CHAPTER II

THERMOSOLUTAL INSTABILITY OF A PARTIALLY IONIZED PLASMA IN POROUS MEDIUM
A. THERMOSOLUTAL INSTABILITY IN POROUS MEDIUM
IN A PARTIALLY IONIZED PLASMA

INTRODUCTION

A detailed account of thermal convection problem, under varying assumptions of hydrodynamics and hydromagnetism, has been given by Chandrasekhar (1981). The conditions under which convective motions are important in astrophysical and geophysical situations are usually far removed from the consideration of single component fluid and therefore it is desirable to consider fluid acted on by a solute gradient. In such situation buoyancy forces can arise not only from density differences due to variations in temperature, but also from those due to variations in solute concentration. Veronis (1965) has investigated the problem of thermohaline convection in a layer of fluid heated and salted from below. In the stellar case, the physics is quite similar in that helium acts like salt in raising the density and in diffusing more slowly than heat.

A partially ionized plasma represents a state which often exists in the universe and there are several situations in which the interaction between the ionized and the neutral gas components becomes

important in cosmic physics. Strömgren (1939) has reported that ionized hydrogen in limited to certain rather sharply bounded regions in space surrounding, for example O-type stars and clusters of such stars and that the gas outside these regions is essentially non-ionized. Others examples of the existence of such situations are given by Alfvén's (1954) theory of the origin of the planetary system, in which a high ionisation rate is suggested to appear from collisions between a plasma and a neutral gas cloud and by the absorption of plasma waves due to ion-neutral collisions such as in the solar photosphere and chromosphere and in cool interstellar clouds [Piddington (1954), Lehnert (1959)]. The effect of collisions with neutral particles on the instability of the plane interface which separates two uniform superposed composite hydromagnetic streaming systems have been studied by Hans (1968) and Bhatia (1970). In all the above studies the medium has been considered to be non-porous.

Keeping in mind the astrophysical (interplanetary and stellar atmospheres) and geophysical (stability of Earth's core and geothermal regions) situations, the thermosolutal convection in porous medium in a partially ionized plasma has been
FORMULATION OF THE PROBLEM AND PERTURBATION EQUATIONS

Consider an incompressible composite plasma layer consisting of an infinitely conducting hydromagnetic fluid of density \( \rho \) permeated with neutrals of density \( \rho_n \) in porous medium and acted on by gravity force \( \vec{g}(0,0,-g) \) and by uniform vertical magnetic field \( \vec{H}(0,0,H) \). This layer is heated and soluted from below so that the temperatures and solute concentrations at the bottom surface \( z=0 \) are \( T_0 \) and \( C_0 \) and at the upper surface \( z=d \) are \( T_1 \) and \( C_1 \) respectively. Assume that both the ionized fluid and the neutral gas behave like continuum fluids and that effects on the neutral component resulting from the fields of gravity, pressure and Darcy resistance are neglected. The magnetic field interacts with the hydromagnetic component only. Then the equations governing the motion of the composite plasma in porous medium, following Boussinesq approximation, are

\[
\frac{1}{\rho} \left[ \frac{\partial \vec{u}}{\partial t} + \vec{u} \cdot \nabla \vec{u} \right] = \frac{1}{\rho} \nabla p + \frac{1}{\rho} \left( \frac{\partial \vec{u}}{\partial t} + \vec{u} \cdot \nabla \vec{u} \right) \frac{\mu_0}{\mu_n} (\nabla \times \vec{H}) \times \vec{H} + \frac{\rho \mu_0}{\rho_n \mu_0} (C_{\text{ion}}-C) , \quad (2.1)
\]

\[
\nabla \cdot \vec{u} = 0 , \quad (2.2)
\]
\[
\frac{d \mathbf{q}_d}{dt} + \frac{1}{\varepsilon} \left( \Theta \mathbf{v}_d \right) \mathbf{q}_d = -\varepsilon \left( \mathbf{v}_d - \bar{\mathbf{v}} \right), \quad (2.A.3)
\]

\[
\varepsilon \frac{d \mathbf{v}}{dt} + \left( \mathbf{q} \cdot \mathbf{v} \right) \mathbf{v} = \chi \mathbf{v}, \quad (2.A.4)
\]

\[
\varepsilon \frac{d \mathbf{c}}{dt} + \left( \mathbf{q} \cdot \mathbf{c} \right) \mathbf{c} = \chi' \mathbf{c}, \quad (2.A.5)
\]

\[
\mathbf{v} \cdot \mathbf{b} = 0, \quad (2.A.6)
\]

\[
\varepsilon \frac{d \mathbf{H}}{dt} = \left( \mathbf{A} \mathbf{v} \right) \mathbf{H} + \varepsilon \mathbf{c} \mathbf{v} \mathbf{b}, \quad (2.A.7)
\]

where \( \mathbf{q}_d, \mathbf{v}, \mathbf{c}, \mu, \mu_k, \eta, \varepsilon \) and \( k_d \) stand for velocity of the neutral gas, density of neutral gas, mutual collisional (frictional) frequency between the two components of the composite medium, viscosity of hydromagnetic fluid, magnetic permeability, electrical resistivity, medium porosity, and medium permeability respectively. \( \chi, \chi' \) denote respectively the thermal diffusivity, analogous solute diffusivity and \( E = \varepsilon + (\mathbf{A} - \mathbf{b}) \frac{f_c}{P_c}, \) where \( f, c \) and \( f_s, c_s \) stand for density and heat capacity of fluid and solid matrix respectively. \( E' \) is an analogous solute parameter.

The equation of state is

\[
f = f_0 \left[ 1 - \alpha (T - T_0) + \alpha' (C - C_0) \right], \quad (2.A.8)
\]
where $\alpha, \alpha'$ denote respectively the coefficient of thermal expansion and analogous coefficient of solvent expansion and $\rho_0$ is the density at the surface $z=0$.

