CHAPTER 6
ANALYSIS OF A BATCH ARRIVAL SINGLE SERVICE QUEUEING SYSTEM WITH UNRELIABLE SERVER AND SINGLE VACATION

6.1 INTRODUCTION


This chapter concentrated on the steady state analysis of a $M^X/G/1$ queueing system with unreliable server and single vacation. Customers arrive according to the compound Poisson process with random arrival size. Arrival rate varies according to the server’s status: busy or broken down. Single service is considered. The server starts the service if one or more customers waiting in the queue and continues the service until the system becomes empty. At a service completion epoch, if there are no customers waiting in the queue, the server goes for a vacation. When he returns from a vacation, if there are one or more customers waiting, he serves until the system becomes empty; otherwise, he stays in the system waiting (dormant period) for the first one to arrive. The service is interrupted if unpredicted break down occurs, and the server is immediately repaired. When the repair is completed, the server immediately resumes service. During the service, if breakdown occurs, the customer who is under service has to remain in the system until the repair process is completed. When the repair is over, the server continues the service. Breakdown times are exponentially distributed and the repair times follow general distribution. The model is studied by the embedded Markov chain technique and level crossing analysis. The model under study is schematically represented in the Figure 6.1.

Figure 6.1 Schematic Representation of the Queueing Model
($Q$ - Queue length)
The motivation for the model comes from a practical situation that exists in a Globe Valve manufacturing industry, after turning operation the components arrive from job shop in batches to CNC turning center for facing and turning processes. The operator of CNC turning center starts the processes immediately. After processing these components, if no components arrive, the operator will start other work (vacation state) such as arranging the tooling, writing the coding, removing the chips, changing the coolant, etc. When the operator returns from other work and finds one or more components waiting for service in the queue, he immediately begins to serve them until the system becomes empty. On the other hand, if the operator finds an empty system at the end of the vacation, he remains idle until a component arrives. The service of components may be interrupted when operator encounters unpredicted breakdowns such as accident event, blunt tool, troubles in coolant, etc. The above situation can be modeled as a \( \text{M}^X/\text{G}/1 \) queueing system with unreliable server and single vacation.

For the proposed model, probability generating functions (PGF) of the number of customers in the system at the completion epoch of the idle period or at an initiation epoch of the busy period, at the departure point epoch are obtained using embedded Markov chain technique and PGF of the number of customers in the system at an arbitrary time using level crossing analysis is obtained. Various performance measures are derived. Particular cases and some special cases are discussed. A cost model for the queueing system is developed and a numerical illustration is provided.

6.2 MATHEMATICAL MODEL

Let \( X \) be the group size random variable of the arrival, \( \lambda \) be the Poisson arrival rate when the server is busy (server is up) and \( \lambda_0 \) be the Poisson arrival rate when the server is broken down (server is down), \( g_k \) be
the probability that ‘k’ customers arrive in a batch. The failure time is exponentially distributed with parameter $\gamma$. When service interruptions occur (breakdowns), it is emergently recovered with a random time. Let $S(x)$ ($s(x)$) ($\tilde{S}(0)$) be the cumulative distribution function (probability density function) {Laplace-Stieltjes transform} of service time. Let $G(x)$ ($g(x)$) ($\tilde{G}(0)$) be the cumulative distribution function (probability density function) {Laplace-Stieltjes transform} of repair time. Let $V(x)$ ($v(x)$) ($\tilde{V}(0)$) be the cumulative distribution function (probability density function) {Laplace-Stieltjes transform} of vacation time. $N(t)$ denotes the number of customers in the system at time $t$.

6.2.1 System Size Distribution at a Busy Period Initiation Epoch

In this section, the steady state queue size distribution at a busy period initiation epoch is developed. $\alpha_n$ ($n \geq 1$) is defined as the steady state probability that an arbitrary (tagged) customer finds ‘n’ customers in the system at the busy period initiation epoch. If $T_l$ ($l = 0, 1, 2, \ldots$) is the initiation epoch of the busy period and $N(T_l)$ is the number of customers in the system at the time instant $T_l$, then $\alpha_n = \lim_{l \to \infty} \text{Prob}[N(T_l) = n], \ n \geq 1$.

Conditioning on the number of customers which arrive during the first vacation after a busy period, the following state equation is obtained.

$$\alpha_n = \sum_{k=1}^{n} a_k g_n^{(k)} + a_0 g_n, \ n \geq 1, \quad (6.1)$$

where

$$g_k = \text{Prob}(X = k); \ k = 1, 2, 3, \ldots.$$
\( g^{(k)}_j = \text{Prob}(Y_k = j) \) is the \( k \)-fold convolution of \( \{g_j\} \) with itself

and \( g^{(0)}_j = 1 \), \( Y_k = X_1 + X_2 + \ldots + X_k \),

\[
a_k = \text{P('k' individual units arrive during a vacation time)} = \int_0^\infty e^{-\lambda t} \left( \frac{\lambda t}{k!} \right) dV(t)
\]

The probability generating function \( \alpha_n \) is defined as \( \alpha(z) = \sum_{n=1}^{\infty} \alpha_n z^n \).

