Chapter 4

Ageing Concepts*

4.1 Introduction

The notion of ageing plays an important role in reliability analysis and in identifying life distributions. Ageing represents the phenomenon by which the residual life of a unit is affected by its age in some probabilistic sense. Most of the ageing concepts exist in the literature are described on the basis of measures defined in terms of the distribution function. It is seen in Chapter 3 that many quantile functions can be utilized for the lifetime data analysis. When one wishes to analyze the ageing properties of such models, the existing definitions based on distribution function are not adequate. Thus, as a follow up to quantile-based analysis, in the present chapter, we review the existing definitions and express them in terms of quantile functions to facilitate a quantile-based analysis. The definitions and the properties of the basic ageing classes using the distribution function have been taken from Lai and Xie (2006) and the references for others are given in the text at the appropriate places. The ageing concepts are studied in three broad heads, those based on hazard quantile function, residual quantile functions and quantile functions. We also illustrate the various ageing concepts in the case of quantile functions.

*Part of the contents of this chapter has been published in Nair and Vineshkumar (2011), Statistics and Probability Letters (see reference no 92).
The objectives of the work in the present chapter are manifold. Firstly it enables understanding of the failure mechanism of the unit under observation, through a distribution modelled by an appropriate quantile function. Secondly various concepts generate classes of life distributions, so that the identification of the model can be limited to that particular class. Lastly we have a new methodology that paves way for different kinds of analysis.

4.2 Ageing concepts based on hazard quantile function

Recalling the notations introduced in Chapter 2, a random variable $X$ or its distribution function represented by $F(x)$ is said to be increasing hazard rate (IHR) (decreasing hazard rate (DHR)) if the hazard function $h(x)$ is increasing (decreasing). In terms of survival function, $F$ is IHR (DHR) if and only if for all $t$ the survival function of the residual life,

$$\tilde{F}_t(x) = \frac{F(x+t)}{F(t)}$$

is decreasing in $t$ for all $x \geq 0$. The following proposition describes this concept based on quantile function. Since we are seeking equivalent conditions as in the conventional definition, the same names for the various concepts will be retained for the ageing classes under the quantile approach also.

**Proposition 4.1**

*The random variable $X$ has increasing hazard quantile function (IHR) (decreasing hazard quantile function (DHR)) if and only if any of the following equivalent conditions hold*

(i) $H(u_2) \geq (\leq) H(u_1)$ for all $u_2 \geq u_1, \ 0 \leq u_1, u_2 < 1$

(ii) $Q(v + (1-v)u) - Q(v)$ is decreasing (increasing) function of $v$
(iii) $H'(u) \geq (\leq) 0$, provided $H(u)$ is differentiable.

**Proof:** (i) follows from the condition $h(x_2) \geq (\leq) h(x_1)$ for all $x_2 \geq x_1$ for IHR (DHR), by setting $x_i = Q(u_i), i = 1, 2$. To prove (ii) we first note that $X$ is IHR (DHR) if and only if the survival function

$$F_t(x) = \frac{F(x + t)}{F(t)}$$

of the residual life $X_t = (X - t | X > t)$ is decreasing in $t$. The quantile function of $X_t$ is $Q(v + (1 - v)u) - Q(v)$. Assuming (ii), $Q(v + (1 - v)u) - Q(v)$ is decreasing in $v$

$$\Leftrightarrow q(v + (1 - v)u)(1 - u) \leq q(v)$$

$$\Leftrightarrow \frac{1}{q(v)} \leq \frac{1}{(1-u)q(v + (1 - v)u)}$$

$$\Leftrightarrow \frac{1}{(1-v)q(v)} \leq \frac{1}{[1 - (v + (1 - v)u)]q(v + (1 - v)u)}$$

$$\Leftrightarrow H(v) \leq H(v + (1 - v)u)$$

$$\Leftrightarrow X$$

is IHR, by (i)

Condition (iii) is obvious from

$$\frac{d}{dx} h(x) \geq 0 \Leftrightarrow \frac{d}{dQ(u)} h(Q(u)) \geq 0 \Leftrightarrow \frac{H'(u)}{q(u)} \geq 0$$

and $q(u) > 0$ since $Q(u)$ is an increasing function.

**Remark 4.1:** When $H'(u) = 0$ for all $u$, $X$ is exponential.

**Remark 4.2:** The distribution $F$ is bathtub failure rate distribution (BT) (upside down bathtub failure rate distribution (UBT)) if and only if $h'(x) < (>) 0$ for $x$ in $(0, x_0)$, $h'(x_0) = 0$ and $h'(x) > (<) 0$ in $(x_0, \infty)$. In terms of quantile function, if $H'(u) < (>) 0$ in $[0, u_0)$, $H'(u_0) = 0$ and
$H'(u) > (<) 0$ in $(u_0,1)$, we say that the hazard quantile function is bathtub-shaped (BT) (upside-down bathtub-shaped (UBT)).