Let $\delta \rho, \delta p, \delta \rho, \delta \gamma, \vec{u}(u,v,w)$ and $\vec{H}(h_x, h_y, h_z)$ denote the perturbations in density $\rho$, pressure $p$, temperature $T$, solute concentration $C$, velocity $(0,0,0)$ and magnetic field $\vec{H}(0,0,H)$ respectively. Then the linearized perturbation equations governing the motion of the composite plasma in porous medium, following Boussinesq approximation, are

$$
\begin{align*}
\frac{1}{\varepsilon} \frac{\partial \vec{p}}{\partial t} &= -\frac{1}{\rho_0} \nabla \delta p - \frac{\rho_0}{\varepsilon} \frac{\partial \delta T}{\partial z} - \frac{1}{\rho_0} \nabla (\alpha_0 - \alpha' \gamma), \\
\nabla \delta \gamma &= 0, \\
\frac{\partial \vec{u}}{\partial t} &= -\vec{c} (\vec{\omega} - \vec{\gamma}) , \\
\frac{\partial \omega}{\partial t} &= \beta \omega + \chi \gamma \vec{z} , \\
\frac{\partial \vec{h}}{\partial t} &= 0 , \\
\frac{\partial \vec{H}}{\partial t} &= (\vec{H} \cdot \vec{\gamma}) \vec{H} + \vec{\gamma} \vec{H}^2 ,
\end{align*}
$$

where $\rho(=d\tau/dz)$ and $\rho'(=dC/dz)$ denote uniform adverse temperature and solute concentration gradients respectively. In writing equation (2.A.9), use has been
made of the equation of state (2.1.8) wherefrom the change in density \( \rho' \), caused by the perturbations \( \Theta \) and \( \Upsilon \) in temperature and concentration, is given by

\[
\rho' = -\rho_0 C(\Theta - \Upsilon) . \tag{2.1.16}
\]

Equations (2.1.9) - (2.1.15), on eliminating \( \tilde{q} \) between (2.1.9) and (2.1.11), give

\[
\begin{aligned}
&\left( \frac{1}{\varepsilon} \left[ 1 + \frac{\alpha_0 \Delta \varepsilon}{\varepsilon^2} \right] \frac{\partial}{\partial t} + \frac{\nu}{k_f} \right) \nabla \cdot \mathbf{\psi} - \frac{g}{\partial z} \left( \frac{\partial^2 \Theta}{\partial z^2} + \frac{\partial^2 \Upsilon}{\partial y^2} \right) (\Theta - \Upsilon) \\
&- \frac{\mu \gamma}{\eta \mu_0} (\frac{\partial h_z}{\partial z}) = 0 , \tag{2.1.17}
\end{aligned}
\]

\[
\left( E \frac{\partial}{\partial t} - \kappa \nabla^2 \right) \Theta = \beta \mathbf{w} \; , \tag{2.1.18}
\]

\[
\left( E' \frac{\partial}{\partial t} - \kappa' \nabla^2 \right) \Upsilon = \beta' \mathbf{w} \; , \tag{2.1.19}
\]

\[
\epsilon \left( \frac{\partial}{\partial t} - \gamma \nabla^2 \right) h_z = \frac{H_0 \omega}{\rho} . \tag{2.1.20}
\]

Here we consider the case in which both the boundaries are free and the medium adjoining the fluid is a perfect (electrical) conductor. If the temperatures and solute concentrations at a boundary are kept fixed, then the boundary conditions appropriate to the problem are

\[
\frac{\partial \omega}{\partial z} = 0 , \; \Theta = \Upsilon = h_z = 0 \quad \text{at} \; z=0 \; \text{and} \; z=d . \tag{2.1.21}
\]
Analyzing the disturbances into normal modes, we assume that the perturbations quantities are of the form

\[ \begin{bmatrix} \omega, \theta, \psi, h_x \end{bmatrix} = \begin{bmatrix} W(x), \Theta(x), \Psi(x), K_{xx} \end{bmatrix} \exp \left( ik_x x + ik_y y + \text{int} \right), \quad (2.A.22) \]

where \( k_x \) and \( k_y \) are the wave numbers in the \( x \) and \( y \) directions, respectively, \( k = \left( k_x^2 + k_y^2 \right)^{1/2} \) is the resultant wave number, and \( n \) is the growth rate, which is, in general, a complex constant.

Letting \( x/d = \tilde{x} \), \( y/d = \tilde{y} \), \( z/d = \tilde{z} \), \( a =kd \), \( \sigma = \text{no} \nu \) and \( D = d |d\tilde{x}| \), equations (2.A.17) - (2.A.20) with the help of expression (2.A.22), in non-dimensional form become

\[ \begin{align*}
\left( \frac{1}{e} \left[ 1 + \chi \frac{2\tilde{x}}{\sigma + \chi^2 \alpha |\sigma|} \right] \sigma + \frac{1}{\nu} \right) (\tilde{D} - \tilde{a}) W + \frac{\nu^2 \tilde{x}^2}{\sigma^2} (\alpha \Theta - \alpha' \Gamma) \\
- \frac{\mu \nu \tilde{H} \tilde{d}}{4 \pi \rho_0 \nu} (\tilde{D} - \tilde{a}) (DK) = 0, \quad (2.A.23) \\
(\tilde{D} - \tilde{a} - \tilde{b}_1 \sigma) \Theta = - (\frac{\sigma d^2}{\chi}) W, \quad (2.A.24) \\
(\tilde{D} - \tilde{a} - \tilde{b}_2 \sigma) \Gamma = - (\frac{\sigma d^2}{\chi}) W, \quad (2.A.25) \\
(\tilde{D} - \tilde{a} - \tilde{b}_2 \sigma) K = - (\frac{\tilde{H} d}{\chi}) DW, \quad (2.A.26)
\end{align*} \]

where \( \tilde{b}_1 = \frac{\sigma}{\chi} \) is the Prandtl number, \( \tilde{b}_2 = \frac{\sigma}{\chi} \) is the magnetic Prandtl number, \( \tilde{q} = \frac{\sigma}{\chi} \) is the Schmidt number,
and \( \rho = k_0 / d^2 \) is the dimensionless permeability. The boundary conditions (2. A. 21) transform to
\[
W = \frac{2}{3} W = 0, \quad \Theta = \gamma = K = 0 \text{ at } z = 0 \text{ and } z = 1. \quad (2. A. 27)
\]

