CHAPTER - III

TORSION OF AN ORTHOTROPIC CYLINDER

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3.1 Introduction

Creep torsion of circular cylinder for an isotropic material has been discussed by Bhatnagar [3], Marin [67], Finnie and Heller [28]. For non-circular cylinder, the problem has been discussed by Mordfin [71]. These authors have analysed the problem after making some simplifying assumptions. Firstly, the deformations are assumed to be small enough to make the infinitesimal strain theory applicable. Secondly, the constitutive equations of the material are simplified by assuming incompressibility of the material together with an yield condition and creep strain law like that of Norton.

Seth's transition theory discussed in section 1.3, which does not require these assumptions has been utilized to calculate the plastic and creep stresses for a circular cylinder. For vanishing anisotropy, as a particular case, the results obtained are the same as given by Seth [125] for fully plastic case and for creep theory as given by Marin [67].

3.2 Governing Equations

Consider a circular cylinder of radius 'a' subjected to finite twist. For torsion of circular shaft Saint Venant gives the components of displacement [111] as
Where $\zeta$ is the angle of twist per unit length.

In equation (3.2.1), it is assumed that there is no radical displacement in a section of the cylinder, but for finite twist in a circular cylinder radial displacement should be assumed and the components of displacement become \[111\],

\[
\begin{align*}
U &= -y\beta \sin \alpha \zeta + \kappa \left(1 - \beta \cos \alpha \zeta \right), \\
\psi &= \kappa \beta \sin \alpha \zeta + y \left(1 - \beta \cos \alpha \zeta \right), \\
\omega &= d\zeta.
\end{align*}
\]

where $\beta$ is a function of $\kappa \left(\kappa^2 = \kappa^2 + y^2\right)$ only and $d$ is a constant.

The components of displacement in cylindrical coordinates become \[125,135\],

\[
\begin{align*}
U &= \kappa \left(1 - \beta \right) \\
\psi &= \alpha \kappa \zeta \\
\omega &= d\zeta
\end{align*}
\]
Substituting the values of $u, v, w$ in equation (1.2.6), we get the finite components of strain as,

\[
\begin{align*}
\varepsilon_{xx} &= \frac{1}{\varepsilon} \left[ 1 - (\xi \beta' + \beta)^2 \right], \\
\varepsilon_{yy} &= \frac{1}{\varepsilon} \left[ 1 - \beta^2 \right], \\
\varepsilon_{zz} &= \frac{1}{\varepsilon} \left[ 1 - (1-d)^2 - (\alpha \xi \beta)^2 \right], \\
\varepsilon_{xy} &= \frac{1}{\varepsilon} \alpha \xi \beta^2, \\
\varepsilon_{xz} &= \varepsilon_{yz} = 0.
\end{align*}
\]

(3.2.4)

The generalized components of strain from equation (1.4.2) are given by,

\[
\begin{align*}
\varepsilon_{xx} &= \frac{1}{\eta m} \left[ 1 - (\xi \beta' + \beta)^n \right]^m, \\
\varepsilon_{yy} &= \frac{1}{\eta m} \left[ 1 - \beta^n \right]^m, \\
\varepsilon_{zz} &= \left( \frac{\beta}{\eta} \right)^m \left[ 1 - \left( \alpha \xi \beta \right)^n \right]^m, \\
\varepsilon_{xy} &= \frac{1}{\eta m} \left[ \alpha \xi \beta \eta^n \beta^n \right]^m, \\
\varepsilon_{xz} &= \varepsilon_{yz} = 0.
\end{align*}
\]

(3.2.5)

where $\beta' = \frac{d \beta}{d \xi}$ and $D^n = \left[ 1 - (1-d)^n \right]$.

Using equation (3.2.5) in equation (1.5.2) the components of stress are
The equations of equilibrium are \([145]\),

\[
\frac{1}{2} \frac{\partial}{\partial \theta} \left( \frac{\partial \gamma_{44}}{\partial \theta} \right) + \frac{1}{2} \frac{\partial}{\partial z} \left( \frac{\partial \gamma_{44}}{\partial z} \right) + \frac{\partial}{\partial \theta} \left( \frac{\partial \gamma_{44}}{\partial \theta} \right) + \frac{\partial}{\partial z} \left( \frac{\partial \gamma_{44}}{\partial z} \right) + \frac{\partial}{\partial \theta} \left( \frac{\partial \gamma_{44}}{\partial \theta} \right) = 0,
\]

\[
\frac{1}{2} \frac{\partial}{\partial \theta} \left( \frac{\partial \gamma_{40}}{\partial \theta} \right) + \frac{1}{2} \frac{\partial}{\partial z} \left( \frac{\partial \gamma_{40}}{\partial z} \right) + \frac{\partial}{\partial \theta} \left( \frac{\partial \gamma_{40}}{\partial \theta} \right) + \frac{\partial}{\partial z} \left( \frac{\partial \gamma_{40}}{\partial z} \right) + \frac{\partial}{\partial \theta} \left( \frac{\partial \gamma_{40}}{\partial \theta} \right) = 0, (3.2.7)
\]

\[
\frac{1}{2} \frac{\partial}{\partial \theta} \left( \frac{\partial \gamma_{42}}{\partial \theta} \right) + \frac{1}{2} \frac{\partial}{\partial z} \left( \frac{\partial \gamma_{42}}{\partial z} \right) + \frac{\partial}{\partial \theta} \left( \frac{\partial \gamma_{42}}{\partial \theta} \right) + \frac{\partial}{\partial z} \left( \frac{\partial \gamma_{42}}{\partial z} \right) + \frac{\partial}{\partial \theta} \left( \frac{\partial \gamma_{42}}{\partial \theta} \right) = 0.
\]

Substituting the values of stresses from equation (3.2.6) in equation (3.2.7), we find the equation of equilibrium, are all satisfied except

\[
\frac{1}{2} \frac{\partial}{\partial \theta} \left( \frac{\partial \gamma_{44}}{\partial \theta} \right) + \frac{\partial}{\partial \theta} \left( \frac{\partial \gamma_{44}}{\partial \theta} \right) + \frac{\partial}{\partial z} \left( \frac{\partial \gamma_{44}}{\partial z} \right) + \frac{\partial}{\partial \theta} \left( \frac{\partial \gamma_{44}}{\partial \theta} \right) + \frac{\partial}{\partial z} \left( \frac{\partial \gamma_{44}}{\partial z} \right) + \frac{\partial}{\partial \theta} \left( \frac{\partial \gamma_{44}}{\partial \theta} \right) = 0 \quad (3.2.8)
\]