**Example 4.1** From (3.7), the hazard quantile function of the lambda distribution by Ramberg and Schmeiser (1974) has the simple expression

$$H(u) = \lambda_2 (1 - u)^{-1} \left[ \lambda_3 u^{\lambda_3 - 1} + \lambda_4 (1 - u)^{\lambda_4 - 1} \right]^{-1}.$$  

Its derivative is

$$H'(u) = \frac{\lambda_2 \left[ \lambda_3 u^{\lambda_3 - 2}(\lambda_3 u - \lambda_3 + 1) + \lambda_4^2 (1 - u)^{\lambda_4 - 1} \right]}{\left[ \lambda_3 u^{\lambda_3 - 1}(1 - u) + \lambda_4 (1 - u)^{\lambda_4} \right]^2},$$

the sign of which depends on

$$g(u) = \lambda_2 \left[ \lambda_3^2 (1 - u)^{\lambda_3 - 1} + \lambda_3 u^{\lambda_3 - 2}(\lambda_3 u + 1 - \lambda_3) \right].$$

The distribution accommodates increasing, decreasing, BT and UBT shaped hazard quantile functions. To illustrate this we consider some special cases. When $\lambda_3 = 0$, the distribution is IHR if $\lambda_2 > 0$ and DHR if $\lambda_2 < 0$ subject to $\lambda_1 - \frac{1}{\lambda_2} > 0$. Setting $\lambda_4 = 0$,

$$g(u) = \lambda_2 \left[ \lambda_3^2 (1 - u)^{\lambda_3 - 1} \right].$$

In this case $H(u)$ is increasing for points in $(\lambda_2 > 0, 0 < \lambda_3 < 1)$ and BT for values in $(\lambda_2 > 0, \lambda_3 > 1)$ with change point $u_0 = \frac{\lambda_3 - 1}{\lambda_3}$. Finally let $\lambda_3 = 2$, $\lambda_4 = 1$,

$$g(u) = \lambda_2 (4u - 1),$$

so that when $\lambda_2 > 0$, $g(u) = 0$ at $u = \frac{1}{4}$ and the distribution is UBT with change point at $u_0 = \frac{1}{4}$. In Figure 4.1 the shapes of hazard quantile function for some selected values of parameters are presented. The
shapes of hazard quantile function for arbitrary choice of parameter values of other quantile models are given in the appendix of this chapter.

Figure 4.1-Shapes of hazard quantile function when (1) $\lambda_1 = 1$, $\lambda_2 = 100$, $\lambda_3 = 0.05$, $\lambda_4 = 0.5$, (2) $\lambda_1 = 0$, $\lambda_2 = -1000$, $\lambda_3 = 0, \lambda_4 = -2$ (3) $\lambda_1 = 1$, $\lambda_2 = 10$, $\lambda_3 = 2$, $\lambda_4 = 0$, (4) $\lambda_1 = 0$, $\lambda_2 = -1000$, $\lambda_3 = -2$, $\lambda_4 = -1$.

Another basic concept is increasing (decreasing) average hazard rate- IHRA (DHRA) defined by the condition $\frac{1}{x} \int_0^x h(t)dt$ is increasing (decreasing) in $x$. Equivalently the distribution is said to be IHRA (DHRA) if $-\left(\frac{1}{x}\right) \log F(x)$ is increasing (decreasing) in $x \geq 0$. In this connection we have the following proposition.

**Proposition 4.2**

We say that $X$ is IHRA (DHRA) if and only if any one of the following conditions are satisfied

(i) $\frac{\int_0^u H(p)q(p)dp}{\int_0^u q(p)dp} = \frac{1}{Q(u)} \int_0^u H(p)q(p)dp = \frac{-\log(1-u)}{Q(u)}$ is increasing (decreasing)

(ii) $H(u) \geq \left(\leq\right) \frac{-\log(1-u)}{Q(u)}$.  

Proof: Recall that $X$ is IHRA if and only if \( \frac{1}{x} \int_0^x h(t)dt \) is increasing. Now

\[
\frac{1}{x} \int_0^x h(t)dt \text{ is increasing } \iff \frac{1}{Q(u)} \int_0^u H(p)q(p)dp \text{ is increasing},
\]

by setting $x = Q(u)$ and $t = Q(p)$, completes the proof of (i).

To prove (ii),

\[
\frac{1}{Q(u)} \int_0^u H(p)q(p)dp \text{ is increasing } \iff Q(u)H(u)q(u)q(u)\int_0^u H(p)q(p)dp \geq 0
\]

\[
\iff Q(u)H(u) - \int_0^u (1 - p)^{-1} dp \geq 0
\]

\[
\iff H(u) \geq - \frac{1}{Q(u)} \log(1 - u).
\]

Remark 4.3 In the quantile formula IHRA takes into consideration a weighted average of $H(p)$ with weight $\frac{q(p)}{Q(u)}$ for $p$ in $(0, u)$.

Some other notions which are less frequently used in analysis are new better than used in hazard rate (NBUHR), increasing hazard rate of order 2 (IHR(2)), new better than used in hazard rate average (NBUHRA) and IHRA* $t_0$, and their duals. A lifetime $X$ is

(i) NBUHR if and only if $h(0) \leq h(x)$ for all $x$ (Loh (1984)).

(ii) NBUHRA if and only if $h(0) \leq \frac{1}{x} \int_0^x h(t)dt$, $x > 0$

(Loh (1984)).

(iii) IHR (2) if and only if $\int_0^x F_u(t)dt \geq \int_0^x F_s(t)dt$

for all $x \geq 0$, $u \geq s$ (Deshpande et al. (1986)).

(iv) IHRA* $x_0$ if and only if for all $x \geq x_0$,

\[
\frac{1}{x} \int_0^x h(t)dt \leq \frac{1}{x_0} \int_0^{x_0} h(t)dt \text{ (Li and Li (1998)).}
\]
From the above definitions the following results are straight forward.

