\beta = 1$) and no bulk service, (i.e $a = b = 1$), then the Equation (4.35) becomes

$$P(z) = \frac{\left( \tilde{V}(\lambda - \lambda X(z)) - 1 \right) (z-1) c_0}{\left( z - S(\lambda - \lambda X(z)) \right) \left( \lambda X(z) - \lambda \right)}$$

which coincides with the result $M^X / G/1$ queueing system and multiple vacations without N-Policy of Lee et al (1994).

Case (iii): Instead of bulk service, if single service is considered (i.e. $a = b = 1$), and all arrivals are allowed to join the system (i.e. $\alpha = 1$ and $\beta = 1$), then the probability that the server is in the busy period and the probability that the server is on vacation at time $t$ is obtained by

$$P(B) = \frac{E(S)f(1)}{1 - \rho}$$

$$= \frac{E(S)\lambda E(X)E(V)c_0}{1 - \rho}$$

$$= \lambda E(S) E(X)$$

$$= \rho$$, is the probability that the server is busy

and

$$P(V) = E(V)c_0$$

$$= 1 - \rho$$, is the probability that the system is in vacation period
where the unknown \( c_0 \) is obtained from the Equation (4.35) by using the condition \( P(1) = 1 \), which is \( c_0 = \frac{1 - \rho}{E(V)} \), where \( f'(1) = \lambda E(X) E(V) c_0 \) and \( \rho = \lambda E(X) E(S) \). This coincides with the result of the \( M^X/G/1 \) queueing system with multiple vacations of Sun Hur and Suneung Ahn (2005) without setup times.

### 4.5.1 Special Cases

The model so developed is general in nature as the service time and vacation time are arbitrary. But for practical purposes, service time and vacation time with particular distribution is required. In this section, some special cases of the proposed model by specifying vacation time random variable as exponential distribution and bulk service time random variable as hyper exponential and Erlangian distributions are discussed.

**Case (i):** Single server batch arrival queue with **hyper exponential bulk service time** and restricted admissibility policy

If the service time is assumed as hyper exponential with probability density function \( s(x) = cu e^{-ux} + (1 - c)we^{-wx} \), where \( u \) and \( w \) are the parameters, then,

\[
\tilde{S}(\alpha(\lambda - \lambda X(z))) \left( \frac{uc}{u + \alpha \lambda (1 - X(z))} \right) \left( \frac{w(1 - c)}{w + \alpha \lambda (1 - X(z))} \right).
\]

Substituting this expression for \( \tilde{S}(\alpha(\lambda - \lambda X(z))) \) in (4.35) and after some algebra, the PGF of the queue size distribution of this special case of the queueing model is obtained as,
Case (ii): Single server batch arrival queue with k-Erlangian bulk service time and restricted admissibility policy

In case of k - Erlang service time random variable with probability density function $s(x) = \frac{(ku)^k x^{k-1} e^{-kux}}{(k-1)!}$, $k > 0$; where u is the parameter, then, $\tilde{S}(\alpha(\lambda - \lambda X(z))) = \frac{\mu k}{\mu k + \alpha \lambda (1 - X(z))}$. Substituting this expression for $\tilde{S}(\alpha(\lambda - \lambda X(z)))$ in (4.35) and after some algebra, the PGF of the queue size distribution of this special case of the queueing model is obtained as,

$$P(z) = \frac{\beta \sum_{i=0}^{b-1} \left[ \left( \frac{\mu k}{\mu k + \alpha \lambda (1 - X(z))} \right)^k \right] (z^b - z^i) c_i}{\alpha \beta \left( z^b \left( \frac{\mu k}{\mu k + \alpha \lambda (1 - X(z))} \right)^k \right) (\lambda X(z) - \lambda)}$$
Case (iii): Single server batch arrival queue with **exponential vacation time** and restricted admissibility policy

If the vacation time is assumed as exponential with probability density function \( v(x) = \gamma e^{-\gamma x} \), where \( \gamma \) is the parameter, then,

\[
\hat{V}(\beta(\lambda - \lambda X(z))) = \left( \frac{\gamma}{\gamma + \beta\lambda(1-X(z))} \right).
\]