Using equation (3.2.6) in equation (3.2.8) we have a non-linear differential equation satisfied by \( \beta \) as
\[
\begin{align*}
P(p+1)^{n-1} \frac{d\beta}{dp} \left\{ 1 - \beta^n(p+1)^n \right\}^{m-1} + P(p+1)^n \left\{ 1 - \beta^n(p+1)^n \right\}^{m-1} + \frac{c_{12}}{c_n} P \left\{ 1 - \beta^n \right\}^{m-1} + \\
+ \frac{c_{13}}{c_n} (p+1)(x\xi) (\omega)^{n-1} \left\{ 1 - \left( \frac{x\xi}{D} \right)^n \right\}^{m-1} - \\
- \frac{1}{m n c_n \beta^n} \left\{ (c_{n1} - c_{n2}) \left\{ 1 - \beta^n(p+1)^n \right\}^{m} + (c_{n2} - c_{n3}) \left\{ 1 - \beta^n \right\}^{m} \right\} + (c_{n3} - c_{n2}) (\omega)^{n} \left\{ 1 - \left( \frac{x\xi}{D} \right)^n \right\}^{m}
\end{align*}
\]

\( (3.2.9) \)

where \( \lambda \beta = \rho \beta \).

If \( m = 1 \), which holds for secondary state of creep, equation (3.2.9) reduces to

\[
\begin{align*}
\left\{ (p+1)^n \left\{ P + \frac{c_{n1} - c_{n2}}{c_n} \right\} + \frac{c_{12}}{c_n} P + (p+1) \frac{c_{13}}{c_n} (x\xi)^{n} + \frac{1}{n c_n} \left\{ (c_{n2} - c_{n3}) + (c_{n3} - c_{n2}) (x\xi)^{n} \right\} - \\
- \frac{1}{n c_n \beta^n} \left\{ (c_{n1} - c_{n2}) + (c_{n2} - c_{n3}) \right\} (x\xi)^{n} \right\} \frac{d\beta}{dp} + \beta P(p+1)^{n-1} \left(3.2.10\right)
\end{align*}
\]

Equation (3.2.9) shows that the transition points of \( \beta \) are,

\( P = 0 \) and \( P = -1 \)

Transition through \( P \rightarrow 0 \) gives nothing of importance. The critical point of interest is \( P \rightarrow -1 \), and asymptotic transition through \( P \rightarrow -1 \) gives plastic or creep stresses \([35-44, 57, 125, 135, 140, 141]\), depending upon the transition functions used. The twisting couple \( M \) is given by

\[
M = 2n \int_{0}^{2\pi} \xi \gamma_{0z} \, d\alpha
\]

\( (3.2.11) \)
The boundary conditions are:

(i) There is no normal traction across the curved surface of the cylinder, i.e.

\[ \gamma_{\lambda \lambda} = 0 \quad \text{at} \quad \lambda = a. \quad (3.2.12) \]

(ii) The resultant force normal to the plane \( Z = \text{constant} \) must vanish is,

\[ \int_0^a \xi \gamma_{zz} \, d\xi = 0. \quad (3.2.13) \]

3.3 **Solution Through Principal Stress**

In this section it is shown that the asymptotic solution through the stress leads from elastic to plastic and then to creep states at the transition point \( \rho \to -1 \) whereas the classical treatment uses different constitutive equations for each state.

We define the transition function \( R \) as

\[ R = \frac{(c_{11} + c_{12})}{c_{11}} + \frac{c_{13}}{c_{11}} \nu - \frac{\eta \gamma_{\lambda \lambda}}{c_{11}} \quad (3.3.1) \]
Substituting the value of \( q'_{\lambda} \) from equation (3.2.6) in equation (3.3.1), we have for \( n=1 \),

\[
R = \frac{c_{11} + c_{12}}{c_{11}} + \frac{c_{13}}{c_{11}} D^n \left[ c_{11} \{1 - R^n(P+1)^n\} + c_{12} \{1 - R^n\} + c_{13} \{1 - \frac{(c_{12} - c_{23})}{c_{12}} ( ) \} \right]
\]

(3.3.2)

Taking logarithmic differentiation of equation (3.3.2), with respect to \( \lambda \), we get

\[
\frac{d \log R}{d \lambda} = \frac{n\beta^n}{\lambda c_{11} R} \left[ c_{11} (P+1)^n \frac{dP}{d\beta} + c_{11} P (P+1)^n + c_{12} P + \frac{c_{13}}{\lambda} \lambda^n (P+1) \right]
\]

(3.3.3)

Substituting the value of \( \frac{dP}{d\beta} \) from equation (3.2.10) in equation (3.3.3), we get

\[
\frac{d \log R}{d \lambda} = \frac{n\beta^n}{\lambda c_{11} R} \left[ -(c_{11} - c_{21})(P+1)^n - \frac{1}{n} \{ c_{12} - c_{23} \} \lambda^n \right] + \frac{1}{n\beta^n} \left[ c_{11} (c_{12} + c_{13} - c_{23}) \lambda^n \right]
\]

(3.3.4)

Taking the asymptotic value of equation (3.3.4) as \( P \to -1 \), we have

\[
\frac{d \log R}{d \lambda} = \frac{[c_{11} (c_{12} - c_{23}) + (c_{12} - c_{23}) D^n] \lambda^n}{H^n c_{12} (1 + \kappa \lambda^n \hat{\lambda}^n)} - \frac{c_{12} - c_{23}}{c_{12} \lambda^2 (1 + \kappa \lambda^n \hat{\lambda}^n)} (3.3.5)
\]
where \( k = \frac{c_{12}}{c_{12}} \) and \( \beta = \frac{H}{L} \), \( H \) is a constant.