**Proposition 4.3**

(i) $X$ is NBUHR (NWUHR) if and only if

$$H(0) \leq (\geq) \frac{\int_0^u H(p)q(p)dp}{\int_0^u q(p)dp}, \text{ for all } u.$$ 

(ii) $X$ is NBUHRA (NWUHRA) if and only if

$$H(0) \leq (\geq) \frac{\int_0^u H(p)q(p)dp}{\int_0^u q(p)dp}, \text{ for all } u.$$ 

(iii) $X$ is IHR(2) if and only if

$$\int_0^u [Q(t+(1-t)v) - Q(t)]q(v)dv \geq \int_0^u [Q(s+(1-s)v) - Q(s)]q(v)dv$$

for all $u \geq 0$ and $t \geq s$.

(iv) $X$ is IHRA* $u_0$ if and only if

$$\frac{\int_0^u H(p)q(p)dp}{Q(u)} \leq \frac{\int_0^{u_0} H(p)q(p)dp}{Q(u_0)} \text{ for all } u \geq u_0.$$ 

**Example 4.2** It has been of interest in reliability theory to find distributions whose hazard rates $h(x)$ have simple functional forms like linear, quadratic, reciprocal linear, etc. in $x$. In a similar fashion we seek distributions for which hazard quantile function is linear, that is

$$H(u) = a + bu.$$ 

From the representation (equation (2.73))

$$Q(u) = \int_0^u \frac{dp}{(1-p)H(p)},$$

we have

$$Q(u) = \frac{1}{a+b} \log \frac{a+bu}{1-u} + C. \quad (4.1)$$
Setting $u = 0$, the constant $C$ is determined as $C = -(a + b)^{-1} \log a$. Thus the distribution of $X$ with linear $H(u)$ is

$$Q(u) = \log \left( \frac{a + bu}{a(1-u)} \right)^{1/a+b}, \ 0 \leq u \leq 1, \ a > 0$$

(4.2)

with quantile density function

$$q(u) = \left(1-u(a+bu)\right)^{-1}.$$ 

(4.3)

In fact (4.2) represents a family of distributions. When $a > 0, b = 0$

$$Q(u) = -\frac{1}{a} \log(1-u),$$

which is exponential with mean $a$. When $a = b > 0$

$$Q(u) = \frac{1}{2a} \log \frac{1+u}{1-u}$$

corresponds to the half logistic distribution.

Taking $a > 0, b = -ap, 0 < p < 1$,

$$Q(u) = \frac{1}{\lambda} \log \frac{1-pu}{1-u}, \ \lambda = a(1-p)$$

is the quantile function of the exponential-geometric distribution of Adamidis et al. (2005). Notice that the family is IHR, IHRA and NBUHR for $b > 0$.

**Example 4.3** For the Govindarajulu model discussed in Section 3.5, as illustrated in Section 3.5.5, we conclude that $X$ is IHR for $\beta \leq 1$ and bathtub-shaped for $\beta > 1$ with change point at $u = \frac{\beta - 1}{\beta + 1}$.

### 4.3 Concepts based on residual function

First we discuss the concepts based on the mean of the residual life. Recall the definitions given in (2.48) and (2.75). Based on the distribution functions, $F$ is said to be in decreasing (increasing) mean
residual life –DMRL (IMRL) class if \( m(x) \) is a decreasing (increasing) function in \( x > 0 \). That is, \( m(s) \geq (\leq) m(t) \) for \( 0 \leq s \leq t \). Equivalent conditions of this ageing concept in terms of quantile function are given in the following proposition.

**Proposition 4.4**

We say that \( X \) is decreasing mean residual quantile function (DMRL) (increasing mean residual quantile function (IMRL)) if and only if one of the following equivalent conditions hold

(i) \( M(u_1) \leq (\geq) M(u_2) \), \( u_1 \geq u_2 \)

(ii) \( \int_0^1 \left\{ Q(u + (1-u)p) - Q(u) \right\} dp \) is decreasing (increasing) in \( u \)

(iii) \( M'(u) \leq (\geq) 0 \)

(iv) \( M(u) \leq (\geq) \frac{1}{H(u)} \)

**Proof:** To prove (i), from the condition

\( X \) is DMRL \( \Rightarrow m(s) \geq (\leq) m(t) \) for \( 0 \leq s \leq t \)

\( \Rightarrow M(u_1) \leq M(u_2) \), \( u_1 \geq u_2 \)

by setting \( t = Q(u_1) \) and \( s = Q(u_2) \).

We have from equation (2.75)

\[ M(u) = (1-u)^{-1} \int_u^1 [Q(p) - Q(u)] dp. \]

Substituting \( t = u + (1-u)p \), we have

\[ \int_0^1 [Q(u + (1-u)p) - Q(u)] dp = M(u). \]

Now assume (i), which means

\( M(u) \) is decreasing \( \iff \int_0^1 [Q(u + (1-u)p) - Q(u)] dp \) is decreasing

\( \Rightarrow (ii) \).
Clearly (iii) ⇔ (ii). We have from (2.77)

\[ M(u) = (1 - u)^{-1} \int_u^1 (H(p))^{-1} \, dp. \]

Differentiating

\[ (1 - u)M'(u) - M(u) = -\frac{1}{H(u)} \]

or

\[ M'(u) = \frac{1}{(1 - u)} \left( M(u) - \frac{1}{H(u)} \right). \]

Now

\[ M'(u) \leq 0 \Leftrightarrow \frac{1}{(1 - u)} \left( M(u) - \frac{1}{H(u)} \right) \leq 0 \]

\[ \Leftrightarrow M(u) \leq \frac{1}{H(u)}. \]

Hence (iii) ⇔ (iv), which completes the proof.