Eliminating \( \Theta, \gamma \) and \( K \) between equations (2. A. 23) - (2. A. 26), we obtain
\[
R \frac{d^2}{d \sigma^2} \left( \frac{d^2}{d \tau^2} - \gamma \right) \left( \frac{d^2}{d \eta^2} - \gamma \right) \left( \frac{d^2}{d \tau^2} - \gamma \right) W = \left[ \frac{1}{\gamma^2} \left( 1 + \frac{\alpha_0 \eta d / \gamma}{\sigma + \frac{1}{2} d / \gamma} \right) \sigma + \frac{1}{\rho} \right] \cdot \left[ (d^2 + \gamma) (d^2 + \gamma - \gamma \eta^2) (d^2 + \gamma - \gamma \eta^2) \right] W
\]
\[
+ \sigma \frac{\partial}{\partial \eta} \left( \frac{d^2}{d \eta^2} - \gamma \right) \left( \frac{d^2}{d \eta^2} - \gamma \right) \frac{\partial W}{\partial \eta}
\]
\[
+ \frac{Q}{\gamma} \left( \frac{d^2}{d \eta^2} \right) \left( \frac{d^2}{d \eta^2} - \gamma \eta^2 \right) \left( \frac{d^2}{d \eta^2} - \gamma \eta^2 \right) \frac{\partial W}{\partial \eta}, \quad (2. A. 28)
\]
where \( R = \gamma \beta d / \nu \chi \) is the thermal Rayleigh number,
\( S = \gamma \alpha_0 \beta d / \nu \chi \) is the analogous solute Rayleigh number and \( Q = \eta_0 \gamma d / \nu \pi \tau \) is the Chandrasekhar number.

Dropping the stars for convenience and using the boundary conditions (2. A. 27), it can be shown with the help of equations (2. A. 23) - (2. A. 26) that all the even order derivatives of \( W \) vanish at the boundaries and, hence, the proper solution of equation (2. A. 28) characterizing the lowest mode is
\[
W = W_0 \sin \pi z, \quad (2. A. 29)
\]
where \( W_0 \) is a constant. Substituting (2. A. 29) in (2. A. 28) and letting \( \chi = \alpha_0 d / \gamma \), \( R_1 = R / \pi^4 \), \( S_1 = S / \pi^4 \), \( Q = Q / \pi \) and \( P = \pi^2 \rho \), we obtain the dispersion relation
\[
R_1 = \frac{(1 + \chi)(1 + \chi + \gamma \eta^2 \sigma / \rho)}{2 \chi} \frac{1}{\sigma^2} \left( 1 + \frac{\alpha_0 \eta d / \gamma}{\sigma + \frac{1}{2} d / \gamma} \right) \sigma + \frac{1}{P} \nabla
\]
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Here we examine the possibility of oscillatory modes, if any, on the stability problem due to the presence of magnetic field and solute gradient. Multiplying equation (2.A.23) by $W^*$, the complex conjugate of $W$, and using equations (2.A.24) - (2.A.26) together with boundary conditions (2.A.27), we obtain

\[
\begin{align*}
\frac{1}{\varepsilon} \left[ 1 + \frac{\alpha_0 \beta_0 \sigma \beta}{\sigma + \beta_0} \right] & + \frac{1}{\beta} \left[ I_1 + \frac{g \alpha x \alpha_i^2}{\beta} (I_4 + E q \sigma \gamma I_5) \right] \\
& + \frac{\mu_0 e \gamma}{4 \pi \beta} C (I_6 + \frac{1}{2} \sigma \gamma I_7) \\
= \frac{g \alpha x \alpha_i^2}{\beta} (I_2 + E q \sigma \gamma I_3),
\end{align*}
\]

where

\[
\begin{align*}
I_1 &= \int_0^1 (|D'\psi|^2 + |\psi|^2) \, dz, \\
I_2 &= \int_0^1 (|D\overline{\psi}|^2 + |\overline{\psi}|^2) \, dz, \\
I_3 &= \int_0^1 |\overline{\psi}|^2 \, dz, \\
I_4 &= \int_0^1 (|\overline{\gamma}|^2 + \overline{\alpha}|\gamma|^2) \, dz, \\
I_5 &= \int_0^1 |\gamma|^2 \, dz, \\
I_6 &= \int_0^1 (|D'k|^2 + 2 \alpha^2 |D'k|^2 + \alpha |k|^2) \, dz, \\
I_7 &= \int_0^1 (|Dk|^2 + \alpha^2 |k|^2) \, dz.
\end{align*}
\]
which are all positive definite. Substituting \( \sigma = \sigma_\tau + i \sigma_\omega \) and equating real and imaginary parts of equation (2.A.31) we obtain

\[
\sigma \left\{ \frac{1}{\varepsilon} \left[ 1 + \frac{\alpha_0 \delta d}{\sigma_\tau + \delta d^2} \right] I_1 - \frac{g \alpha \chi a^2}{\nu \beta} E \frac{1}{d} I_3 \right\} + \frac{g \alpha' \chi d^2}{\nu \beta} E q I_5 + \frac{\mu_\varepsilon \varepsilon \eta}{4 \pi \rho_\omega} b I_7
\]

\[-\frac{4 \frac{1}{\beta} I_1 + \frac{4}{\varepsilon} \left[ 1 + \frac{\sigma_\tau^2}{(\sigma_\tau + \delta d^2)^2 + \sigma_\omega^2} \right] I_1 - \frac{g \alpha \chi a^2}{\nu \beta} I_2 \right\}, \quad (2.A.33)

and

\[
\sigma \left\{ \frac{1}{\varepsilon} \left[ 1 + \frac{\alpha_0 \delta d}{\sigma_\tau + \delta d^2} \left( \frac{\sigma_\tau + \delta d^2}{\sigma_\tau + \delta d^2} \right) - \sigma_\tau^2 \right] I_1 + \frac{g \alpha \chi a^2}{\nu \beta} E \frac{1}{d} I_3 \right\} + \frac{g \alpha' \chi d^2}{\nu \beta} E q I_5 - \frac{\mu_\varepsilon \varepsilon \eta}{4 \pi \rho_\omega} b I_7 \right\} = 0. \quad (2.A.34)