Integrating equation (3.3.5), yields,

\[
R = k_1 \left( 1 + k \kappa \gamma \frac{\alpha}{\lambda} \right)^A \left( \frac{\alpha}{1 + k \kappa \gamma \frac{\alpha}{\lambda}} \right)^B
\]

where \( k_1 \) is the constant of integration; \( B = -\frac{c_{12} - c_{22}}{H c_{12}} \), and

\[
A = \left[ \frac{(c_{12} - c_{21}) + (c_{12} - c_{22}) + C_{13} - c_{23} \gamma - (c_{13} - c_{23}) H \kappa}{H^{n} c_{12} k \kappa \gamma} \right]
\]

The asymptotic value of \( \beta \) is obtained from equation (3.3.2) and (3.3.6) as,

\[
\beta = \frac{c_{ii}}{c_{12}} k_1 \left( 1 + k \kappa \gamma \frac{\alpha}{\lambda} \right)^A \left( \frac{\alpha}{1 + k \kappa \gamma \frac{\alpha}{\lambda}} \right)^B
\]

Using equation (3.3.6) in equation (3.3.1), we get,

\[
\gamma_2 k = \frac{c_{ii}}{H} \left[ \frac{c_{ii} + c_{12} + c_{13}}{c_{ii}} - k_1 \left( 1 + k \kappa \gamma \frac{\alpha}{\lambda} \right)^A \left( \frac{\alpha}{1 + k \kappa \gamma \frac{\alpha}{\lambda}} \right)^B \right]
\]

From equation (3.2.8) and (3.3.8), we have

\[
\gamma_2 k - \gamma_0 = \frac{c_{ii} k_1}{H} \left( 1 + k \kappa \gamma \frac{\alpha}{\lambda} \right)^A \left( \frac{\alpha}{1 + k \kappa \gamma \frac{\alpha}{\lambda}} \right)^B
\]

\[
\times \left[ \frac{(c_{12} - c_{21}) + (c_{12} - c_{22}) + c_{13} - c_{23} \gamma - (c_{13} - c_{23}) H \kappa}{H^{n} c_{12} k \kappa \gamma} \right]^{\gamma \gamma \frac{\alpha}{\lambda}}
\]

(3.3.9)
Substituting the value of \( y_{e_2} \) from equation (3.3.8) in equation (3.3.9), we have

\[
\gamma_{e_0} = \frac{c_{ll}}{n} \left[ \frac{c_{l2} + c_{12}}{c_{l2}} + \frac{c_{l3} \eta}{c_{l2}} - k_1 (1 + \kappa \alpha_L \eta)^n \right] \left[ A^n \left( \frac{A^n}{1 + \kappa \alpha_L \eta} \right)^\beta \right. 
\]

\[
\times \left. \left[ \frac{1}{1 + \left( (c_{l1} - c_{s1}) + (c_{s2} - c_{s2}) + (c_{s3} - c_{s3}) D - (c_{s3} - c_{s3}) H \alpha + (c_{s2} - c_{s2}) H \alpha \kappa \alpha_L \eta \right)^n} \eta \right] \right] 
\]

(3.3.10)

The asymptotic value of \( \gamma_{zz} \) for \( n = 1 \), is obtained from equations (3.2.6) and (3.3.7) as,

\[
\gamma_{zz} = \frac{1}{n} \left[ c_{31} + c_{32} + c_{33} D^n - k_1 c_{11} (c_{32} + c_{33} \alpha_L \eta)^n (1 + \kappa \alpha_L \eta)^n \left( \frac{A^n}{1 + \kappa \alpha_L \eta} \right)^\beta \right] 
\]

(3.3.11)

The constants \( k_1 \) and \( D^n \) in equation (3.3.8) and (3.3.11) are to be determined from boundary conditions (3.2.12) and (3.2.13).

Using boundary condition (3.2.12) in equation (3.3.8), we get

\[
k_1 = \left( \frac{c_{l1} + c_{l2}}{c_{l1}} + \frac{c_{l3} \eta}{c_{l1}} \right) (1 + \kappa \alpha_L \eta)^{-A} \left( \frac{A^n}{1 + \kappa \alpha_L \eta} \right)^{-\beta} 
\]

(3.3.12)

Applying the boundary condition (3.2.13) in equation (3.3.11), we have
The value of material constant in the transition range is given by [136]

\[ \gamma = \frac{1}{n} \left[ \frac{C_{11}C_{22}C_{33} - C_{11}C_{23}^2 - C_{33}C_{23}^2 + 2C_{12}C_{23} - C_{13}C_{23}^2}{C_{22}C_{33} - C_{23}^2} \right]. \] (3.3.14)

where \( \gamma \) is the yield stress.

The corresponding values of the stresses are given by

\[ \\gamma_{\text{eq}} = \frac{C_{11} \gamma \left( C_{22}C_{33} - C_{23}^2 \right)}{\left( C_{11}C_{22}C_{33} - C_{11}C_{23}^2 - C_{33}C_{23}^2 + 2C_{12}C_{23} - C_{13}C_{23}^2 \right)} \times \]

\[ \times \left\{ \frac{C_{11} + C_{12}}{C_{11}} + \frac{C_{13}D^\gamma}{C_{11}} \left( 1 + k \alpha D^\gamma \right)^\frac{A}{(1 + k \alpha D^\gamma)^\beta} \right\} , \]
\[ \tau_{0\theta} = \frac{c_{11} y (c_{22} c_{33} - c_{23}^2)}{(c_{11} c_{22} c_{33} - c_{11} c_{23}^2 - c_{33} c_{12} + 2 c_{12} c_{23} - c_{13} c_{22})} \times \left[ \frac{c_{11} + c_{12}}{c_{11}} + \frac{c_{13} \lambda}{c_{11}} - k_1 \left( 1 + k_{\alpha} \lambda \right)^{\eta} \right] \times \left[ 1 + \frac{(c_{11} - c_{21}) + (c_{42} - c_{43}) + (c_{13} - c_{23}) \lambda \eta - (c_{12} - c_{22}) \lambda \eta} {1 + k_{\alpha} \lambda^{\eta}} \right] \right] \]