Multiplying (6.1) by appropriate powers of \( z \) and then taking summation over all possible values of ‘\( n \)’, we get

\[
\alpha(z) = \tilde{V}(\lambda - \lambda X(z)) + \tilde{V}(\lambda)[X(z) - 1] \quad (6.2)
\]

which represents the PGF of the number of customers in the system at the completion epoch of the idle period or at the initiation epoch of the busy period

\textbf{6.2.2 System Size Distribution at a Departure Epoch}

In this section, the probability generating function of the departure point system size distribution is derived. \( \pi_j \) (\( j = 0, 1, 2, 3 \ldots \)) is defined as the steady-state probability that ‘\( j \)’ customers are left in the system at a departure epoch of a customer. The probability \( \pi_j \) is obtained by embedded Markov chain technique. Here \( \{ \pi_j ; j = 0, 1, 2, 3 \ldots \} \) forms a Markov chain with transition probability matrix,
Also \( \pi_j, j = 0, 1, 2, 3 \ldots \) satisfies the following steady-state equation:

\[
\pi_j = \pi_0 \sum_{k=1}^{j+1} \alpha_k r_{j-k+1} + \sum_{k=1}^{j+1} \pi_k r_{j-k+1}; j \geq 0 \quad \text{with} \quad \sum_{j=0}^{\infty} \pi_j = 1 \quad (6.3)
\]

where \( r_i \) is the probability of ‘\( i \)’ customers arrive during the period starting with the initiation of a service of a customer and ending with the completion of its service, includes break down, if any.

 Conditioning on the actual service length of this customer and the number of breakdowns during this service, we get

\[
r_i = \sum_{m=0}^{\infty} \sum_{k=0}^{\infty} \left( \sum_{i=0}^{j} \frac{e^{-\lambda t} (\lambda t)^k}{k!} \frac{e^{-\lambda_0 y} (\lambda_0 y)^{m-k}}{(m-k)!} g_i^{(m)} \right) \frac{e^{-\gamma t} (\gamma t)^j}{j!} dG(j)(y) dS(t) \quad (6.4)
\]

Now, the probability generating function of \( r_i, i = 0, 1, 2, 3, \ldots \) is obtained as

\[
r(z) = \sum_{i=0}^{\infty} r_i z^i = \sum_{i=0}^{\infty} \gamma + \lambda (1 - X(z)) - \gamma G(\lambda_0 (1 - X(z))) \quad (6.5)
\]

Let \( \Pi(z) \) be the PGF of \( \pi_j, j > 0 \), then

\[
\Pi(z) = \sum_{j=0}^{\infty} \pi_j z^j = \frac{\pi_0 r(z)(1 - \alpha(z))}{r(z) - z} \quad (6.7)
\]
Now using (6.5) and (6.2) in (6.7) and solving this equation, we get finally

\[
\Pi(z) = \frac{\pi_0 \left(1 - \tilde{V}(\lambda - \lambda X(z)) - \tilde{V}(\lambda) [X(z) - 1] \right) \bar{S} \left(\gamma + \lambda (1 - X(z)) - \gamma \tilde{G}(\lambda_0 (1 - X(z))) \right)}{\bar{S} \left(\gamma + \lambda (1 - X(z)) - \gamma \tilde{G}(\lambda_0 (1 - X(z))) \right) - z}
\]  

(6.8)

Now, since \(\sum_{j=0}^{\infty} \pi_j = 1\), by taking the limit of \(\Pi(z)\) as \(z \to 1\) is unity, we get

\[
\pi_0 = \frac{1 - E(X) E(S) [\lambda + \gamma \lambda_0 E(G)]}{\lambda E(X) E(V) + \tilde{V}(\lambda) E(X)}
\]

\[
\pi_0 = \frac{1 - \rho}{\lambda E(X) E(V) + \tilde{V}(\lambda) E(X)}, \text{ where } \rho = E(X) E(S) [\lambda + \gamma \lambda_0 E(G)]
\]

and \(\rho < 1\) is the condition to be satisfied for the existence of steady state for the model under consideration.

Substituting \(\pi_0\) in (6.8), the PGF of departure point system size is obtained as

\[
\Pi(z) = \frac{\left(1 - \rho\right) \left(1 - \tilde{V}(\lambda - \lambda X(z)) - \tilde{V}(\lambda) [X(z) - 1] \right) \bar{S} \left(\gamma + \lambda (1 - X(z)) - \gamma \tilde{G}(\lambda_0 (1 - X(z))) \right)}{\bar{S} \left(\gamma + \lambda (1 - X(z)) - \gamma \tilde{G}(\lambda_0 (1 - X(z))) \right) - z \left[\lambda E(X) E(V) + \tilde{V}(\lambda) E(X)\right]}
\]

(6.9)