It is clear from equation (2.A.33) that \( \sigma_\tau \) may be positive or negative, i.e. there may be stability or instability in the presence of solute gradient, magnetic field and collisional effects for porous medium, which is also true in their absence. Equation (2.A.34) implies that \( \sigma_\omega = 0 \) or \( \sigma_\omega \neq 0 \) which
means that the modes may be non-oscillatory or oscillatory. In the absence of stable solute gradient, magnetic field, and for \( \gamma > \gamma_0 \), \( \gamma = 0 \) which means that oscillatory modes are not allowed and the principle of exchange of stabilities is satisfied for porous medium in the absence of magnetic field, stable solute gradient, and for \( \gamma > \gamma_0 \). The presence of each magnetic field and stable solute gradient, brings oscillatory modes which were nonexistent in their absence. The collisional effects may also bring in oscillatory modes.

THE STATIONARY CONVECTION

For the stationary convection \( \sigma = 0 \) and equation (2.A.30) reduces to

\[
R_1 = \frac{G + \kappa^2}{x p} + \frac{Q}{\epsilon x} (G + \kappa^2).
\]

(2.A.35)

To study the effect of stable solute gradient, magnetic field, and medium permeability on \( R_1 \) we examine the nature of \( \frac{dR_1}{ds_1} \), \( \frac{dR_1}{dQ_1} \), and \( \frac{dR_1}{d\rho} \) analytically. Equation (2.A.35) yields

\[
\frac{dR_1}{ds_1} = +1,
\]

(2.A.36)

\[
\frac{dR_1}{dQ_1} = \frac{G + \kappa^2}{\epsilon x},
\]

(2.A.37)
The stable solute gradient and the magnetic field have stabilizing effects whereas medium permeability has a destabilizing effect on thermosolutal instability in porous medium in a partially ionized plasma.

THE OVERSTABLE CASE

Here we discuss the possibility of whether instability may occur as an overstability. Let us put \( \sigma \mid \lambda = i \sigma \) remembering that \( \sigma \) may be complex. Since for overstability, we wish to determine the critical Rayleigh number for the onset of instability via a state of pure oscillations, it suffices to find conditions for which (2.A.30) will admit of solutions with \( \sigma \) real. Equation (2.A.30) becomes

\[
\frac{dR_1}{d\lambda} = \frac{(1+x)}{\lambda} \left[ i \sigma \left( 1 + \alpha_0 \frac{\partial f_1}{\partial \lambda} \right) + \frac{1}{p} \right]
\]

\[
+ \frac{\alpha_0 (1+x) + i \epsilon E_1 \sigma_1}{\lambda} \left[ \frac{Q_1 (1+x)}{\epsilon (1+x + i \epsilon E_1 \sigma_1)} \right] \cdot (2.A.39)
\]

It follows from equation (2.A.39) that

\[
\frac{dS_1}{d\lambda} = \frac{(1+x + i \epsilon E_1 \sigma_1)}{(1+x + i \epsilon E_1 \sigma_1)}.
\]
The imaginary part of equation (2.A.40) equated to zero yields $E_{b1} = E_{l1}$. Substituting $E_{b1} = E_{l1}$ in the real part of equation (2.A.40) gives (2.A.36). Similarly, it can be easily shown that equation (2.A.39) yields (2.A.37) and (2.A.38).

Thus the stable solute gradient and the magnetic field have stabilizing effects whereas medium permeability has a destabilizing effect on thermosolutal convection in porous medium in a partially ionized plasma also for the overstable case.

Equation (2.A.39) also yields

$$\frac{d R_4}{d \psi} = - \left( \frac{\sigma_2^2 \alpha_0 \Phi}{J} \right) \left( 1 + \chi \right) \left[ (1+\chi) \left( \frac{\Phi_2}{\Pi \sigma_2^2} + \frac{\Phi_3 \delta \nu_2}{\sigma_2^2} + 2 E_{l1} \sigma_2^2 \frac{\Phi_1}{\nu_2^2} \right) \right]$$

$$+ i \sigma_1 \left[ E_{b1} \left( - \frac{\Phi_4}{\Pi \sigma_1^2} + \frac{\delta \nu_2}{\nu_2^2} \right) - 2 (1+\chi) \frac{\Phi_3 \delta \nu_2}{\nu_2^2} \right]$$

$$\times \left[ \left( - \frac{\Phi_4}{\Pi \sigma_1^2} + \frac{\delta \nu_2}{\nu_2^2} \right) + \frac{L_1 \lambda_1 \lambda_2 \sigma_2^2}{\nu_2^2} \sigma_1^2 \right]^{-1}. \quad (2.A.41)$$

Substituting the imaginary part of equation (2.A.41) equated to zero in the real part of equation (2.A.41) yields

$$\frac{d R_4}{d \psi} = - \left( \frac{\sigma_2^2 \alpha_0 \Phi}{J} \right) \left( \frac{\sigma_2^2 \alpha_0 \Phi}{J} \right).$$
It is clear from equation (2.4.2) that the collisional frequency has a destabilizing effect on the thermosolutal convection in porous medium in a partially ionized plasma.

\[
\begin{aligned}
\cdot \left(1 + \frac{1}{\epsilon x}\right) \\
\frac{(1+\epsilon^2)}{\left(-\pi \sigma_1^2 + \frac{2}{\lambda_1^2} \right)} \\
\frac{\frac{1}{\delta_1^2}}{\left(-\pi \sigma_1^2 + \frac{2}{\lambda_1^2} \right)} + \frac{4\pi \sigma_1^2}{\lambda_1^2} \delta_1^2 \sigma_1^2
\end{aligned}
\]

(2.4.2)
INTRODUCTION

The thermal instability in hydromagnetics has been treated in detail by Chandrasekhar (1981). The thermohaline convection in layer of fluid heated and salted from below has been investigated by Veronis (1965). The physics is quite similar in the stellar case in that helium acts like salt in raising the density and in diffusing more slowly than heat. The Boussinesq approximation has been used throughout, which states that the density may be treated as a constant in all the terms in the equation of motion except the one in the external force. The approximation is well justified in the case of incompressible fluids but when the fluids are compressible, the equations governing the system become quite complicated. To simplify the set of equations governing the flow of compressible fluids, Spiegel and Veronis (1960) have made the following assumptions:

a) The fluid layer depth is much smaller than the scale height as defined by Spiegel and Veronis (1960) and
b) The variations in temperature, pressure and density, introduced by the motion, do not exceed their total static variations.