\[ \tau_{zz} = \tau_{\alpha \lambda} + \frac{1}{\eta} \left[ c_{51} - c_{11} + c_{32} - c_{12} + (c_{33} - c_{12}) (D^n \lambda \eta \rho^n - \beta^n (c_{32} - c_{42})) \right] \] (3.3.15)

where

\[ A = \frac{(c_{11} - c_{21}) + (c_{42} - c_{43}) + (c_{13} - c_{23}) \lambda \eta - (c_{12} - c_{22}) \lambda \eta} {1 + k_{\alpha} \lambda^{\eta}} \cdot \frac{\lambda \eta \rho^n} {c_{12} \cdot k_{\alpha} \eta} \]

\[ B = - \frac{c_{12} - c_{22}} {n c_{12}} \]

\[ k_{\alpha} \] and \[ D^n \] are given by equations (3.3.12) and (3.3.13). The shearing stress from equation (3.2.6) for \( m = 1 \) is,

\[ \tau_{0z} = \frac{c_{11} y (\lambda \eta \rho^n)} {\eta} \] (3.3.16)

Substituting the value of \( \beta^n \) from equation (3.3.7) in equation (3.3.16), we get

\[ \tau_{0z} = \frac{c_{11} y \cdot c_{11} \cdot \lambda \eta \rho^n + \eta B} {c_{12} \cdot k_1 \lambda \eta} \left( 1 + k_{\alpha} \lambda^{\eta} \right)^{\eta - 1 - 1 - 1} \] (3.3.17)
The twisting couple \( M \) is,

\[
M = 2\pi \int_0^a \frac{e}{x} \gamma_{0z} \, dx
\]

\[
= 2\pi \frac{C_{14}}{\eta} \frac{C_{21}}{C_2} \frac{\eta_2 \gamma_{0z} + 3 + n \beta}{\eta_2 + 3 + n \beta} \left[ \frac{1}{\eta_2 + 3 + n \beta} + \frac{\kappa \eta_1 \eta_1 \eta_2^2 (A-B-1)}{\frac{3n}{2} + 3 + n \beta} \right]
\]

Eliminating \( C_{14} \) between equations (3.3.17) and (3.3.18), we have for first approximation,

\[
\gamma_{0z} = \frac{2m}{\eta a^3 \frac{1}{4}} \left( 1 + \kappa \eta_1 \eta_1^2 \right)^{(A-B-1)} \left[ \frac{1}{\eta_2 + 3 + n \beta} + \frac{\kappa \eta_1 \eta_1 \eta_2^2 (A-B-1)}{\frac{3n}{2} + 3 + n \beta} \right]
\]

The maximum shearing stress occurs at \( \frac{\eta}{2} = \alpha \) where the yielding first sets in. Denoting its greatest value by \( \gamma' \), equation (3.3.19) becomes,

\[
\gamma' = \frac{2m}{\eta a^3 \frac{1}{4}} \left( 1 + \kappa \eta_1 \eta_1^2 \right)^{(A-B-1)} \left[ \frac{1}{\eta_2 + 3 + n \beta} + \frac{\kappa \eta_1 \eta_1 \eta_2^2 (A-B-1)}{\frac{3n}{2} + n \beta + 3} \right]
\]
For \( n = 2 \), elastic-plastic transition occurs. Seth [125].

Taking \( n = 2 \) in equation (3.3.15)-(3.3.17) and (3.3.20), the transitional stresses are

\[
\sigma_{z0} = \frac{C_{11} \left( C_{22} C_{33} - C_{23}^2 \right) \gamma}{\left( C_{11} C_{22} C_{33} - C_{11} C_{23}^2 - C_{33} C_{12}^2 + 2 C_{12} C_{23} - C_{13} C_{22} \right)} \times \left\{ \frac{C_{11} + C_{12}}{C_{11}} + \frac{C_{13}^2 \gamma^2}{C_{11}} - K_1 \left( 1 + K \kappa^2 \lambda^2 \right) \left( \frac{\lambda^2}{1 + K \kappa^2 \lambda^2} \right) \right\} \]

\[
\tau_{\theta 0} = \frac{C_{11} \gamma \left( C_{22} C_{33} - C_{23}^2 \right)}{\left( C_{11} C_{22} C_{33} - C_{11} C_{23}^2 - C_{33} C_{12}^2 + 2 C_{12} C_{23} - C_{13} C_{22} \right)} \times \left\{ \frac{C_{11} + C_{12}}{C_{11}} + \frac{C_{13}^2 \gamma^2}{C_{11}} - K_1 \left( 1 + K \kappa^2 \lambda^2 \right) \left( \frac{\lambda^2}{1 + K \kappa^2 \lambda^2} \right) \times \left\{ 1 + \frac{(C_{11} - C_{21}) + (C_{12} - C_{22}) + (C_{31} - C_{32}) \gamma^2 - (C_{13} - C_{23}) H \lambda - (C_{31} - C_{32}) H \lambda^2 \kappa \lambda}{(1 + K \kappa^2 \lambda^2) H \kappa} \right\} \right\} \]

\[
\tau_{zz} = \tau_{z0} + \frac{1}{2} \left\{ c_{31} - c_{11} + c_{32} - c_{12} + (c_{31} - c_{11}) (\lambda^2 - \kappa^2 \lambda^2 \beta^2) - \beta^2 (c_{32} - c_{12}) \right\} \]

\[
\tau_{\theta z} = \frac{C_{11} C_{12} \cdot \kappa}{2} \times \left( 1 + K \kappa^2 \lambda^2 \right) \left( 1 - \beta - 1 \right) \]

\[
\tau = \frac{2m}{n \alpha^3} \cdot \frac{A - B - 1}{4} \left( \frac{1 + K \kappa^2 \lambda^2}{1 + 2 \beta} + \frac{K \kappa^2 (A - B - 1) \alpha^2}{6 + 2 \beta} \right) \]

(3.3.21)
where
\[
A = \frac{(C_{11} - C_{22}) + (C_{44} - C_{22}) + (C_{44} - C_{22})D^2 - (C_{11} - C_{22})H^2X^2)}{2H^2X^2C_{12}K},
\]
and
\[
\beta = \frac{(C_{11} - C_{22})}{2C_{12}}
\]