Substituting this expression for \( \hat{V}(\beta(\lambda - \lambda X(z))) \) in (4.35) and after some algebra, the PGF of the queue size distribution of this special case of the queueing model is obtained as,

\[
P(z) = a \left( \sum_{n=0}^{b} \left( \frac{\beta}{(\gamma + \beta\lambda(1-X(z)))} \right)^{-1} \left( z^b - z^i \right) c_i \right)
\]

\[+ \left( \sum_{n=0}^{\infty} \left( \frac{\beta}{(\gamma + \beta\lambda(1-X(z))} \right)^{-1} \left( z^b - z^i \right) c_i \right)
\]

\[= \alpha \beta \left( z^b - \hat{S}(\alpha(\lambda - \lambda X(z))) \right) \left( \lambda X(z) - \lambda \right)
\]

### 4.6 COST MODEL

Cost analysis is the most important phenomenon in any practical situation at every stage. Cost involves startup cost, operating cost, holding cost and reward cost. It is quite natural that the management of the system desires to minimize the total average cost and to optimize the cost. Addressing this, in this section, the cost model for the proposed queueing system is developed and the total average cost is obtained with the following assumptions:

- \( C_s \): Startup cost per cycle
- \( C_h \): Holding cost per customer per unit time
- \( C_o \): Operating cost per unit time
- \( C_r \): Reward cost per cycle due to vacation
Since the length of the cycle is the sum of the idle period and busy period, from the Equations (4.40) and (4.41), the expected length of cycle, $E(T_c)$ is obtained as

$$E(T_c) = E(\text{length of the Idle Period}) + E(\text{length of the Busy Period})$$

$$= \frac{E(V)}{1 - \sum_{n=0}^{a-1} \sum_{i=0}^{n} \omega_i p_{n-i}} + \frac{E(S)}{\sum_{i=0}^{a-1} p_i}$$

Now, the total average cost (TAC) per unit time is obtained as,

Total average cost = Start-up cost per cycle + holding cost of number of customers in the queue per unit time + Operating cost per unit time * $\rho$ – reward cost due to vacation per cycle.

$$\text{TAC} = \left[ C_s - C_i E(I) \right] \frac{1}{E(T_c)} + C_h E(Q) + C_o \rho$$ \hspace{1cm} (4.44)

It is difficult to have a direct analytical result for the optimal value $a^*$ (minimum batch size in $M^X/G(a,b)/1$ queueing system) to minimize the total average cost. The simple direct search method to find optimal policy for a threshold value $a^*$ to minimize the total average cost, is defined.

**Step 1:** Fix the value of maximum batch size ‘b’

**Step 2:** Select the value of ‘a’ which will satisfy the following relation

$$\text{TAC}(a^*) \leq \text{TAC}(a), \hspace{0.5cm} 1 \leq a \leq b$$

**Step 3:** The value $a^*$ is optimum, since it gives minimum total average cost.

Using the above procedure, the optimal value of ‘a’ can be obtained, which minimizes the total average cost function. Some numerical example to illustrate the above procedure is presented in the next section.
4.7 NUMERICAL ILLUSTRATION

In this section, the consistency of the theoretical results obtained in the sections 4.3 – 4.4 are justified numerically with the following assumptions and notations:

- Service time distribution is 2- Erlang with parameter $\mu$
- Batch size distribution of the arrival is geometric with mean 2
- Vacation time is exponential with parameter $\gamma$
- Minimum service capacity $a$
- Maximum service capacity $b$
- Probability of arriving batch will be allowed to join the system during the busy period $\alpha$
- Probability of arriving batch will be allowed to join the system during the vacation period $\beta$

4.7.1 Effects of Various Parameters on the Performance Measures

The effects of various parameters such as arrival rates, expected queue length, expected idle period, expected busy period, probability that the server is on vacation, probability that the server is busy, different probabilities of admitting arrivals to join the system during busy period, during the vacation period and threshold value ‘a’ are analyzed numerically and presented in Tables 4.1 – 4.4. All numerical results are obtained using Mat Lab software.