It has been shown by Spiegel and Veronis (1960) under the above assumptions, that the flow equations are the same those for incompressible fluids except that the static temperature gradient is replaced by its excess over the adiabatic and \( c_v \) is replaced by \( c_p \). The thermal instability of compressible fluids in the presence of uniform rotation and magnetic field, separately has been studied by Sharma (1977).

A partially ionized plasma represents a state which often exists in the Universe and there are several situations in cosmic physics in which the interaction between the ionized hydrogen is limited to certain sharply bounded regions in space surrounding, for example, O-type stars and clusters of such stars and that the gas outside these regions is essentially non-ionized. Other examples of the existence of such situations are given by Alfvén (1954) theory of the origin of planetary system, in which a high ionization rate is suggested to appear from collisions between a plasma and a neutral gas cloud and by the absorption of plasma waves due to ion-neutral collisions such as in the solar photosphere and chromosphere and in cool
interstellar clouds [Lehnert (1959), Piddington (1954)]. Hans (1968) has shown that these collisions stabilize the Rayleigh-Taylor instability. For the Kelvin-Helmholtz configuration, Rao and Kalra (1967) and Hans (1968) have found that the collisional effects are in fact destabilizing for a sufficiently large collision frequency. The thermal instability of a compressible and partially ionized plasma has been studied by Sharma and Misra (1986). The medium has been considered to be non-porous in all the above studies.

The physical properties of meteorites, comets and interplanetary dust strongly suggest the importance of porosity in astrophysical context [McDonnel (1978)]. The stability of convective flow in hydromagnetics in a porous medium using Rayleigh's procedure has been studied by Lapwood (1948). A macroscopic equation which describes flow of a fluid through a homogeneous and isotropic porous medium is the Darcy's equation. As a result, the usual viscous term in the equation of fluid motion is replaced by the resistance term \(-\left(\frac{\mu}{k}\right)\mathbf{q}\) where \(\mu\), \(k\) and \(\mathbf{q}\) stand for fluid viscosity, medium permeability, and filter velocity of the fluid.

Keeping in mind the relevance and importance in astrophysics (interplanetary and stellar atmospheres) and geophysics (stability of Earth's core and geothermal regions), the present section deals with the compressibility and collisional effects on
thermosolutal instability of a partially ionized plasma in porous medium.

**FORMULATION OF THE PROBLEM AND PERTURBATION EQUATIONS**

Consider an infinite, horizontal, compressible and composite plasma layer of thickness $d$ consisting of an electrically conducting hydromagnetic fluid of density $\rho$ permeated with neutrals of density $\rho_0$ in porous medium and acted on by gravity field $\mathbf{g}(0,0,-g)$ and magnetic field $\mathbf{H}(0,0,H)$. This layer is heated and soluted from below such that uniform adverse temperature gradient $\beta (\equiv \frac{dT}{dz})$ and uniform adverse solute gradient $\beta (\equiv \frac{DC}{dz})$ are maintained. Assume that the ionized fluid and the neutral gas behave like continuum fluids and that effects on the neutral component resulting from the fields of pressure, gravity and Darcy resistance are neglected. The magnetic field interacts with the hydromagnetic component only. Then the equations governing the motion of the composite plasma in porous medium, following Boussinesq approximation, are

\[
\frac{1}{c} \left[ \frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} \right] = -\frac{1}{\rho_m} \nabla p + \mathbf{g} (1 + \frac{\rho_p}{\rho_m}) + \frac{\mu e}{4 \pi \eta_m} (\nabla \times \mathbf{H}) \times \mathbf{H} + \frac{\kappa}{\rho_m c} \left( \frac{\nabla \cdot \mathbf{v}}{c} - \mathbf{v} \right), \quad (2.B.1)
\]

\[
\nabla \cdot \mathbf{v} = 0, \quad (2.B.2)
\]

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\begin{align*}
\frac{\partial q_d}{\partial t} + (\mathbf{v} \cdot \nabla) q_d &= -\varepsilon (\varphi_d - \varepsilon), \\
E \frac{\partial T}{\partial t} + (\mathbf{v} \cdot \nabla) T &= \chi T, \\
E \frac{\partial C}{\partial t} + (\mathbf{v} \cdot \nabla) C &= \chi C, \\
\nabla \cdot \mathbf{H} &= 0, \\
E \frac{\partial H}{\partial t} &= (\mathbf{H} \cdot \nabla) \mathbf{H} + \varepsilon \eta \nabla \times \mathbf{H},
\end{align*}

where \( q_d, \varphi_d, \mathbf{v}, \mu, \lambda, \gamma, \varepsilon, \chi, \chi' \) and \( k_1 \) stand for velocity of the neutral gas, density of the neutral gas, mutual collisional (frictional) frequency between the two components of the composite medium, viscosity of hydromagnetic fluid, magnetic permeability, electrical resistivity, medium porosity, thermal diffusivity, analogous solute diffusivity and medium permeability respectively. \( \varepsilon = \varepsilon + (1 - \varepsilon) \frac{F_m}{F_p} \) where \( F_n \) and \( F_p, C_s \) stand for density and heat capacity of fluid and solid matrix respectively. \( E' \) is a solute parameter analogous to \( E \).

Spiegel and Veronis (1960) defined \( f \) as one of the state variables \([\text{density } \rho, \text{ pressure } p \text{ or temperature } T]\) and expressed these in the form

\[ f(x, y, z, t) = f_m + f_0(z) + f'(x, y, z, t), \quad (2.8) \]
where \( f_m \) is the constant space average of \( f \), \( f_o \) is the variation in the absence of motion and \( f' \) is the fluctuation resulting from motion.