For fully plastic state, we have the following relations between the constants [see equation (1.5.13)] i.e.

\[C_{12} = C_{13} = C_{11}, \quad C_{23} = C_{23} = C_{22}, \quad C_{31} = C_{32} = C_{33}.\]

Equation (3.3.21) becomes,

\[
\gamma_{\theta\theta} = \frac{C_{22} \gamma_{\theta} (C_{33} - C_{22})}{(C_{22} - C_{22} - C_{11}C_{33} + C_{22}C_{11})} \left\{ 2 - \kappa_1 \left(1 + \alpha^2 \lambda^2\right) \left(\frac{\lambda^2}{1 + \alpha^2 \lambda^2}\right) \right\}
\]

\[
\gamma_{zz} = \gamma_{\theta\theta} + \frac{1}{2} \left[ 2 - \alpha^2 \lambda^2 \beta^2 - \beta^2 \right] (C_{33} - C_{22}),
\]

\[
\gamma_{\theta z} = \frac{C_{44}}{2} \kappa_1 \alpha^2 \left(1 + \alpha^2 \lambda^2\right)^{A - \beta - 1}.
\]
\[ q^\prime = \frac{2m}{\pi a^3} \cdot \frac{1}{4} \cdot \frac{(1 + \alpha^2 a^2)^{A-\beta-1}}{\left[ \frac{1}{4+2\beta} + \frac{\alpha^2 a^2 (A-\beta-1)}{6+2\beta} \right]} \]

where

\[ A = \frac{C_{11} - C_{22}}{C_{11}} \cdot \left( \frac{2 - H \alpha^2 + D^2}{2 H^2 \alpha^2} \right), \quad \beta = -\frac{(C_{11} - C_{22})}{2 C_{11}} \]

and

\[ \kappa_1 = (2 + D^2) \left( 1 + \alpha^2 a^2 \right)^{-A} \frac{\left( \frac{a^2}{1 + \alpha^2 a^2} \right)^{-B}}{\left( \frac{a^2}{1 + \alpha^2 a^2} \right)^B} \]

**Particular Case**

For isotropic materials, the constants reduce to two only

\[ [145] \text{ i.e. } C_{11} = C_{22} = C_{33}, \quad C_{12} = C_{21} = C_{31} = C_{13} = C_{23} = C_{32} = C_{11} - 2C_{16}, \quad C = C_{12}, \quad \mu = \frac{C_{11} - C_{12}}{\epsilon} \quad \text{and} \quad \epsilon = \frac{C_{11} - C_{12}}{C_{11}}. \]

Equation (3.3.21) reduce to,

\[ \gamma_{\epsilon \epsilon} = \frac{\gamma (2-c)}{C (3-2c)} \left\{ (3-2c) - (1-c)(1-\alpha^2) - \kappa_1 \left( \frac{\alpha^2}{1 + \alpha^2 \alpha^2} \right) \frac{c}{2(1-c)} \right\}, \]

\[ \gamma_{\theta \theta} = \frac{\gamma (2-c)}{C (3-2c)} \left\{ (3-2c) - (1-c)(1-\alpha^2) - \kappa_1 \left( \frac{\alpha^2}{1 + \alpha^2 \alpha^2} \right) \frac{c}{2(1-c)} \right\} \times \left[ \frac{(1-c) \alpha^2 \alpha^2 + 1}{(1-c)(1+\alpha^2 \alpha^2)} \right], \]
\[ \gamma_{zz} = \gamma_{xx} + \gamma \frac{(2-c)}{(3-2c)} \left[ \frac{2}{1+(\alpha_0^2 a^2) \left(1 + \frac{c}{\alpha_0^2 (1-c)}\right)} \right], \]

\[ \gamma_{\theta z} = \frac{\gamma (2-c)}{2} \frac{1}{(3-2c)} \frac{1}{(1-c)} \left[ \frac{1}{2} \left(1 + \frac{c}{\alpha_0^2 (1-c)}\right) \right] - \frac{(1+\frac{c}{\alpha_0^2 (1-c)})}{1+c^2}, \quad (3.3.23) \]

\[ \gamma = \frac{8m}{\pi a^3} \left[ \frac{1}{4+\frac{c}{1-c}} - \frac{\alpha^2 a^2}{6+\frac{c}{1-c}} \right], \]

\[ D = \frac{2\alpha^2}{(1-c)} \kappa_1 \alpha^2 + \frac{c}{1-c} \left[ \frac{1}{4+\frac{c}{1-c}} - \frac{\alpha^2 a^2}{6+\frac{c}{1-c}} \right], \]

and

\[ K_1 = \left[ (2-c) - (1-c) D^2 \right] \left( \frac{\alpha^2}{1+\alpha^2 a^2} \right) - \frac{c}{\alpha_0^2 (1-c)}. \]

where

\[ \gamma = \frac{2m}{\pi a^3} \left( \frac{3-2c}{2(2-c)} \right) \]

For fully plastic state i.e. \( c \to 0 \), equation (3.3.23) become,

\[ \gamma_{zz} = \frac{\gamma}{3} \left( 2 + D^2 \right) \log \left\{ \left( \frac{a_0^2}{\alpha_0^2} \right)^2 \left( \frac{1+\alpha^2 a^2}{1+\alpha^2 a^2} \right) \right\}, \]

\[ \gamma_{\theta \theta} = \frac{\gamma}{3} \left[ (2 + D^2) \log \left( \frac{a_0^2}{\alpha_0^2} \left( \frac{1+\alpha^2 a^2}{1+\alpha^2 a^2} \right) \right) - \frac{2}{(1+\alpha^2 a^2)} \right], \]

\[ \gamma_{zz} = \gamma_{xx} + \frac{2\gamma}{3} \left[ D^2 - (\alpha_0^2 \beta)^2 \right], \quad (3.3.24) \]
These equations are the same as obtained by Gupta and Rana [41].