**6.2.3 System Size Distribution at an Arbitrary Time**

Let \(p_n\) be the probability distribution of ‘n’ customers in the system at an arbitrary time epoch. Let \(q_n^0\) and \(q_n^0\) be the steady state
probabilities that there are ‘n’ customers in the system at an arbitrary time
epoch when the server is up and down, respectively. Then \( p_n = q_n + q_n^0 \)
and \( P(z) = q(z) + q^0(z) \), \( |z| \leq 1 \), where \( P(z) \), \( q(z) \) and \( q^0(z) \)
are the probability generating functions of \( p_n \), \( q_n \) and \( q_n^0 \), respectively. Instead of
relating \( p_n \) to \( \pi_n \), we relate \( q_n \) and \( q_n^0 \) to \( \pi_n \). This is achieved by relating the
rates of up and down crossings of the process \( \{M(t), t \geq 0\} \), where \( M(t) \) is the
number customers in the system at time \( t \). Let \( \tilde{\lambda} \) be the effective arrival rate
in the steady state.

The rate of down crossings to level ‘n’ is given by (Shanthikumar
and Chandra (1982)),

\[
r_d(n) = \lambda \sum_{i=0}^{n} g_{i+1} \pi_{n-i}, \quad n \geq 0
\]

(6.10)

The rates of up crossings over level \( n \) are

\[
r_u(0) = \lambda g_1 q_0, \quad n = 0
\]

\[
r_u(n) = \lambda \sum_{i=0}^{n} g_{i+1} q_{n-i} + \sum_{i=1}^{n} q_i \left(1 - \sum_{j=0}^{n-i} b_j\right), \quad n \geq 1
\]

(6.11)

where

\[
b_i = \int_0^\infty \sum_{m=0}^{\infty} \frac{(-\lambda t)^m e^{-\lambda t}}{m!} g_i^{(m)} dG(t), \quad i \geq 0
\]

is the probability that ‘\( i \)’ customers arrive during a down time of the server.

Taking the PGF of (6.10) and (6.11) and equating them, we get

\[
\tilde{\lambda} X(z) \prod(z) = \left(\frac{\lambda X(z) q(z)}{z} + \gamma q(z) \left(1 - b(z)\right)\right) - \gamma q_0 \left(1 - \frac{b(z)}{1 - z}\right)
\]

(6.12)
where \( b(z) = \tilde{G}(\tilde{\lambda}_0(1-X(z))) \) is the PGF of \( b_i, i \geq 0 \).

Solving (6.12) for \( q(z) \), we get

\[
q(z) = \frac{\tilde{\lambda}_0 X(z)(1-z)\Pi(z) + q_0 z\gamma(1-b(z))}{\lambda(1-z)X(z) + z\gamma(1-b(z))}, \quad |z| \leq 1 \tag{6.13}
\]

Substituting \( n = 0 \) in (6.10) and (6.11), and equating them,

we get \( q_0 = \frac{\tilde{\lambda}_0 \pi_0}{\tilde{\lambda}} \tag{6.14} \)

Alternate equation for rate of up crossings over level \( n \) is

\[
r_u(n) = \lambda \sum_{i=0}^{n} g_{i+1} q_{n-i} + \lambda_0 \sum_{i=0}^{n} g_{i+1} q_{0}^{n-i} \tag{6.15}
\]

Taking the PGF of (6.10) and (6.15) and equating them, we get

\[
\tilde{\lambda} q(z) + \lambda_0 q^0(z) = \tilde{\lambda} \Pi(z) \tag{6.16}
\]

Therefore \( q^0(z) = \frac{\tilde{\lambda} \Pi(z) - \lambda q(z)}{\lambda_0} \tag{6.17} \)

Using Equation (6.13) in (6.17), we get

\[
q^0(z) = \frac{\tilde{\lambda}_0 \gamma z(1-b(z))\Pi(z) - \lambda q_0 z\gamma(1-b(z))}{\lambda_0 (\lambda(1-z)X(z) + z\gamma(1-b(z)))}, \quad |z| \leq 1 \tag{6.18}
\]

The PGF of \( p_n, n = 0,1,2,3, \ldots \) is defined as \( P(z) = \sum_{n=0}^{\infty} p_n z^n \).
Using Equations (6.13) and (6.18) in \( P(z) = q(z) + q^0(z) \), after some algebra we get

\[
P(z) = \frac{\hat{\lambda}}{\lambda_0} \cdot \frac{\lambda_0 X(z)(1-z) + z\gamma(1-b(z))}{\lambda(1-z) X(z) + z\gamma(1-b(z))} \]

(6.19)

\[
\frac{\left(\lambda_0 - \lambda \right)q_0 z\gamma(1-b(z))}{\lambda_0 (\lambda(1-z) X(z) + z\gamma(1-b(z)))}, \quad |z| \leq 1
\]

Using \( \lim_{z \to 1} P(z) = 1 \) and solving for \( \hat{\lambda} \), we get

\[
\hat{\lambda} = \frac{\lambda (\lambda + \gamma \lambda_0 E(G) E(X))}{\lambda (1 + \gamma E(G) E(X)) - (\lambda - \lambda_0) \pi_0^G \gamma E(G) E(X)}
\]

(6.20)

Thus, (6.19) gives the probability generating function of the number of customers in the system at an arbitrary time.