**Example 4.4** For the linear hazard quantile function distribution (4.2)

\[ M(u) = \frac{1}{1 - u} \int_u^1 (Q(p) - Q(u)) \, dp \]

\[ = \frac{1}{b(1 - u)} \log \frac{a + b}{a + bu}. \]

Note that

\[ \frac{1}{b(1 - u)} \log \frac{a + b}{a + bu} \leq \frac{a + b}{(a + bu)b(1 - u)} \leq \frac{1}{a + bu} = \frac{1}{H(u)} \]

for \( b < 0 \) and hence \( X \) is DMRL by (iv) of Proposition 4.4.

Four other ageing properties involving mean residual life are net decreasing (increasing) mean residual life (NDMRL (NIMRL)) defined by \( m(x) \leq (\geq) m(0) \), used better (worse) than aged (UBA (UWA)) defined by (Alzaid (1994))
\[ \overline{F}(x) \geq (\leq) \exp \left[ -\frac{x}{m(\infty)} \right], \quad m(\infty) < \infty, \]

used better (worse) than aged in expectation, UBAE (UWAE) (Alzaid (1994)) satisfying

\[ m(x) \geq (\leq)m(\infty) \]

and decreasing (increasing) mean residual life in harmonic average, DMRLHA (IMRLHA) that satisfies the condition (Deshpande et al. (1986))

\[ \frac{1}{x} \int_0^x \frac{dt}{m(t)} \]

decreasing (increasing) in \( x \).

In the following proposition, the above definitions are expressed based on quantile functions.

**Proposition 4.5**

A lifetime random variable \( X \) with \( M(1) = \lim_{u \to 1^-} M(u) < \infty \) is

(i) net decreasing (increasing) mean residual function, NDMRL (IDMRL) if and only if \( M(u) \leq (\geq)M(0) \)

(ii) UBA (UWA) if and only if

\[ Q(u + (1-u)v) - Q(u) \geq (\leq) - \frac{1}{M(1)} \log(1-u) \text{ for all } 0 \leq u, \ v < 1. \]

(iii) UBAE (UWAE) if and only if \( M(u) \geq (\leq)M(1) \) for all \( 0 < u < 1 \)

(iv) DMRLHA (IMRLHA) if and only if \( \frac{\int_0^u q(p) \, dp}{\int_0^u q(p) \, dp} \) is

(decreasing) in \( u \).

**Proof:** To prove (i), we know that \( X \) is NDMRL if and only if
\[ m(x) \leq m(0). \]

Setting \( x = Q(u) \) and noting \( Q(0) = 0 \), we have \( X \) is NDMRL if and only if

\[ M(u) \leq M(0), \]

which proves the assertion (i). The random variable \( X \) is UBA if and only if

\[ \overline{F}_t(x) \geq \exp\left[-\frac{x}{M(\infty)}\right], \quad m(\infty) < \infty. \]

In the above inequality, left side is the survival function of \( X_t = X - t | X > t \) and right side is the survival function of the exponential distribution with mean \( m(\infty) < \infty \). Recalling from Section 2.1.2, the quantile function of \( X_t \) is

\[ Q_t(u) = Q(v + (1-v)u) - Q(v) \]

by setting \( F(t) = v \) and \( F_t(x) = u \). The quantile function of the exponential distribution mentioned above is

\[ Q_E(u) = \frac{-1}{M(1)} \log(1 - u), \]

where \( m(\infty) = \lim_{u \to 1} M(u) = M(1) \). Thus \( X \) is UBA if and only if

\[ 1 - \left[ Q(v + (1-v)u) - Q(v) \right] \leq 1 - \frac{-1}{M(1)} \log(1 - u) \]

or

\[ \left[ Q(v + (1-v)u) - Q(v) \right] \geq \frac{-1}{M(1)} \log(1 - u). \]

The proof of (iii) is straightforward from the definition of UBAE. To prove (iv), note that

\[ X \text{ is DMRLHA } \Rightarrow \left[ \frac{1}{x} \int_0^x \frac{dt}{m(t)} \right]^{-1} \text{ is decreasing}. \]
Setting $x = Q(u)$, the above condition is equivalent to

$$\left[ \frac{1}{Q(u)} \int_0^u \frac{1}{M(p)} q(p) dp \right]^{-1}$$

is decreasing

$$\Rightarrow \left[ \frac{\int_0^u \frac{1}{M(p)} q(p) dp}{\int_0^u q(p) dp} \right]^{-1}$$

is decreasing

$$\Rightarrow \frac{\int_0^u \frac{1}{M(p)} q(p) dp}{\int_0^u q(p) dp}$$

is increasing.

This completes the proof.

Other ageing concept based on residual quantile function is in connection with the variance of residual function (VRL) discussed in Section 2.2.3 and its quantile version given in Section 2.4.3. The random variable $X$ has decreasing (increasing) variance residual life, abbreviated as DVRL (IVRL) if and only if $\sigma^2(x)$ is decreasing (increasing) in $x$.

Recall the quantile-based definitions of VRL given in (2.70) through (2.72) and the coefficient of variation in (2.73). We have the following proposition for DVRL (IVRL) in terms of quantile function.

**Proposition 4.6**

The following conditions are equivalent

(i) $X$ is DVRL (IVRL)

(ii) $C(u) \leq (\geq) 1$, where $C^2(u) = \frac{V(u)}{M^2(u)}$ \hspace{1cm} (4.4)

**Proof:** Assume that (i) is true. To prove (ii) we have

$$V(u) = \frac{1}{1-u} \int_u^1 M^2(p) dp.$$

Differentiating
\[ V(u) - (1 - u)V'(u) = M^2(u) \]

or

\[ (1 - u)V'(u) = V(u) - M^2(u), \]

since \( X \) is DVRL,

\[ V'(u) \leq 0 \iff V(u) - M^2(u) \leq 0 \]
\[ \iff C(u) \leq 1, \]

which proves the assertion.