As a numerical example, we have taken a circular shaft of 10 ft. long and 2 inch in diameter made of Barytes material. The values of material constants are taken as [64]

\[ c_{11} = 907 \quad c_{12} = 273 \quad c_{13} = 275 \quad c_{22} = 800 \quad c_{23} = 468, \]
\[ c_{33} = 1074 \quad c_{44} = 128 \quad c_{55} = 293 \quad c_{66} = 283. \]

Curves have been drawn in figure (3.1) between shear stress ratio \( \frac{\tau_{0z}}{\tau_{pe}} \) (where \( \tau_{pe} = \frac{2m}{\pi a^2} \)) and the relative distance from the centre. The torque required for the orthotropic transitional yielding at an angle of twist per length equal to 1° is equal to 680 In.lbf. and torque required at fully plasticity is 226 In.lbf. At the transitional state, the shear stress is maximum for cylinder made of isotropic material as compared to orthotropic material whereas just reverse is the case at fully plastic state.
3.4 Solution Through the Stress-Difference

The transition function through the stress-difference at the transition point \( p \rightarrow -1 \), gives the creep stresses \([35-44, 57, 140, 141]\). The transition function \( R_1 \) is taken as

\[
R_1 = \frac{\tau_{\theta \theta} - \tau_{\theta \theta}^0}{\tau_{\theta \theta}^0} = \frac{1}{\eta m} \left[ \left( C_{\theta \theta} - C_{\theta \theta}^0 \right) \{1 - \beta (P + 1)^n\}^m + \left( C_{\theta \theta} - C_{\theta \theta}^0 \right) \{1 - \beta \}^m \right] + \left( C_{\theta \theta} - C_{\theta \theta}^0 \right) \{1 - \left( \frac{\alpha \epsilon B}{D} \right)^n\}^m \right] \quad (3.4.1)
\]

Taking the logarithmic differentiation of equation (3.4.1) with respect to \( \theta \), we get

\[
\frac{d \log R_1}{d \theta} = \frac{-m n \beta}{\eta m R_1 \theta} \left[ \left( C_{\theta \theta} - C_{\theta \theta}^0 \right) \{1 - \beta (P + 1)^n\}^m \left( P + 1 \right) \left( \beta \frac{d P}{d \beta} + (P + 1) \right) + \left( C_{\theta \theta} - C_{\theta \theta}^0 \right) \{1 - \beta \}^m \left( \beta \frac{d P}{d \beta} + (P + 1) \right) \right] \quad (3.4.2)
\]

Substituting the value of \( \beta \frac{d P}{d \beta} \) from equation (3.2.9) in equation (3.4.2), we have

\[
\frac{d \log R_1}{d \theta} = \frac{-m n \beta}{\eta m R_1 \theta} \left[ \left( C_{\theta \theta} - C_{\theta \theta}^0 \right) \{1 - \beta (P + 1)^n\}^m \left( P + 1 \right) \left( \beta \frac{d P}{d \beta} + (P + 1) \right) + \left( C_{\theta \theta} - C_{\theta \theta}^0 \right) \{1 - \beta \}^m \left( \beta \frac{d P}{d \beta} + (P + 1) \right) \right] \quad (3.4.3)
\]
Taking the asymptotic value as $\rho \to -1$ in equation (3.4.3), we get

$$\frac{d \ln R_1}{d \rho} = - \frac{m \eta \beta (C_{11}C_{22} - C_{12}C_{21}) (1 - \rho^n)^{m-1}}{\kappa C_{11} \left[ (C_{11} - C_{21}) + (C_{12} - C_{22}) (1 - \rho^n)^m \right] + (C_{13} - C_{23})(\omega^n)^m \left( 1 - (\alpha \lambda \beta)^m \right)^m} - \frac{(C_{11} - C_{21})}{C_{11} \lambda} \tag{3.4.4}$$

Integrating equation (3.4.4), yields

$$R_1 = \frac{A \lambda}{C_{11} \bar{c}_n} \left[ (C_{11} - C_{21}) + (C_{12} - C_{22}) (1 - \rho^n)^m + (C_{13} - C_{23})(\omega^n)^m \left( 1 - (\alpha \lambda \beta)^m \right)^m \right] \tag{3.4.5}$$

where $A$ is the constant of integration. The asymptotic value of $\beta$ as $\rho \to -1$ is $\frac{A}{\lambda}$, $A$ being a constant. For secondary state of creep (i.e. for $m = 1$) equation (3.4.5) become

$$R_1 = \gamma_{\lambda \lambda} - \gamma_{\theta \theta} = A \frac{\lambda}{C_{11} \bar{c}_n} \left[ S + \bar{c}_n \right] \tag{3.4.6}$$

where

$$S = \frac{(C_{11} - C_{21}) + (C_{12} - C_{22}) + (C_{13} - C_{23})(\omega^n)^m}{C_{22} - C_{12} \cdot H^n}$$

$$H = \frac{C_{21}C_{12} - C_{11}C_{22}}{C_{11} \cdot (C_{12} - C_{22})}$$
Substituting equation (3.4.6) in equation (3.2.8) and integrating we get

\[ \tau_{\alpha\xi} = -A \int \frac{(c_{11} - c_{21})}{c_{11}} \cdot (s + \alpha^n) \cdot d\alpha + \beta. \]  

(3.4.7)

where \( \beta \) is a constant of integration. To evaluate the integral we apply the Binomial expansion and then integrating term by term, equation (3.4.7) become

\[ \tau_{\alpha\xi} = -A \frac{c_{11} - c_{21}}{c_{11}} \left[ \frac{b}{c_{12} + c_{21}} + \frac{(c_{12}c_{21} - c_{11}c_{22})}{c_{12} - c_{22}} \cdot \frac{-\eta L}{c_{11}(c_{12} - c_{22}) \left( \frac{c_{12}c_{21} - c_{11}c_{22}}{(n+1)c_{11}c_{22} + (1-n)c_{12}c_{21} - c_{11}c_{12} - c_{21}c_{22}} \right)} \right] + \beta \]  

(3.4.8)

The asymptotic values of other stresses can be written from equations (3.4.6), (3.4.7), (3.2.5) and (3.2.6) as

\[ \tau_{\theta\phi} = -A \frac{c_{11} - c_{21}}{c_{11}} \left[ \frac{b}{c_{12} + c_{21}} + \frac{(c_{12}c_{21} - c_{11}c_{22})}{c_{12} - c_{22}} \cdot \frac{-\eta L}{c_{11}(c_{12} - c_{22}) \left( \frac{c_{12}c_{21} - c_{11}c_{22}}{(n+1)c_{11}c_{22} + (1-n)c_{12}c_{21} - c_{11}c_{12} - c_{21}c_{22}} \right)} \right] + \beta \]  