### 6.3 PERFORMANCE MEASURES

In this section, some useful performance measures of the proposed model like, expected number of customers in the system at a busy period initiation epoch \( L_{BI} \), expected system size at a departure epoch \( L_D \), expected system size at an arbitrary time epoch \( L_A \), expected length of idle period \( E(I) \) and expected length of busy period \( E(B) \) are derived which are useful to find the total average cost of the system.

#### 6.3.1 Expected System Size at a Busy Period Initiation Epoch

The expected system size \( L_{BI} \) (i.e. mean number of customers in the system) at a busy period initiation epoch, is obtained by differentiating \( \alpha(z) \) at \( z = 1 \) and is given by
\[ \lim_{z \to 1} \alpha(z) = L_{BI} \]

\[ L_{BI} = \lambda E(X)E(V) + \tilde{V}(\lambda)E(X) \]  \hspace{1cm} (6.21)

Let \( E(\alpha) \) and \( E[\alpha(\alpha - 1)] \) be the first two factorial moments of the distribution of the system size at a busy period initiation epoch, then

\[ E(\alpha) = \left( \frac{d\alpha(z)}{dz} \right)_{z=1} = \lambda E(X)E(V) + \tilde{V}(\lambda)E(X) \]

\[ E(\alpha(\alpha - 1)) = \left( \frac{d^2\alpha(z)}{dz^2} \right)_{z=1} = (\lambda E(X))^2 E(V^2) + E(X(X-1)) \left( \lambda E(V) + \tilde{V}(\lambda) \right) \]

Thus the variance of the busy period initiation queue size distribution is given by

\[ \text{Var}(\alpha) = E(\alpha(\alpha - 1)) + E(\alpha) - (E(\alpha))^2 \]

\[ \text{Var}(\alpha) = (\lambda E(X))^2 E(V^2) + E(X(X-1)) \left( \lambda E(V) + \tilde{V}(\lambda) \right) \]

Thus the variance of the busy period initiation queue size distribution is given by \( \text{Var}(\alpha) = E(\alpha(\alpha - 1)) + E(\alpha) - (E(\alpha))^2 \)

6.3.2 Expected System Size at a Departure Epoch

The expected number of arrivals during the service period of a customer is given by

\[ E(r) = \lim_{z \to 1} r(z) \]

Using (6.5), we have,

\[ E(r) = E(X)E(S)[\lambda + \gamma \lambda_0 E(G)] \]

The expected system size \( L_D \) (i.e. mean number of customers waiting in the system) at a departure epoch, is obtained by differentiating \( \Pi(z) \) at \( z = 1 \) and is given by
\[
\lim_{z \to 1} P(z) = L_D
\]

\[
L_D = \frac{1}{2} \left( \frac{\alpha^2}{\alpha} - 1 \right) + \rho
\]

\[
+ \left( \frac{E(S) X^*(1) (\lambda + \gamma \lambda_0 E(G)) + \gamma \lambda_0^2 E(G^2) X^2 (X) E(S) + E(S^2) E^2 (X) \left[ \lambda + \gamma \lambda_0 E(G) \right]^2}{2(1 - \rho)} \right)
\]

(6.22)

where \( \alpha^2 = (\lambda E(X))^2 E(V^2) + E(X^2) \lambda E(V) + \tilde{V}(\lambda) E(X^2) \)

### 6.3.3 Expected System Size at an Arbitrary Time

The expected system size \( L_A \) (i.e. mean number of customers waiting in the system) at an arbitrary time epoch, is obtained by differentiating \( P(z) \) at \( z = 1 \) and is given by

\[
\lim_{z \to 1} P(z) = L_A
\]

\[
L_A = \frac{\tilde{\lambda} (1 + \gamma E(G)E(X))}{\lambda + \gamma \lambda_0 E(G)E(X)} L_D
\]

\[
+ \frac{\tilde{\lambda} (\lambda - \lambda_0)}{\lambda + \gamma \lambda_0 E(G)E(X)} \left[ 2E(X)E(G) + E(G) X^*(1) + E^2 (X) \left[ \lambda_0 E(G^2) - 2E(G) \right] \right]
\]

\[
+ \frac{(\lambda_0 - \lambda) \gamma q_0 E(G) E(X)}{\lambda + \gamma \lambda_0 E(G)E(X)}
\]

\[
+ \frac{(\lambda_0 - \lambda) \gamma q_0 \left[ (\lambda + \gamma \lambda_0 E(G)E(X)) V_1 - E(G)E(X) V_2 \right]}{2(\lambda + \gamma \lambda_0 E(G)E(X))^2}
\]

(6.23)
where $L_D$ is given by the Equation (6.22), $q_0$ is given by the Equation (6.14),

$V_1 = \lambda_0 E^2(X) E(G^2) + E(G) X^*(1)$ and

$V_2 = 2(\lambda + \gamma \lambda_0 E(G)) E(X) + \gamma \lambda_0^2 E^2(X) E(G^2) + \gamma \lambda_0 E(G) X^*(1)$

### 6.3.4 Expected Length of Idle Period

Let $I$ be the idle period random variable. Another random variable $J$ is defined as,

- $J = 0$, if the server finds at least one customer in the queue at the end of a vacation
- $J = 1$, if the server finds no customer in the queue at the end of a vacation

Let $D$ be the random variable “Dormant Period” and $E(D)$ be the expected length of dormant period.