Another version of mean residual life is obtained by considering the mean of the asymptotic distribution of residual life given survival beyond age \( x \), called the renewal mean residual life (RMRL) (Nair and Sankaran (2010)) defined as

\[ m_R(x) = \frac{\int_x^\infty (t-x)\bar{F}(t)dt}{\int_x^\infty \bar{F}(t)dt}. \]

Setting \( x = Q(u) \), we have the renewal mean residual quantile function

\[ M_R(u) = \frac{\int_u^1 (Q(p) - Q(u))(1-p)q(p)dp}{\int_u^1 (1-p)q(p)dp}. \quad (4.5) \]

A lifetime variable \( X \) is decreasing (increasing) RMRL, DRMRL (IRMRL) if and only if \( m_R(x) \) or equivalently \( M_R(u) \) is decreasing (increasing) in \( x \) (\( u \)).

**Proposition 4.7**

\( X, DRMRL \ (IRMRL) \iff M_R(u) \leq (\geq) M(u) \) for all \( u \).

**Proof:** We have
Ageing concepts

\[ M_R(u) = \frac{\int_u^1 (Q(p) - Q(u))(1 - p)q(p)dp}{\int_u^1 (1 - p)q(p)dp} \]

or

\[ M_R(u) \int_u^1 (1 - p)q(p)dp = \int_u^1 (Q(p) - Q(u))(1 - p)q(p)dp. \]

Differentiating, we get

\[ -(1 - u)q(u)M_R(u) + M'_R(u)(1 - u)M(u) = -q(u)(1 - u)M(u) \]

or

\[ M(u) = \frac{M_R(u)q(u)}{q(u) + M'_R(u)} \]

Thus \( X \) is DRMRL (IRMRL) \( \Leftrightarrow M_R(u) \leq (\geq) M(u) \).

Various researchers have used percentile residual life (PRL) defined in Section 2.2.4 to define ageing classes. Important ageing concepts based on PRL are decreasing \( \alpha \)-percentile residual life (DPRL-\( \alpha \)) and new better than used with respect to the \( \alpha \)-percentile residual life (NBUP-\( \alpha \)) and their duals. We say that \( F \) is DPRL-\( \alpha \) if \( p_\alpha(t) \) is decreasing in \( t \) and NBUP-\( \alpha \) if \( p_\alpha(0) \geq p_\alpha(t) \) for all \( t \). For the earlier development of these ageing classes we refer to Haines and Singpurwalla (1974) and Joe and Prochan (1984). Recently Franco-Pereira et al. (2010) have proved the equivalence of the following conditions for \( F \) to be DPRL-\( \alpha \).

(i) \( p_\alpha(t) \) is decreasing

(ii) \( (1 - \alpha)f(t) \leq f\left[ \frac{1}{\alpha} \left(1 - \alpha \right)F(t) \right] \]

(iii) \( (1 - \alpha)f\left[ F^{-1}(u) \right] \leq f\left( \frac{1}{\alpha} \left(1 - \alpha \right)u \right), \quad 0 < u < 1 \)
(iv) \( h(t) \leq h(t + p_\alpha(t)), \ t \in (0, T) \).

Thus we have the following proposition.

**Proposition 4.8**

If \( X \) is DPRL-\( \alpha \), then the following conditions are equivalent.

(i) \( P_\alpha(u) \) is decreasing.

(ii) \( q(u) \geq (1-\alpha)q[1-(1-\alpha)(1-u)], \ 0 < u < 1 \)

(iii) \( H(u) \leq H(1-(1-\alpha)(1-u)) \)

**Proof:** Note that

\( X \) is DPRL-\( \alpha \) \( \Rightarrow \) \( p_\alpha(t) \) is decreasing

\[ \Rightarrow \frac{d}{dt} p_\alpha(t) \leq 0 \]

\[ \Rightarrow \frac{d}{dQ(u)} p_\alpha(Q(u)) \leq 0 \]

\[ \Rightarrow \frac{P'(u)}{q(u)} \leq 0 \]

\[ \Rightarrow P_\alpha(u) \) is decreasing

Assuming (i), we have

\[ Q(1-(1-\alpha)(1-u)) - Q(u) \) is decreasing

\[ \Leftrightarrow q(1-(1-\alpha)(1-u))(1-\alpha) - q(u) \leq 0 \]

Thus (i) \( \Rightarrow \) (ii).

Also from (ii)

\( q(u) \geq (1-\alpha)q[1-(1-\alpha)(1-u)] \)

\[ \Leftrightarrow (1-u)q(u) \geq (1-\alpha)(1-u)q[1-(1-\alpha)(1-u)] \]

\[ \Leftrightarrow (1-u)q(u) \geq [1-(1-(1-\alpha)(1-u))]q[1-(1-\alpha)(1-u)] \]

\[ \Rightarrow H(u) \leq H[1-(1-\alpha)(1-u)] \]

means (ii) \( \Leftrightarrow \) (iii), which completes the proof.
In quantile terminology, $X$ is *new better than used with respect to the $\alpha -$ percentile residual life* (NBUP-$\alpha$) if $P_{\alpha}(0) \geq P_{\alpha}(u)$.

### 4.4 Concepts based on survival functions

The ageing properties in this class are obtained by comparing survival at different points of time. Most important among them are the new better (worse) than used, NBU (NWU) and those generated from it like NBUE, HNBUE, etc. We say that $X$ is NBU (NWU) if and only if

\[ \overline{F}(x+t) \leq (\geq) \overline{F}(x)\overline{F}(t), \text{ for all } x, t > 0. \]

Based on this definition we have the following proposition.