(3.4.9)
\( \gamma_{zz} = \frac{c_{33}}{c_{13} + c_{23}} (\gamma_{\alpha\alpha} + \gamma_{\theta\theta}) + \left[ c_{31} - \frac{c_{33} (c_{11} + c_{21})}{c_{13} + c_{23}} \right] \frac{1}{\eta} + \frac{[c_{32} - \frac{c_{33} (c_{12} + c_{22})}{c_{13} + c_{23}}]}{[1 - \beta^n]} \) (3.4.10)

Substituting the values of \( \gamma_{\alpha\alpha} \) and \( \gamma_{\theta\theta} \) from equations (3.4.8) and (3.4.9) in equation (3.4.10), the transitional value of \( \gamma_{zz} \) is given by

\[
\gamma_{zz} = -A \frac{c_{33}}{c_{13} + c_{23}} \left[ \frac{S}{c_{11}} (c_{11} + c_{21}) \right] + \left[ \frac{c_{12} c_{21} - c_{11} c_{22}}{c_{13} + c_{23}} \right] \frac{\eta}{\lambda} + \left[ \frac{c_{31} - \frac{c_{33} (c_{11} + c_{21})}{c_{13} + c_{23}}}{\eta} \right] + \frac{2B c_{33}}{c_{13} + c_{23}} \left[ \frac{c_{32} - \frac{c_{33} (c_{12} + c_{22})}{c_{13} + c_{23}}}{\eta} \right] \]

(3.4.11)

The constants \( A \) and \( B \) have to be determined from the boundary conditions (3.2.12) and (3.2.13). Using (3.2.12) in equation (3.4.8), we get

\[
B = A \frac{\frac{c_{21} - c_{22}}{c_{11}}}{c_{11}} \left[ \frac{S}{c_{11}} \right] + \left[ \frac{c_{12} c_{21} - c_{11} c_{22}}{c_{13} + c_{23}} \right] \frac{\eta}{\lambda} \left[ \frac{1}{\eta} \right] + \left[ \frac{c_{31} - \frac{c_{33} (c_{11} + c_{21})}{c_{13} + c_{23}}}{\eta} \right] \frac{[1 - \beta^n]}{\eta} \]

(3.4.12)
Using the value of constant $\beta$ in equations (3.4.8), (3.4.9) and (3.4.11), we get the transitional creep stresses as,

\[ \tau_{\theta\theta} = -A \alpha \left[ \frac{(c_{11} - c_{22})}{c_{11} - c_{22}} \left( \frac{\gamma_0}{\eta} \right) \frac{c_{11}}{c_{11} - c_{22}} - 1 \right] \]

\[ + \frac{c_{11} (c_{12} - c_{22})}{(\eta + 1) c_{11} c_{22} + (1 - \eta) c_{21} c_{22}} \frac{-\eta \lambda}{\eta} \left\{ \frac{\gamma_0}{\eta} \right\} \]

\[ \left( \frac{c_{11} (c_{12} - c_{22})}{(\eta + 1) c_{11} c_{22} + (1 - \eta) c_{21} c_{22}} \right) \frac{-\eta \lambda}{\eta} \left\{ \frac{\gamma_0}{\eta} \right\} \]

(3.4.13)

\[ \tau_{\theta\theta} = -A \alpha \left[ \frac{(c_{11} - c_{22})}{c_{11} - c_{22}} \left( \frac{\gamma_0}{\eta} \right) \frac{c_{11}}{c_{11} - c_{22}} - 1 \right] \]

\[ + \frac{c_{11} (c_{12} - c_{22})}{(\eta + 1) c_{11} c_{22} + (1 - \eta) c_{21} c_{22}} \frac{-\eta \lambda}{\eta} \left\{ \frac{\gamma_0}{\eta} \right\} \]

\[ \left( \frac{c_{11} (c_{12} - c_{22})}{(\eta + 1) c_{11} c_{22} + (1 - \eta) c_{21} c_{22}} \right) \frac{-\eta \lambda}{\eta} \left\{ \frac{\gamma_0}{\eta} \right\} \]

(3.4.14)

\[ \tau_{zz} = -A \frac{c_{33}}{c_{13} + c_{23}} \frac{(c_{11} - c_{22})}{c_{11} - c_{22}} \left( \frac{\gamma_0}{\eta} \right) \frac{c_{11}}{c_{11} - c_{22}} - 1 \]

\[ + \frac{c_{11} (c_{12} - c_{22})}{(\eta + 1) c_{11} c_{22} + (1 - \eta) c_{21} c_{22}} \frac{-\eta \lambda}{\eta} \left\{ \frac{\gamma_0}{\eta} \right\} \]

\[ \left( \frac{c_{11} (c_{12} - c_{22})}{(\eta + 1) c_{11} c_{22} + (1 - \eta) c_{21} c_{22}} \right) \frac{-\eta \lambda}{\eta} \left\{ \frac{\gamma_0}{\eta} \right\} \]

(3.4.15)
Where the value of $A$ is obtained from equation (3.2.13) and (3.4.15) as,

$$A = \frac{1}{\eta} \left[ \left( \frac{G_{12} - G_{22}}{G_{11} + G_{22}} \right) + \frac{1}{2} \left[ \frac{G_{12} - G_{22}}{G_{12} + G_{22}} \right] \right] - \frac{C_{12} - C_{22}}{C_{11} - C_{22}} \eta \frac{\eta}{\eta + 1} \left[ \frac{G_{11} - G_{22}}{C_{12} + C_{33}} \right] \left[ 2 + \frac{1}{2} \left( \frac{G_{12} - G_{22}}{G_{12} + G_{22}} \right) \right] \right]$$

(3.4.16)

The asymptotic value of $\beta$ is obtained from equation (3.4.1) and (3.4.5) for $m = 1$ is,

$$\beta = \left[ \frac{G_{12} - G_{22}}{(n \eta) C_{12} (C_{11} - C_{22}) \lambda C_{12} - k_2} \right]$$