Now,

- $E(I) = E(I|J=0)P(J=0) + E(I|J=1)P(J=1)$
- $= E(V) P(J=0) + (E(V) + E(D)) P(J=1)$

Thus, the expected length of idle period is obtained as

$$E(I) = E(V) + \frac{1}{\lambda} \tilde{V}(\lambda) \quad (6.24)$$

### 6.3.5 Expected Length of Busy Period

Let $B$ be the busy period random variable. Another random variable $U$ is defined as
\[ U = \begin{cases} 
0, & \text{if the server finds no customer in the queue at a service completion epoch} \\
1, & \text{if the server finds at least one customer in the queue at a service completion epoch} 
\end{cases} \]

Let \( S_r \) be the time taken to serve a customer including the repair times if any.

Hence, \( E(S_r) = E(S)(1 + \gamma E(G)) \)

Now,
\[
E(B) = E(B/U=0)P(U=0) + E(B/U=1)P(U=1)
= E(S_r)P(U=0) + (E(S_r) + E(B)) P(U=1)
= E(S_r) \pi_0 + (E(S_r) + E(B)) (1- \pi_0 )
\]
solving for \( E(B) \), the expected length of busy period is obtained as

\[
E(B) = \frac{E(S)(1+\gamma E(G))E(X)\left(\lambda E(V) + \ddot{V} (\lambda)\right)}{1-\rho} \quad \text{(6.25)}
\]

### 6.4 PARTICULAR CASES

In this section, some of the existing models are deduced as a particular case of the proposed model.

**Case (i):** If we consider a model without server breakdowns (i.e., \( \gamma = 0 \)), then (6.9) becomes

\[
\Pi(z) = \frac{(1-\rho) \left(1 - \ddot{V} (\lambda - \lambda X(z)) - \ddot{V} (\lambda)[X(z)-1] \right) \ddot{S} (\lambda (1 - X(z)))}{\ddot{S} [\lambda (1 - X(z))] - z \left(\lambda E(X)E(V) + \ddot{V} (\lambda) E(X)\right)} \quad \text{and} \quad \text{(6.25)}
\]

becomes

\[
E(B) = \frac{E(S)E(X)[\lambda E(V) + \ddot{V} (\lambda)]}{1-\rho}, \quad \text{where} \quad \rho = \lambda E(S)E(X), \quad \text{which agree with the result obtained by Choudhury (2002).}
\]
Case (ii): If we consider a model without vacation and break down (i.e., $Pr[V = 0] = 1$ and $\gamma = 0$), our model is reduced to the ordinary $MX / G / 1$ queueing system. Therefore (6.22) becomes,

$$L_D = \frac{1}{2} \left( \frac{E(X^2)}{E(X)} - 1 \right) + \rho + \left( \frac{E(S)X^* (1) \lambda + E(S)E^2 (X) \lambda^2}{2(1-\rho)} \right),$$

where $\rho = \lambda E(S) E(X)$. In this case, the result coincides with those of Takagi’s system (1991).

6.4.1 Special Cases

The model so developed is general in nature as the service time, repair time and vacation time are arbitrary. But for practical purposes, service time, repair time and vacation time with particular distribution is required. In this section, some special cases of the proposed model by specifying vacation time random variable as exponential distribution and repair time random variable as Erlangian distribution are discussed.

Case (i): Single server batch arrival queue with unreliable server and exponential vacation time

If the vacation time is assumed as exponential with probability density function $\nu(x) = \omega e^{-\omega x}$, where $\omega$ is the parameter, then,

$$\tilde{V}(\lambda - \lambda X(z)) = \left( \frac{\omega}{\omega + \lambda (1 - X(z))} \right).$$

Substituting this expression for $\tilde{V}(\lambda - \lambda X(z))$ in (6.2) and after some algebra, the PGF of the system size distribution at the initiation epoch of the busy period of this special case of the queueing model is obtained as,

$$\alpha(z) = \left( \frac{\omega}{\omega + \lambda (1 - X(z))} \right) + \left( \frac{\omega}{\omega + \lambda} \right) (X(z) - 1)$$
Case (ii): Single server batch arrival queue with unreliable server, single vacation and \( k \)-Erlangian repair time

In case of \( k \)-Erlang repair time random variable with probability density function

\[
g(x) = \frac{(ku)^k x^{k-1} e^{-kux}}{(k-1)!}, \quad k > 0; \text{ where } u \text{ is the parameter,}
\]

then,

\[
\tilde{G}(\lambda_0(1-X(z))) \left( \frac{\beta k}{\beta k + \lambda_0(1-X(z))} \right)^k.
\]