**Proposition 4.9**

A lifetime variable $X$ with quantile function $Q(u)$ is NBU (NWU) if and only if

\[ Q(u+v-uv) \leq (\geq) Q(u) + Q(v) \text{ for all } u, v. \quad (4.6) \]

**Proof:** The random variable $X$ is NBU (NWU) if and only if

\[ \overline{F}(x+t) \leq (\geq) \overline{F}(x)\overline{F}(t), \text{ for all } x, t > 0. \]

Setting $x = Q(u)$ and $t = Q(v)$, so that $x + t = Q(u) + Q(v)$. Now

\[
\overline{F}(x+t) \leq \overline{F}(x)\overline{F}(t) \Rightarrow 1 - F(x+t) \leq (1-u)(1-v) \\
\Rightarrow F(x+t) \geq u + v - uv \\
\Rightarrow Q(u) + Q(v) \geq Q(u+v-uv),
\]

as asserted.

The equality sign in (4.6) holds good when

\[ Q(1-(1-u)(1-v)) = Q(1-(1-u)) + Q(1-(1-v)) \]

or

\[ Q(1-(1-u)(1-v)) = Q(1-(1-u)) + Q(1-(1-v)), \quad (4.7) \]

which reduces to the form
The last equation transforms to the Cauchy functional equation
\[ a(xy) = a(x) + a(y) \]
with \( a(x) = Q(1 - x) \).
The only continuous solution to the above functional equation is
\[ a(x) = k \log x. \]
Thus
\[ Q(u) = k \log(1 - u), \]
and for this to be a quantile function we must have \( k = -c, c > 0 \). Hence
\[ Q(u) = -c \log(1 - u), \]
represents the exponential law.

**Example 4.5** The power-Pareto law in Section 3.3, specified by the quantile function
\[ Q(u) = Cu^{\lambda_1} (1-u)^{-\lambda_2} \quad C, \lambda_1, \lambda_2 > 0, \quad 0 < u < 1 \]
contains both NBU and NWU distributions. An obvious case is when \( \lambda_2 = 0 \), the model represents the power distribution, which is NBU. On the other hand when \( \lambda_1 = 0 \), \( Q_3(u) \) becomes Pareto and hence NWU. In general, by (4.6) the NBU (NWU) cases are sorted out from the inequality
\[ (u + v - uv)^{\lambda_1} \leq (\geq) u^{\lambda_1} (1-v)^{\lambda_2} + (1-u)^{\lambda_2} v^{\lambda_2} \text{ for all } u, v. \]
For example, when \( \lambda_1 - \lambda_2 = 1 \), we have log-logistic distribution that is NWU.

There are some generalizations of the NBU concepts in the form of
NBU* \( t_0 \), NBU- \( t_0 \), NBU(2) and NBU(2)-\( t_0 \) along with the corresponding dual classes. A distribution is a NBU-\( t_0 \) (NWU-\( t_0 \)) class of life distribution if it satisfies
Ageing concepts

\[
F(t_0 + x) \leq (\geq) \bar{F}(t_0) \bar{F}(x), \quad x \geq 0. 
\]

See Hollander et al. (1985) for details. We say that, \(X\) is NBU* \(_{t_0}\) (NWU* \(_{t_0}\)) if

\[
F(x + y) \leq (\geq) F(x)F(y), \quad x \geq 0, 0 < t_0 < y \quad \text{(Li and Li (1998))}
\]

The difference between NBU- \(_{t_0}\) and NBU* \(_{t_0}\) is that in the former \(t_0\) is a fixed time while in the later it extends beyond \(t_0\). A lifetime random variable \(X\) is said to be NBU(2) (NWU(2)) if and only if

\[
\int_0^x F(y)dy \geq (\leq) \int_0^{t+y} \frac{\bar{F}(t)}{\bar{F}(t)}dy 
\]

for all \(t, x \geq 0\). This class is obtained by Deshpande et al. (1986) and more details are available there. Elabatal (2007) studied the extension of NBU(2) class at specific age \(t_0\). The identification of these classes of quantile functions is as follows. The proof follows from Proposition 4.9.

**Proposition 4.10**

We say that \(X\) is

(i) \(\text{NBU-} u_0 (\text{NWU-} u_0)\) for some \(0 \leq u_0 < 1\),

\[
Q(u + u_0 - uu_0) \leq (\geq) Q(u) + Q(u_0) \quad \text{for all} \quad 0 \leq u \leq 1 
\]

(ii) \(\text{NBU*} u_0 (\text{NWU*} u_0)\) if and only if

\[
Q(u + v - uv) \leq (\geq) Q(u) + Q(v) \quad \text{for all} \quad 0 \leq u < 1 \quad \text{and} \quad v \geq u_0
\]

(iii) \(\text{NBU(2)} (\text{NWU (2)})\) if and only if

\[
\frac{1}{1-u} \int_0^u [1 - Q^{-1}(Q(p) + Q(v))]q(p)dp \leq \int_0^u (1 - p)q(p)dp, \quad \text{for all} \quad 0 \leq u, v \leq 1
\]

(iv) \(\text{NBU(2)} u_0\) if and only if

\[
\frac{1}{1-u_0} \int_0^u [1 - Q^{-1}(Q(p) + Q(u_0))]q(p)dp \leq \int_0^u (1 - p)q(p)dp
\]
for some \( u_0 \).