The effects of various performance measures for a fixed ‘a’ and ‘b’ with respect to different probabilities of admitting arrivals to join the system during busy period are obtained numerically. These results are tabulated in Table 4.1. It is observed that, if the probability of admitting customers during busy period increases, then
• the expected queue length, the expected busy period and the probability that the server is busy increase

• the probability that the server is on vacation and the expected idle period decrease.

In Table 4.2, for different arrival rates, the effects of various performance measures for a fixed ‘a’ and ‘b’ are presented. From the table, it is clear that, if the arrival rate increases, then

• the mean queue size, the mean busy period and the probability that the server is busy increase

• the mean idle period and the probability that the server is on vacation decrease.

The effects of threshold value ‘a’ on the expected queue length for various probabilities of admitting customers during the busy period are obtained numerically, and these results are tabulated in Table 4.3. From the table, the following observations are made:

• For a fixed threshold value ‘a’, when the probability of allowing customers during busy period increases, the expected queue length increases.

• For a fixed probability ($\alpha$) of allowing customers during busy period, when threshold value increases, the expected queue length increases.

The effects of threshold value ‘a’ on the expected queue length for various probabilities of admitting customers during the vacation period are obtained numerically, and these results are tabulated in Table 4.4. From the table, the following observations are made.
• For a fixed threshold value ‘a’, when the probability of allowing customers during vacation period increases, the expected queue length increases.

• For a fixed probability (β) of allowing customers during vacation period, when threshold value increases, the expected queue length increases.

### 4.7.2 Optimal Cost

In this section, a numerical example is analyzed to illustrate how the management of an electroplating processing system can effectively use the results obtained in the sections 4.3, 4.4 and 4.6 to make the decision regarding the threshold value to minimize the total average cost.

It is assumed that, the maximum capacity of an electroplating process is 12 units (i.e. \( b = 12 \) pieces). If the management of an electroplating process allows the operator to start the process even for a single piece (i.e. \( a = 1 \)) without waiting for further arrival, clearly, the operating cost will increase. On the other hand, if they start the process until all 12 pieces arrive, the holding cost may increase; hence, there must be some value between 1 and 12 that will optimize the cost. An optimal policy regarding the threshold value ‘a’ which will minimize the total average cost is wished to be obtained.

The total average costs are obtained numerically with the following assumptions:

- **Startup cost**: 4.00
- **Holding cost per customer**: 0.25
- **Operating cost per unit time**: 7.00
- **Reward cost per unit time due to vacation**: 1.00
The effects of the threshold value ‘a’ on the total average cost with b = 12 are reported in Tables 4.3 and 4.4, and represented in Figures 4.1, 4.2 and 4.3.

From the Table 4.3 and the Figure 4.2, it is clear that, for an electroplating process center with the capacity of 12 pieces (i.e. b = 12) at a time, the **management has to fix the threshold value a = 5 to minimize the total average cost** for the probability of admitting pieces during the busy period 0.2 (i.e. $\alpha = 0.2$) and the probability of admitting pieces during the vacation period 0.6 (i.e. $\beta = 0.6$). From the Table 4.4 and the Figures 4.1 and 4.3, the **management has to fix the threshold value a = 4 to minimize the total average cost** for the probability of admitting pieces during the busy period 0.7 (i.e. $\alpha = 0.7$) and the probability of admitting pieces during the vacation period 0.2 (i.e. $\beta = 0.2$).

Similarly, the management has to fix the threshold value ‘a’ to minimize the total average cost for various probabilities of admitting pieces during the vacation and non-vacation periods.

**4.8 CONCLUSION**

In this chapter, “a bulk queueing system with multiple vacations under restricted admissibility policy of arriving batches” is analyzed. The probability generating function for the queue size at an arbitrary time epoch is derived. Various performance measures are also obtained. Some particular cases and special cases are also discussed. The theoretical development of the model is justified with numerical results. The results so obtained in this chapter can be used for managerial decision to optimize the overall cost and search for the best operating policy in a waiting line system.
Table 4.1 Probability of Admitting Customers during Busy Period
(Vs) Performance Measures
(For $\lambda=3.5; \mu=2.0; a = 3; b = 4; \gamma =15; \beta =0.9$)