Substituting this expression for \( \tilde{G}(\lambda_0(1-X(z))) \) in (6.9) and after some algebra, the PGF of the system size distribution at the departure point of this special case of the queueing model is obtained as,

\[
\Pi(z) = \frac{(1-p) \left( 1 - \tilde{V}(\lambda - \lambda X(z)) - \tilde{V}(\lambda[X(z)-1]) \right) S \left( \gamma + \lambda(1-X(z)) - \gamma \left( \frac{\beta k}{\beta k + \lambda_0(1-X(z))} \right) \right) - z \left( \lambda E(X) E(V) + \tilde{V}(\lambda) E(X) \right) }{S \left( \gamma + \lambda(1-X(z)) - \gamma \left( \frac{\beta k}{\beta k + \lambda_0(1-X(z))} \right) \right)}.
\]

6.5Cost Model

Cost analysis is the most important phenomenon in any practical situation at every stage. Cost involves startup cost, operating cost, holding cost and reward cost. It is quite natural that the management of the system desires to minimize the total average cost. Addressing this, in this section, the cost model for the proposed queueing system is developed and the total average cost is obtained with the following assumptions:

- \( C_S \): Startup cost per cycle
- \( C_h \): Holding cost per customer per unit time
- \( C_o \): Operating cost per unit time
- \( C_r \): Reward cost per cycle due to vacation
Since the length of the cycle is the sum of the idle and busy periods from the Equations (6.24) and (6.25), the expected length of cycle, $E(T_C)$ is obtained as $E(T_C) = E(I) + E(B)$

Now, the total average cost (TAC) per unit time is obtained as

\[
\text{Total average cost} = \text{Start-up cost per cycle} + \text{Holding cost of number of customers in the queue per unit time} + \text{Operating cost} \cdot \rho - \text{Reward cost due to vacation per cycle}
\]

\[
TAC = C_s \frac{1}{E(T_C)} + C_h L_A + C_o \rho - C_r \frac{E(I)}{E(T_C)}
\] (6.26)

6.6 NUMERICAL ILLUSTRATION

In this section, the consistency of the theoretical results obtained in the sections 6.2 – 6.3 are justified numerically with the following assumptions and notations:

- Service time distribution is 2 – Erlang with parameter $\mu$
- Batch size distribution of the arrival is geometric with mean 2
- Vacation time is exponential with parameter $\omega$
- Failure time is exponential with parameter $\gamma$
- Repair time is exponential with parameter $\beta$

6.6.1 Effects of Various Parameters on the Performance Measures

The effects of various parameters such as arrival rates, service rates, failure rates, expected idle period, expected busy period and expected system size at different time epochs are analyzed numerically and presented in
Tables 6.1 - 6.4 and represented in Figure 6.2. All numerical results are obtained using Mat Lab software.

The effects of different arrival rates for a fixed failure rate on the expected system size at a busy period initiation epoch, at a departure epoch and at an arbitrary time epoch with respect to different service rates are obtained numerically. These results are tabulated in Table 6.1 and represented in Figure 6.2. From the table and the figure, it is clear that,

- when arrival rate increases, the expected system size at a busy period initiation epoch, at a departure epoch and at an arbitrary time epoch increases
- when service rate increases, the expected system size at a departure epoch and at an arbitrary time epoch decreases whereas the expected system size at a busy period initiation epoch doesn’t change

The effects of different failure rates for a fixed service rate on the expected system size at a busy period initiation epoch, at a departure epoch and at an arbitrary time epoch with respect to different arrival rates are obtained numerically. These results are tabulated in Table 6.2. From the table, the following observations are made:

- As failure rate increases, the expected system size at a departure epoch and at an arbitrary time epoch increases whereas the expected system size at a busy period initiation epoch doesn’t change
- As arrival rate increases, the expected system size at a busy period initiation epoch, at a departure epoch and at an arbitrary time epoch increases
The effects of different service rates for a fixed arrival rate on the expected system size at a busy period initiation epoch, at a departure epoch and at an arbitrary time epoch with respect to different failure rates are obtained numerically. These results are tabulated in Table 6.3. From the table, the following points are observed:

- As service rate increases, the expected system size at a departure epoch and at an arbitrary time epoch decreases whereas the expected system size at a busy period initiation epoch doesn’t change.

- As failure rate increases, the expected system size at a departure epoch and at an arbitrary time epoch increases whereas the expected system size at a busy period initiation epoch doesn’t change.

### 6.6.2 Effects of Various Parameters on the Total Average Cost

The total average costs are obtained numerically with the following assumptions:

- **Start up cost**: 4.00
- **Holding cost per customer**: 0.50
- **Operating cost per unit time**: 5.00
- **Reward cost per unit time due to vacation**: 2.00
The effects of different arrival rates on the total average cost for a fixed failure rate with respect to various service rates are discussed numerically and these values are reported in Table 6.4. A graphical representation is also shown in Figures 6.3 – 6.5. From the table and the figures, the following points are observed:

- As arrival rate increases, the total average cost increases
- As service rate increases, the total average cost decreases
- When the server is assigned for secondary job, the total average cost decreases

Thus, the theoretical development of the model is justified with the numerical results which are consistent with the fact that when the server is allotted to secondary job, the idle time is properly utilized and hence the total average cost is minimized.