An integrated version of the NBU (NWU) definition leads to the NBUC (NWUC) class defined by Cao and Wang (1991). We say that \( X \) is NBUC (NWUC) if and only if

\[
\int_x^{\infty} \overline{F_i(y)} \, dy \leq (\geq) \int_x^{\infty} \overline{F(y)} \, dy
\]

and it extends to NBUC-\( t_0 \) for a specific age \( t_0 \). The counterparts of these two classes in quantile form are presented in the following proposition.

**Proposition 4.11**

A lifetime variable \( X \) belongs to the ageing class

(i) \( \text{NBUC} \) (NWUC) if and only if

\[
\frac{1}{1 - u} \int_u^1 [1 - Q^{-1}(Q(p) + Q(v))] q(p) \, dp \leq (\geq) \int_u^1 (1 - p)q(p) \, dp
\]

(ii) \( \text{NBUC-} u_0 \) (NWUC- \( u_0 \)) if and only if for some \( u_0 \) in \( [0, 1) \)

\[
\frac{1}{1 - u_0} \int_u^1 [1 - Q^{-1}(Q(p) + Q(u_0))] q(p) \, dp \leq \int_u^1 (1 - p)q(p) \, dp
\]

**Proof:** To prove (i), setting \( F(t) = v \) and \( F(y) = p \), so that \( t = Q(v) \), \( y = Q(p) \) and \( dy = q(p) \, dp \), the condition

\[
\int_x^{\infty} \overline{F_i(y)} \, dy \leq \int_x^{\infty} \overline{F(y)} \, dy
\]

or

\[
\frac{1}{F(t)} \int_x^{\infty} \overline{F(y + t)} \, dy \leq \int_x^{\infty} \overline{F(y)} \, dy
\]

is equivalent to

\[
\frac{1}{1 - v} \int_v^1 [1 - Q^{-1}(Q(p) + Q(v))] q(p) \, dp \leq \int_v^1 (1 - p)q(p) \, dp,
\]

since \( F(y + t) = 1 - F(y + t) = 1 - Q^{-1}(Q(p) + Q(u)) \).
The proof of (ii) is similar to that of (i) by taking $v = u_0$, a fixed value in $(0, 1)$.

The (ii) part of Proposition 4.11 brings its relationship with NBU(2) (NWU (2)) and NBUC (NWUC). Finally, we have still larger class called harmonically new better (worse) than used in expectation (HNBUE (HNWUE)), which is defined by

$$
\int_x^\infty F(t)dt \leq \mu \exp\left(-\frac{x}{\mu}\right) \text{ for all } x \geq 0.
$$

This leads to the next proposition.

**Proposition 4.12**

The HNBUE (HNWUE) property holds for $X$ if and only if

(i) $\int_u^1 (1-p)q(p)dp \leq (\geq) \mu e^{\frac{Q(u)}{\mu}}$

(ii) $\frac{\int_0^u q(p)}{M(p)} \geq (\leq) \frac{1}{\mu}$.

**Proof:** The proof of (i) is straight forward from the definition of HNBUE by setting $F(t) = p$ and $F(x) = u$. To prove the equivalence of (i) and (ii),

(ii) $\Leftrightarrow \int_0^u q(p)\left(\frac{1}{1-p}\int_p^1 (1-t)q(t)dt\right)^{-1}dp \geq \frac{Q(u)}{\mu}$

$\Leftrightarrow \int_0^u q(p)(1-p)dp \int_p^1 (1-t)q(t)dt \geq \frac{Q(u)}{\mu}$

$\Leftrightarrow -\log\left(\frac{1}{\mu} - \int_0^u (1-t)q(t)dt\right) \geq \frac{Q(u)}{\mu}$

(by noting $\int_p^1 (1-t)q(t)dt = \mu - \int_0^p (1-t)q(t)dt$)

$\Leftrightarrow$ (i)

This completes the proof.
The equilibrium distribution specified by the density function (of a random variable $Z$, say)

$$f_Z(x) = \frac{\overline{F}(x)}{\mu}, \quad x > 0. \quad (4.8)$$

plays an important role in evolving new ageing classes and also in proving relationships between various concepts. Equation (4.8) is obtained as the asymptotic distribution of age or residual life (or of forward and backward recurrence times) in renewal theory. The distribution function of $Z$ becomes

$$F_Z(x) = \frac{\int_0^x \overline{F}(t)\, dt}{\mu}.$$ setting $x = Q(u)$

$$F_Z(Q(u)) = \frac{1}{\mu} \int_0^u (1-p)q(p)\, dp.$$

Note that $T(u) = \int_0^u (1-p)q(p)\, dp$ is the quantile version of the total time on test transform of $X$ and $\phi(u) = \mu^{-1}T(u)$ is the scaled transform (See Section 2.5 for further details). Hence

$$Q(u) = Q_Z(\phi(u))$$

and

$$Q_Z(u) = Q(\phi^{-1}(u)), \quad (4.9)$$

give the relationship between the quantile functions of $Z$ and $X$. Since $T(u)$ is also a quantile function and denoting the random variable corresponding to $T(u)$ as $X_T$, $\phi(u)$ is the quantile function of $\mu^{-1}X_T$.

These results help the analysis of equilibrium distributions in terms of quantile functions. As an example, $\phi(u) = u$ for the exponential
distribution and hence \( X \) and \( Z \) are identically distributed. Also \( \phi^{-1}(u) \) is the distribution function of a uniform random variable over \((0, 1)\).