<table>
<thead>
<tr>
<th>$\alpha$</th>
<th>$P(B)$</th>
<th>$P(V)$</th>
<th>$E(Q)$</th>
<th>$E(B)$</th>
<th>$E(I)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.2</td>
<td>0.5201</td>
<td>0.4799</td>
<td>1.4585</td>
<td>0.6380</td>
<td>0.2336</td>
</tr>
<tr>
<td>0.4</td>
<td>0.5833</td>
<td>0.4167</td>
<td>2.0167</td>
<td>0.7082</td>
<td>0.1845</td>
</tr>
<tr>
<td>0.6</td>
<td>0.6608</td>
<td>0.3392</td>
<td>3.0514</td>
<td>0.8455</td>
<td>0.1423</td>
</tr>
<tr>
<td>0.8</td>
<td>0.7582</td>
<td>0.2418</td>
<td>5.5017</td>
<td>1.1587</td>
<td>0.1086</td>
</tr>
<tr>
<td>1.0</td>
<td>0.8843</td>
<td>0.1157</td>
<td>15.6384</td>
<td>2.3754</td>
<td>0.0820</td>
</tr>
</tbody>
</table>

$P(V)$ - Probability that the server is on multiple vacations; $P(B)$ - Probability that the server is busy; $E(Q)$ – Expected queue length; $E(I)$ – Expected idle period; $E(B)$ – Expected busy period

Table 4.2 Arrival Rate (Vs) Performance Measures
(For $\mu=2.0; a = 3; b = 4; \gamma =15; \alpha =0.8; \beta =0.7$)

<table>
<thead>
<tr>
<th>$\lambda$</th>
<th>$P(B)$</th>
<th>$P(V)$</th>
<th>$E(Q)$</th>
<th>$E(B)$</th>
<th>$E(I)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.5</td>
<td>0.5098</td>
<td>0.4902</td>
<td>2.3981</td>
<td>1.0200</td>
<td>0.1241</td>
</tr>
<tr>
<td>3.0</td>
<td>0.6109</td>
<td>0.3891</td>
<td>3.4015</td>
<td>1.0548</td>
<td>0.1192</td>
</tr>
<tr>
<td>3.5</td>
<td>0.7107</td>
<td>0.2893</td>
<td>5.1706</td>
<td>1.2186</td>
<td>0.1069</td>
</tr>
<tr>
<td>4.0</td>
<td>0.8090</td>
<td>0.1910</td>
<td>8.8425</td>
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<td>0.0945</td>
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</tbody>
</table>

$P(V)$ - Probability that the server is on multiple vacations; $P(B)$ - Probability that the server is busy; $E(Q)$ – Expected queue length; $E(I)$ – Expected idle period; $E(B)$ – Expected busy period
Table 4.3  Threshold Value (Vs) Expected Queue Length and Total Average Cost for different probabilities (\(\alpha\)) of allowing customers during busy period
(For \(\lambda = 1.0, \mu = 2.5, b = 12, \gamma = 2, \beta = 0.6\))

<table>
<thead>
<tr>
<th>(\alpha)</th>
<th>E(Q)</th>
<th>TAC</th>
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<td>(\alpha = 0.2)</td>
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<td>(\alpha = 0.6)</td>
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<td>(\alpha = 1.0)</td>
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</table>

Table 4.4  Threshold Value (Vs) Expected Queue Length and Total Average Cost for different probabilities (\(\beta\)) of allowing customers during vacation
(For \(\lambda = 1.0, \mu = 2.5, b = 12, \gamma = 2, \alpha = 0.7\))

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Figure 4.1 Threshold Value (Vs) Total Average Cost for $\alpha = 0.7$, $\beta = 0.2$

(For $\lambda = 1.0$, $\mu = 2.5$, $b = 12$, $\gamma = 2$)
Figure 4.2 Threshold Value (Vs) Total Average Cost for different probabilities ($\alpha$) of allowing customers during busy period
(For $\lambda=1.0$, $\mu=2.5$, $b=12$, $\gamma=2$, $\beta=0.6$)

Figure 4.3 Threshold Value (Vs) Total Average Cost for different probabilities ($\beta$) of allowing customers during vacation
(For $\lambda=1.0$, $\mu=2.5$, $b=12$, $\gamma=2$, $\alpha=0.7$)