6.7 CONCLUSION

In this chapter, “steady state analysis of a $\text{M}^X/\text{G}/1$ queueing system with unreliable server and single vacation” is analyzed. Probability generating functions (PGF) of the number of customers in the system at the completion epoch of the idle period or at the initiation epoch of the busy period, at a departure point epoch and the number of customers in the system at an arbitrary time are obtained. Various performance measures are derived. Particular cases and some special cases are discussed. A cost model for the queueing system is developed. The theoretical development of the model is justified with numerical results.
Table 6.1 Arrival Rate (Vs) Expected System Size for various Service Rates
(For $\gamma = 0.6$, $\lambda_0 = 0.8$, $\omega = 4$, $\beta = 10$ )

<table>
<thead>
<tr>
<th>$\mu$</th>
<th>$\lambda$</th>
<th>$L_{BI}$</th>
<th>$L_D$</th>
<th>$L_A$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>2.0</td>
<td>1.6667</td>
<td>2.4191</td>
<td>2.4820</td>
</tr>
<tr>
<td>Service rate 10</td>
<td>2.5</td>
<td>2.0162</td>
<td>3.2229</td>
<td>3.2896</td>
</tr>
<tr>
<td></td>
<td>3.0</td>
<td>2.3571</td>
<td>4.3435</td>
<td>4.4110</td>
</tr>
<tr>
<td></td>
<td>3.5</td>
<td>2.6833</td>
<td>6.1002</td>
<td>6.1668</td>
</tr>
<tr>
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<td>4.0</td>
<td>3.0000</td>
<td>9.4850</td>
<td>9.5495</td>
</tr>
<tr>
<td></td>
<td>2.0</td>
<td>1.6667</td>
<td>2.0942</td>
<td>2.1548</td>
</tr>
<tr>
<td>Service rate 12</td>
<td>2.5</td>
<td>2.0162</td>
<td>2.6739</td>
<td>2.7388</td>
</tr>
<tr>
<td></td>
<td>3.0</td>
<td>2.3571</td>
<td>3.3911</td>
<td>3.4574</td>
</tr>
<tr>
<td></td>
<td>3.5</td>
<td>2.6833</td>
<td>4.3245</td>
<td>4.3907</td>
</tr>
<tr>
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<td>4.0</td>
<td>3.0000</td>
<td>5.6350</td>
<td>5.7000</td>
</tr>
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<td>2.0</td>
<td>1.6667</td>
<td>1.8984</td>
<td>1.9575</td>
</tr>
<tr>
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<td>2.0162</td>
<td>2.3681</td>
<td>2.4316</td>
</tr>
<tr>
<td></td>
<td>3.0</td>
<td>2.3571</td>
<td>2.9150</td>
<td>2.9804</td>
</tr>
<tr>
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<td>3.5</td>
<td>2.6833</td>
<td>3.5689</td>
<td>3.6346</td>
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<tr>
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<td>4.0</td>
<td>3.0000</td>
<td>4.3802</td>
<td>4.4453</td>
</tr>
</tbody>
</table>

$L_{BI}$ – Expected system size at a busy period initiation epoch; $L_D$ – Expected system size at a departure epoch; $L_A$ – Expected system size at an arbitrary time epoch.
### Table 6.2 Failure Rate (Vs) Expected System Size for various Arrival Rates

(For $\mu = 15$, $\lambda_0 = 0.8$, $\omega = 4$, $\beta = 10$)

<table>
<thead>
<tr>
<th>$\lambda$</th>
<th>$\gamma$</th>
<th>$L_{BI}$</th>
<th>$L_D$</th>
<th>$L_A$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Arrival rate 2.0</td>
<td>0.4</td>
<td>1.6667</td>
<td>1.8190</td>
<td>1.8590</td>
</tr>
<tr>
<td></td>
<td>0.5</td>
<td>1.6667</td>
<td>1.8230</td>
<td>1.8724</td>
</tr>
<tr>
<td></td>
<td>0.6</td>
<td>1.6667</td>
<td>1.8270</td>
<td>1.8854</td>
</tr>
<tr>
<td></td>
<td>0.7</td>
<td>1.6667</td>
<td>1.8311</td>
<td>1.8983</td>
</tr>
<tr>
<td></td>
<td>0.8</td>
<td>1.6667</td>
<td>1.8351</td>
<td>1.9110</td>
</tr>
<tr>
<td>Arrival rate 3.0</td>
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<td>2.3571</td>
<td>2.7443</td>
<td>2.7890</td>
</tr>
<tr>
<td></td>
<td>0.5</td>
<td>2.3571</td>
<td>2.7501</td>
<td>2.8051</td>
</tr>
<tr>
<td></td>
<td>0.6</td>
<td>2.3571</td>
<td>2.7560</td>
<td>2.8209</td>
</tr>
<tr>
<td></td>
<td>0.7</td>
<td>2.3571</td>
<td>2.7618</td>
<td>2.8365</td>
</tr>
<tr>
<td></td>
<td>0.8</td>
<td>2.3571</td>
<td>2.7677</td>
<td>2.8517</td>
</tr>
<tr>
<td>Arrival rate 4.0</td>
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<td>4.0038</td>
<td>4.0487</td>
</tr>
<tr>
<td></td>
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<td>3.0000</td>
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<td>4.0684</td>
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<tr>
<td></td>
<td>0.6</td>
<td>3.0000</td>
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<td>0.7</td>
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<td>4.1068</td>
</tr>
<tr>
<td></td>
<td>0.8</td>
<td>3.0000</td>
<td>4.0416</td>
<td>4.1256</td>
</tr>
</tbody>
</table>