**Example 4.6** The linear hazard quantile family of distributions discussed in Example 4.2 has the quantile density function

\[
q(u) = [(1-u)(a+bu)]^{-1}
\]

and hence

\[
\phi(u) = \frac{\int_0^u (1-p)q(p)dp}{\mu} = \frac{\log \left( \frac{a+bu}{a} \right)}{\log \left( \frac{a+b}{a} \right)}.
\]

The inverse of \( \phi(u) \) is

\[
\phi^{-1}(u) = \frac{a}{b} \left( \frac{a+b}{a} \right)^u - 1, \quad 0 \leq u \leq 1,
\]

a distribution function on \((0, 1)\). Hence, from (4.9)

\[
Q_z(u) = \log \frac{b}{a} \frac{(a+b)^{u-1}}{a^{u-1} - (a+b)^{u-1}} , \quad 0 \leq u \leq 1
\]

is the quantile function of the equilibrium distribution.

There are few ageing classes that involve \( Z \) and its residual life \( Z = (Z - t | Z > t) \). One is new better (worse) than renewal used (NBRU (NWRU)), which is identical with NBUC (NWUC), discussed earlier. The renewal new is better (worse) than used (RNBU (RNWU)) is defined by (Abouammoh et al. (2000))

\[
\frac{\bar{F}(x+t)}{\bar{F}(t)} \leq (\geq) \frac{1}{\mu} \int_{x}^{x} \bar{F}(u)du, \quad (4.10)
\]
while its integrated version

\[ E(X_t) \leq (\geq) E(Z) \]  \hspace{1cm} (4.11)

is the renewal new better (worse) than used in expectation (RNBUE (RNWUE)). Further we have renewal new is better than renewal used (RNBRU) and its dual RNWRU are defined by

\[ \mu \int_{x+t}^{\infty} \bar{F}(u) du \leq (\geq) \int_{x}^{\infty} \bar{F}(u) du \int_{t}^{\infty} \bar{F}(u) du \]

and the corresponding integrated version in renewal new better (worse) than renewal used in expectation (RNBRUE (RNWRUE)), whenever

\[ E(Z_t) \leq (\geq) E(Z). \]  \hspace{1cm} (4.13)

Alternative expressions for the classes in terms of quantile functions are given below. The proof of this proposition is straightforward by following the steps in the proof of Proposition 4.11.

**Proposition 4.13**

A lifetime variable \( X \) belongs to

(i) RNBU (RNWU) class if and only if

\[ \frac{1}{1 - v} \left[ 1 - Q^{-1}(Q(u) + Q(v)) \right] \leq (\geq) \frac{1}{\mu} \int_{u}^{v} (1 - p) q(p) dp \]

(ii) RNBRU (RNWRU) class if and only if

\[ \mu \int_{u}^{v} \left[ 1 - Q^{-1}(Q(p) + Q(u)) \right] q(p) dp \leq (\geq) \left( \int_{u}^{v} (1 - p) q(p) dp \right) \left( \int_{v}^{1} (1 - p) q(p) dp \right) \]

**Proof:** To prove (i), set \( F(t) = v, F(x) = u \), so that

\[ F(x + t) = Q^{-1}(Q(u) + Q(v)), \]

and take

\[ F(s) = p \Rightarrow \bar{F}(s) ds = (1 - p) q(p) dp, \]

the definition of RNBU given in (4.10) becomes
\[
\frac{1 - Q^{-1}(Q(u) + Q(v))}{1 - v} \leq \frac{1}{\mu} \int_u^1 (1 - p) q(p) dp,
\]
as asserted. With the above substitutions the proof of (ii) is direct from (4.12).

**Remark 4.4** The expectations in (4.11) and (4.13) are obtained by integrating the quantile functions of the variables \( X_t, Z \) and \( Z_i \) between 0 and 1.

In conclusion, we have provided the definitions of various ageing classes in terms of quantile functions. These definitions become essential when we deal with life distributions specified by quantile functions, especially when they do not have tractable forms of distribution functions. We have given several examples that illustrate this situation.
Appendix

**Figure 1** - Shapes of hazard quantile function of Power Pareto model when

1. $C = 0.1, \lambda_1 = 0.5, \lambda_2 = 0.01$
2. $C = 0.5, \lambda_1 = 2, \lambda_2 = 0.01$
3. $C = 0.01, \lambda_1 = 2, \lambda_2 = 0.5$
4. $C = 0.01, \lambda_1 = 0.5, \lambda_2 = 0.5$

**Figure 2** - Shapes of hazard quantile function of Freimer et al. model when

1. $\lambda_1 = 0, \lambda_2 = 100, \lambda_3 = -0.5, \lambda_4 = -0.1$
2. $\lambda_1 = 0, \lambda_2 = 500, \lambda_3 = 3, \lambda_4 = 2$
3. $\lambda_1 = 0, \lambda_2 = 2, \lambda_3 = 10, \lambda_4 = 5$
4. $\lambda_1 = 0, \lambda_2 = 100, \lambda_3 = 2, \lambda_4 = 0.5$
5. $\lambda_1 = 0, \lambda_2 = 250, \lambda_3 = 2, \lambda_4 = 0.001$
Figure 3-Shapes of hazard quantile function of Staden and Loots model when
(1) $\lambda_1 = 0$, $\lambda_2 = 0.01$, $\lambda_3 = 0.5$, $\lambda_4 = -2$, (2) $\lambda_1 = 0$, $\lambda_2 = 100$, $\lambda_3 = 0.5$, $\lambda_4 = 10$
(3) $\lambda_1 = 0$, $\lambda_2 = 1$, $\lambda_3 = 0.6$, $\lambda_4 = 0.5$, (4) $\lambda_1 = 0$, $\lambda_2 = 0.1$, $\lambda_3 = 1$, $\lambda_4 = -5$

Figure 4-Shapes of hazard quantile function of Govindarajulu model when
(1) $\beta = 0.1$, (2) $\beta = 2$