The initial state is, therefore, a state in which the pressure, density, temperature, solute concentration and velocity at any point in the plasma are given by

\[
p = p(z), \quad T = T(z), \quad C = C(z), \quad \delta = 0, \quad (2.B.9)
\]

respectively, where

\[
\begin{align*}
T(z) &= T_0 - \beta(z), \\
T(z) &= T_0 - \beta(z), \\
\beta(z) &= \beta_m - \frac{q}{\Delta} \int \left( f_m + f_o \right) \, dz, \\
p(z) &= p_m \left[ 1 - \alpha_m (T - T_m) + \alpha' (C - C_m) + \kappa_m (b + b_m) \right], \\
K_m &= \left( \frac{\partial \rho}{\partial z} \right)_m, \\
\alpha_m' &= -\left( \frac{\partial \rho}{\partial z} \right)_m.
\end{align*}
\]

\[(2.B.10)\]

Then the linearized perturbation equations governing the motion of the compressible and composite plasma, in porous medium, are

\[
\begin{align*}
\frac{1}{c} \frac{\partial \rho}{\partial t} &= - \frac{1}{p_m} \nabla \delta \rho + \frac{\delta p}{p_m} \left( \frac{\partial p}{\partial \rho} \right)_m - \frac{\rho L}{k} \delta \rho \\
&+ \frac{1}{4n p_m} \left( \nabla \cdot \delta \right) x_{\rho} \rho + \frac{f_d x \epsilon}{p_m} \left( \delta q - \delta \rho \right), \\
\nabla q &= 0,
\end{align*}
\]

\[(2.B.11)\]

\[(2.B.12)\]
\[
\frac{d^2 v_d}{dt^2} = - \gamma (r_d, \varphi^2) ,
\]
(2.8.13)
\[
\frac{\partial \varphi}{\partial t} = \left( \eta - \frac{3}{c_p} \right) \varphi + \chi^2 \varphi ,
\]
(2.8.14)
\[
E \frac{\partial \psi}{\partial t} = \frac{1}{c_p} \psi + \chi^2 \psi ,
\]
(2.8.15)
\[
\frac{\partial \psi}{\partial t} = 0 ,
\]
(2.8.16)
\[
\frac{\partial \psi^2}{\partial t} = \left( \eta^2 \psi \right)^2 + \epsilon \eta^2 \psi ,
\]
(2.8.17)

where \( \delta \rho \), \( \delta p \), \( \bar{q}(u, v, w) \), \( \bar{q}_d(l, m, n) \), \( \bar{h}(h_x, h_y, h_z) \), \( \Theta \) and \( \gamma \) denote, respectively, the perturbations in density \( \rho \), pressure \( p \), fluid velocity \((0,0,0)\), neutral velocity \((0,0,0)\), magnetic field \( \hat{h} \), temperature \( T \) and solute concentration \( C \). \( g/c_p \) stands for adiabatic gradient. \( \alpha_m^{(= \alpha)} \) is the coefficient of thermal expansion and \( \alpha_m^{(= \alpha)} \) is the analogous solute coefficient. The equation of state is
\[
\rho = \rho_m \left[ 1 - \alpha \left( T - T_m \right) + \alpha' \left( C - C_m \right) \right] .
\]
(2.8.18)

The change in density \( \delta \rho \), caused mainly by the perturbations in temperature and solute concentration, is given by
\[
\delta \rho = - \rho_m \left( \alpha \Theta - \alpha' \gamma \right) .
\]
(2.8.19)

Eliminating \( \bar{q}_d \) between (2.8.11) and (2.8.13), equations (2.8.11) - (2.8.17) yield
Consider the case of plasma layer with two free boundaries and the adjoining medium to be electrically nonconducting. The boundaries are assumed to be perfect conductors of both heat and solute. The case of two free boundaries is the most appropriate for stellar atmospheres (Spiegel 1965). The boundary conditions appropriate for the problem are

\[ \begin{align*}
\omega &= 0 = \gamma = \frac{\partial \omega}{\partial z} = \frac{\partial \gamma}{\partial z} = 0 \quad \text{at } z=0 \text{ and } z=1, \\
\xi = 0 \quad \text{and } h_x, h_y, h_z \text{ are continuous}
\end{align*} \tag{2.B.24} \]

\( \xi = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \) is the z-component of vorticity and

\( \eta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \) is the z-component of current density.

In the absence of any surface current, tangential components of magnetic field are continuous. Hence, the boundary conditions in addition to (2.B.24) are

\[ \frac{\partial h_z}{\partial z} = 0 \quad \text{on the boundaries.} \tag{2.B.25} \]
**DISPERSION RELATION**

We assume that the perturbation quantities are of the form

\[
[w,0,0,h_z] = [w(x),0,0,h_z] = \begin{bmatrix} k_x \exp(ik_x x + ik_y y + i\omega t) \end{bmatrix}, \quad (2.B.26)
\]

where \( \omega \) is, in general, a complex constant; \( k_x \) and \( k_y \) are the wave numbers along the \( x \)- and \( y \)-directions and \( k = (k_x^2 + k_y^2)^{1/2} \) is the resultant wave number.