(3.4.17)

where $\lambda = \left[ \frac{G_{11} - G_{22}}{C_{12} + C_{33}} \right] \lambda \left[ \frac{G_{12} - G_{22}}{G_{12} + G_{22}} \right] \left[ 2 + \frac{1}{2} \left( \frac{G_{12} - G_{22}}{G_{12} + G_{22}} \right) \right] \right] \right] \right]$$

(3.4.18)
The twisting couple \( M \) is,

\[
M = 2\pi \int_0^\infty \lambda^2 \varphi_{\theta z} \, d\lambda. \tag{3.4.19}
\]

Substituting the value of \( \varphi_{\theta z} \) from (3.4.18) in equation (3.4.19) and integrating, to the first approximation we have,

\[
M = \frac{2\pi}{\eta} C_{44} \frac{\eta_{12}}{\left(C_{22}-C_{12}\right)} \left(\frac{C_{11} \left(C_{22}-C_{21}\right)}{C_{22} \left(C_{11}-C_{12}\right)} \right) \cdot \left\{ \frac{\eta^2 + 3 - \left(C_{22}-C_{21}\right)}{C_{12}} \right\} \cdot \left[ \frac{1}{\eta^2 + 3 - \left(C_{22}-C_{21}\right)} \right] + \frac{\eta \eta}{\left(C_{22}-C_{12}\right) \left(\frac{\eta^2 + 3 - \left(C_{22}-C_{21}\right)}{C_{12}} \right)} - \frac{K_2 \eta}{C_{12} \left(C_{22}-C_{12}\right) \left(\frac{\eta^2 + 3}{C_{12}} \right)} - \frac{\eta^2 + 3 - \left(C_{22}-C_{21}\right)}{C_{12} \left(C_{22}-C_{12}\right) \left(\frac{\eta^2 + 3}{C_{12}} \right)} \right\} \tag{3.4.20}
\]

From equation (3.4.18) and (3.4.20), we have
The maximum shearing stress occurs at $\lambda = a$, denoting its absolute value by $\eta'$, we have

\[
\eta' = \frac{M}{2 \pi a^3} \left[ 1 - \frac{K_2 \alpha}{(\eta A)} \frac{C_{22} - C_{32}}{C_{12}} \right] \left[ \frac{1}{\eta^3 - \frac{(C_{32} - C_{22}) \alpha \eta^3}{C_{12}}} \right] \frac{\eta}{\eta^3 + \frac{(C_{22} - C_{32}) \alpha \eta^3}{C_{12}}} + \frac{C_{12} - C_{32}}{C_{12}} \left( \frac{\frac{3}{2} \eta + 3 - \frac{(C_{32} - C_{22}) \alpha \eta^3}{C_{12}}} {C_{12} C_{11} - C_{21}} \right)
\]

\[
(3.4.22)
\]
3.5. **Isotropic Steady State of Creep**

For isotropic materials, constants reduce to two only \[145\]

\[ c_{11} = c_{22} = c_{33}, \quad c_{12} = c_{21} = c_{23} = c_{32} = c_{31} = c_{13} = \]
\[ = c_{11} - 2c_{66}, \quad \sigma = c_{12}, \quad \mu = \frac{c_{11} - c_{12}}{2}, \quad \text{and} \quad c = \frac{c_{11} - c_{12}}{c_{11}}. \]

Equation (3.4,6) then becomes,

\[
\tau_{rr} - \tau_{\theta\theta} = \Theta \sigma + 2\eta + c(\eta - 1) \tag{3.5.1}
\]

For Hencky measure i.e. \( n \to 0 \), which holds good for stationary state of creep, equation (3.5.1) reduce to the Tresca type for incompressible material \((c \to 0)\) i.e.

\[
\tau_{rr} - \tau_{\theta\theta} = \Theta \equiv \varpi
\]

where \( \varpi \) is the yield stress in tension.

Creep transitional stresses (3.4.13), (3.4.14), (3.4.15) and (3.4.16) for isotropic material become

\[
\tau_{rr} = \frac{\Theta \sigma}{2n - c(\eta - 1)} \left\{ \frac{\sigma}{\Theta} - 1 \right\} - 2\eta + c(\eta - 1) \]
\[
\tau_{\theta\theta} = \frac{\Theta \sigma}{2n - c(\eta - 1)} \left\{ \frac{\sigma}{\Theta} \left( 1 - 2\eta + c(\eta - 1) \right) - 1 \right\}. \tag{3.5.2}
\]
Shearing stress from equation (3.4.21) reduce to

\[ \tau_{zz} = \frac{Aa}{(2-c) [2n-c(n-1)]} \left\{ \left( \frac{\partial}{\partial z} \right) \left( \frac{-2n+c(n-1)}{2-2n+c(n-1)} \right) - \eta \right\} + \frac{(3-2c) c \mu (2-\beta \eta)}{n(c-1)} \]

where

\[ A = \frac{1}{\eta} a \left[ \mu(2-c) \right]^{1-c} \]

Shearing stress from equation (3.4.21) reduce to

\[ \tau_{zz} = \frac{1}{4} \left[ 3 + \frac{\eta}{2} + \frac{c}{1-c} \right] (\frac{\partial}{\partial z})^{\frac{\eta}{2}} + \frac{c}{1-c}, \frac{\partial^2}{\eta \alpha^2} \] \quad (3.5.3)

Maximum shearing stress is given by

\[ \tau' = \frac{1}{4} \left[ 3 + \frac{\eta}{2} + \frac{c}{1-c} \right] \tau_e \] \quad (3.5.4)

where \( \tau_e \) is the elastic maximum shearing stress equal to \( \frac{\partial M}{\eta \alpha^2} \)

For incompressible material i.e. \((C-\gamma)\), equation (3.5.4) become,

\[ \tau' = \frac{1}{4} \left[ \frac{6+n}{2} \right] \tau_e \] \quad (3.5.5)

The expression for maximum shear stress is the same as given by Marin [67] provided. We put \( \eta = \frac{2}{N} \)
Fig(3.1). Plastic stress distribution of cylinder under torsion.