$L_{BI}$ – Expected system size at a busy period initiation epoch; $L_D$ – Expected system size at a departure epoch; $L_A$ – Expected system size at an arbitrary time epoch.
Table 6.3 Service Rate (Vs) Expected System Size for various Failure Rates

(For $\lambda_1 = 2.5$, $\lambda_0 = 0.8$, $\omega = 4$, $\beta = 10$)

<table>
<thead>
<tr>
<th>$\gamma$</th>
<th>$\mu$</th>
<th>$L_{BI}$</th>
<th>$L_{D}$</th>
<th>$L_{A}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Failure rate 0.4</td>
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<td></td>
</tr>
<tr>
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<td>2.0192</td>
<td>3.1979</td>
<td>3.2437</td>
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</tr>
<tr>
<td>12.0</td>
<td>2.0192</td>
<td>2.6585</td>
<td>2.7030</td>
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</tr>
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<td>2.0192</td>
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</tr>
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</tr>
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<td>Failure rate 0.5</td>
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<td></td>
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<tr>
<td>Failure rate 0.6</td>
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<td>2.0192</td>
<td>3.2229</td>
<td>3.2896</td>
<td></td>
</tr>
<tr>
<td>12.0</td>
<td>2.0192</td>
<td>2.6739</td>
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<td></td>
</tr>
<tr>
<td>14.0</td>
<td>2.0192</td>
<td>2.3681</td>
<td>2.4316</td>
<td></td>
</tr>
<tr>
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<td>2.1727</td>
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<td>2.0192</td>
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<td>2.0984</td>
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</tr>
</tbody>
</table>

$L_{BI}$ – Expected system size at a busy period initiation epoch; $L_{D}$ – Expected system size at a departure epoch; $L_{A}$ – Expected system size at an arbitrary time epoch.
Table 6.4  Arrival Rate (Vs) Performance Measures and Total Average Cost for various Service Rates

(For $\gamma = 0.6, \lambda_0 = 0.8, \omega = 4, \beta = 10$)

<table>
<thead>
<tr>
<th>$\mu$</th>
<th>$\lambda$</th>
<th>$L_A$</th>
<th>$E(I)$</th>
<th>$E(B)$</th>
<th>TAC when server is not assigned for secondary job</th>
<th>TAC when server is assigned for secondary job</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>Service rate 10</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2.0</td>
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<td>0.4167</td>
<td>0.2992</td>
<td>6.0827</td>
<td>5.5006</td>
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</tr>
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<td>0.4038</td>
<td>0.4365</td>
<td>6.5729</td>
<td>6.0923</td>
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</tr>
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<td>3.0</td>
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</tr>
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</tr>
<tr>
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<td></td>
</tr>
<tr>
<td></td>
<td>Service rate 12</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
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</tr>
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<td></td>
</tr>
<tr>
<td></td>
<td>Service rate 14</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2.0</td>
<td>1.9575</td>
<td>0.4167</td>
<td>0.1784</td>
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<td>7.3032</td>
<td>6.8927</td>
<td></td>
</tr>
</tbody>
</table>

$L_A$ – Expected system size at an arbitrary time;  $E(I)$ – Expected idle period;  $E(B)$ – Expected busy period
Figure 6.2 Arrival Rate (Vs) Mean System Size at an Arbitrary Time
(For $\gamma = 0.6, \lambda_0 = 0.8, \omega = 4, \beta = 10$ and different service rates)

Figure 6.3 Arrival Rate (Vs) Total Average Cost for the Service Rate 10
(For $\gamma = 0.6, \lambda_0 = 0.8, \omega = 4, \beta = 10$)
Figure 6.4 Arrival Rate (Vs) Total Average Cost with Effective Utilization of Idle Time
(For $\gamma = 0.6$, $\lambda_0 = 0.8$, $\omega = 4$, $\beta = 10$ and various service rates)

Figure 6.5 Arrival Rate (Vs) Total Average Cost without Utilization of Idle Time
(For $\gamma = 0.6$, $\lambda_0 = 0.8$, $\omega = 4$, $\beta = 10$ and various service rates)