Using (2.B.26), equations (2.B.20) - (2.B.23) in nondimensional form become

\[
\frac{1}{\varepsilon} \left( 1 + \alpha \frac{x \partial}{\sigma + \frac{1}{2} \frac{d}{d\zeta}} \right) W - \frac{1}{\rho c_p} (\partial^2 - \frac{d^2}{d\zeta^2}) \frac{\partial^2 \sigma}{\partial \zeta^2} = 0, \quad (2.B.27)
\]

\[
(\partial^2 - \frac{d^2}{d\zeta^2}) \frac{\partial^2 \sigma}{\partial \zeta^2} = -\frac{\sigma}{\varepsilon} \frac{\partial}{\partial \zeta} (G - 1) W, \quad (2.B.28)
\]

\[
(\partial^2 - \frac{d^2}{d\zeta^2}) \Gamma = -\left( \frac{\partial^2}{\partial \zeta^2} \right) W, \quad (2.B.29)
\]

\[
(\partial^2 - \frac{d^2}{d\zeta^2}) \frac{\partial^2 \sigma}{\partial \zeta^2} = -\left( \frac{\partial^2}{\partial \zeta^2} \right) DW, \quad (2.B.30)
\]

where \( G = (c_p / g) \sigma \). Here we have expressed \( x, y, z \) in the new unit of length \( d \) and put \( \sigma = kd \), \( \varepsilon = \eta d \). The boundary conditions (2.B.24) and (2.B.25) transform to

\[
W = \frac{2}{\gamma} = 0, \quad \Gamma = 0, \quad \text{at } z = 0 \text{ and } 1. \quad (2.B.31)
\]

Using (2.B.31), it can be shown that all the even order derivatives of \( W \) must vanish for \( z = 0 \) and \( z = 1 \).
and hence the proper solution of W characterizing the lowest mode is

\[ W = W_0 \sin \pi z, \quad (2.8.32) \]

where \( W_0 \) is a constant.

Eliminating \( \Theta, \Gamma \) and \( K \) between (2.8.27) - (2.8.30), putting the proper solution (2.8.32) in the resulting equation and letting \( x = \delta/k^2 \), \( R_1 = \frac{g\alpha_0 B^2}{\omega x L^4} \), \( S_1 = \frac{g\alpha_0 B^2}{\omega x L^4} \), \( \sigma_1 = \frac{\sigma}{k^2} \),

\[ Q_1 = \frac{\gamma^2}{4\pi \rho_0 \omega} \pi^2 \]

and \( \rho = \frac{\pi \rho_0}{2} \), we obtain the dispersion relation

\[ R_1 = \frac{G}{G-1} \left[ \frac{(1+x)(1+x+iL\beta x)}{\pi} \right] \left[ \frac{i\sigma_1}{\epsilon} \left( 1 + \frac{\alpha_0^2 \beta^2}{\pi \sigma_1^2 + x^2 \omega^2} \right) + \frac{1}{P} \right]
\]

\[ + S_1 \frac{(1+x+iL\beta x)}{1+x+iL\beta x} + Q_1 \frac{\epsilon}{\pi \rho_0 \omega} \left( 1 + \frac{x}{\pi \sigma_1^2 + x^2 \omega^2} \right) \]

\[ (2.8.33) \]

**THE STATIONARY CONVECTION**

For the stationary convection, \( \sigma = 0 \) and equation (2.8.33) reduces to

\[ R_1 = \frac{G}{G-1} \left[ \frac{(1+x)^2}{\pi P} + S_1 + Q_1 \frac{(1+x)}{\epsilon} \right], \quad (2.8.34) \]

This expresses the modified Rayleigh number \( R_1 \) as a function of dimensionless wave number \( x \) and the parameters \( S_1, Q_1, G \) and \( P \). For fixed values of \( Q_1, S_1 \) and \( P \), let the nondimensional number \( G \) accounting for the compressibility effects be also kept as fixed. Then we find that
\[ R_c = \left( \frac{G}{G-1} \right) R_c \]  

(2.B.35)

where \( R_c \) and \( R_c \) denote, respectively, the critical Rayleigh numbers in the presence and absence of compressibility. Thus the effect of compressibility is to postpone the onset of thermosolutal instability. The case \( G>1 \) is relevant here as \( G<1 \) and \( G=1 \) correspond to negative and infinite Rayleigh numbers. Hence we obtain a stabilizing effect of compressibility. It is evident from equation (2.B.34) that

\[ \frac{dR_i}{dS_i} = \frac{G}{G-1} \]  

(2.B.36)

\[ \frac{dR_1}{dQ_1} = \left( \frac{G}{G-1} \right) \left( \frac{1+\frac{C_1}{\epsilon}}{\epsilon} \right) \]  

(2.B.37)

and

\[ \frac{dR_i}{d\rho} = -\left( \frac{G}{G-1} \right) \left( \frac{1+\frac{C_1}{\epsilon}}{\epsilon} \right) \frac{\lambda^2}{\rho^2} \]  

(2.B.38)

The stable solute gradient and magnetic field, therefore, have stabilizing effects whereas the medium permeability has a destabilizing effect on thermosolutal instability of a compressible, partially ionized plasma in porous medium.
THE OVERSTABLE CASE

Here we discuss the effects of collisional frequency, magnetic field, medium permeability and solute gradient for the overstable case. We have put $\sigma = i\sigma_1$ in equation (2.8.33), it being remembered that $\sigma_1$ is complex. Since for overstability, one wishes to determine the critical Rayleigh number for the onset of instability via a state of pure oscillations, $\sigma_1$ is real in equation (2.8.33) for the overstable case. It follows from equation (2.8.33) that

$$\frac{dR_1}{dx} = - \frac{G}{G-1} \left( \frac{1+x}{\epsilon \chi} \right) \alpha_0 \frac{\partial^2 \phi}{\partial x^2} \left[ \left( 1 + \chi \right) \left( -\frac{\partial^2 \phi}{\partial x^2} + 2 \frac{\partial^2 \phi}{\partial y^2} \right) + \frac{2}{b_1} \frac{\partial^2 \phi}{\partial y^2} \right]$$

$$+ \frac{\partial^2 \phi}{\partial x^2} \left[ \left( -\frac{\partial^2 \phi}{\partial x^2} + \frac{2}{b_1} \frac{\partial^2 \phi}{\partial y^2} \right) - 2 \left( 1 + \chi \right) \frac{2}{b_1} \frac{\partial^2 \phi}{\partial y^2} \right]$$

Substituting the imaginary part of equation (2.8.39) equated to zero in the real part of equation (2.8.39) yields

$$\frac{dR_1}{dx} = - \left[ \frac{G}{G-1} \left( \frac{2 \pi \sigma_1}{\epsilon \chi} \right) \frac{\partial^2 \phi}{\partial x^2} \left( 1 + \chi \right) \frac{2 \sigma_1}{\partial y^2} \right].$$

$$\left[ \left( 1 + \chi \right) + \frac{2}{b_1} \frac{2}{\sigma_1} \frac{\partial^2 \phi}{\partial x^2} \right] \left[ \left( -\frac{\partial^2 \phi}{\partial x^2} + \frac{2}{b_1} \frac{\partial^2 \phi}{\partial y^2} \right) \frac{2}{\sigma_1} \frac{\partial^2 \phi}{\partial y^2} \right]$$

(2.8.40)
The collisional frequency, thus, has a destabilizing effect on the thermosolutal instability of a compressible, partially ionized plasma in porous medium. Equation (2.B.33) also yields

\[ \frac{dR_t}{dp} = -\left( \frac{G}{G-1} \right) \left( \frac{1+x}{x} \right) \left( \frac{1+x+iE}{p^2} \right). \]  

The imaginary part equated to zero gives \( \eta = 0 \). Putting \( \eta = 0 \) in the real part of equation (2.B.41) gives

\[ \frac{dR_t}{dp} = -\frac{G}{G-1} \left( \frac{1+x^2}{x} \right). \]  

Similarly it can be shown that equation (2.B.33) yields (2.B.36) and (2.B.37).

Thus the medium permeability has a destabilizing effect whereas the magnetic field and the stable solute gradient have stabilizing effects on thermosolutal instability of a compressible, partially ionized plasma in porous medium for the overstable case also.

**THE OSCILLATORY MODES**

In this section, we examine the possibility of oscillatory modes, if any, on the stability problem due to the presence of collisions, magnetic field and solute gradient. Multiplying equation (2.B.27) by \( W^* \), the complex conjugate \( W \) and using equations (2.B.28)-
(2.B.30) together with the boundary conditions (2.B.31), we obtain

\[
\left[ \frac{1}{\varepsilon} \left( 1 + \frac{\alpha_0 \lambda d}{\sigma + \alpha_2 d^2} \right) + \frac{1}{P_l} \right] I_1 + \frac{C_l}{4\pi \rho_m \nu} \left( I_3 + \frac{b_2 \sigma^2}{2} I_7 \right) + \frac{\alpha_0^2 \lambda^2}{2 \rho_l^2} \left( I_4 + E_q \sigma^2 I_5 \right) = \frac{C_p x_1 \alpha^2}{2 (G-1)} \left( I_2 + \frac{b_1 \sigma^2}{2} I_3 \right),
\]

where

\[
\begin{align*}
I_1 &= \int_0^1 \left( |Dw|^2 + \tilde{d} |\tilde{w}|^2 \right) dz, \\
I_2 &= \int_0^1 \left( |D\tilde{w}|^2 + \tilde{d} |\tilde{w}|^2 \right) dz, \\
I_3 &= \int_0^1 |\tilde{w}|^2 dz, \\
I_4 &= \int_0^1 \left( |D\tilde{w}|^2 + \tilde{d} |\tilde{w}|^2 \right) dz, \\
I_5 &= \int_0^1 |\tilde{w}|^2 dz, \\
I_6 &= \int_0^1 \left( |\tilde{w}|^2 + \tilde{d} |\tilde{w}|^2 \right) dz, \\
I_7 &= \int_0^1 \left( |\tilde{w}|^2 + \tilde{d} |\tilde{w}|^2 \right) dz,
\end{align*}
\]

which are all positive definite. \( \sigma^* \) is the complex conjugate of \( \sigma \). Putting \( \sigma = \sigma_1 + i \sigma_2 \) in equation (2.B.43) and then equating the real and imaginary parts, we obtain

\[
\sigma_1 \left[ \frac{1}{\varepsilon} \left( 1 + \frac{\alpha_0 \lambda d}{\sigma + \alpha_2 d^2} \right) \right] I_1 + \frac{C_p x_1 \alpha^2}{2 (G-1)} \frac{b_1 \sigma^2}{2} I_3 \]
\begin{align*}
&+ \frac{g \alpha' \delta^2}{\nu \beta^2} \left[ \epsilon \beta I_5 + \frac{\epsilon \eta}{4 \pi \rho_0 m \nu} \right] \beta I_7 \\
&= - \left[ \frac{\alpha}{\nu} + \left\{ 1 + \frac{\sigma^2}{(\sigma^2 + \omega^2)^2} \right\} \right] \frac{I_1}{\epsilon} \\
&+ \frac{c_p \alpha \delta^2}{\nu (1 - G)} I_2 + \frac{g \alpha' \delta^2}{\nu \beta^2} I_4 + \frac{\epsilon \eta}{4 \pi \rho_0 m \nu} I_6 \right], \tag{2.B.45}
\end{align*}

and

\begin{align*}
&\sigma \left\{ 1 + \frac{\omega c d/\omega}{(\sigma^2 + \omega^2)^2} \right\} \frac{I_1}{\epsilon} \\
&+ \frac{c_p \alpha \delta^2}{\nu (1 - G)} \left( \epsilon \beta I_3 - \frac{g \alpha' \delta^2}{\nu \beta^2} \left( \epsilon \beta I_5 - \frac{\epsilon \eta}{4 \pi \rho_0 m \nu} \beta I_7 \right) \right) = 0. \tag{2.B.46}
\end{align*}

It is evident from (2.B.46) that \( \sigma = 0 \) or \( \sigma \neq 0 \), which means that the modes may be non-oscillatory or oscillatory. In the absence of stable solute gradient, magnetic field and for \( \omega > \omega \sqrt{\alpha_0 \delta^2} \), \( \sigma \neq 0 \). This means that oscillatory modes are not allowed and the principle of exchange of stabilities is satisfied for the thermosolutal instability of a compressible, composite plasma in porous medium in the absence of stable solute gradient, magnetic field and for \( \omega > \omega \sqrt{\alpha_0 \delta^2} \).

The presence of each stable solute gradient and magnetic field brings oscillatory modes which were nonexistent in their absence. The collisional effects may also bring oscillatory modes in the system.

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Equation (2.B.45) simply tells that $\sigma_{1}$ may be positive or negative i.e. there may be instability or stability in the presence of stable solute gradient, magnetic field and collisional effects in porous medium, which is also true in their absence.