Chapter 6

L-moments of residual life*

6.1 Introduction

In Section 2.16, we have discussed the basic features of L-moments. We pointed out there that the L-moments are alternative to conventional moments and they have several advantages over the ordinary moments. In reliability analysis, residual life function and related measures are good indicators in describing ageing patterns of a distribution, and these are being used in other disciplines also. Note that most popular measures of residual life that are discussed in the literature are based on ordinary moments, for example the mean of residual life, variance of residual life, etc.. Considering the advantages of L-moments over ordinary moments, it is worthy to study the measures of residual life based on L-moments. In this chapter we investigate the properties of the first two L-moments of residual life and their relevance in various aspects of reliability analysis. This problem does not appear to have been considered in literature.

6.2 Definition and properties

Recall the definition of L-moments of order \( r \) from Section 2.1.6, which is given by

*The discussions in this chapter is based on Nair and Vineshkumar (2010) appeared in the Journal of Statistical Planning and Inference (see reference no. 93)*
\( L_r = \frac{1}{r} \sum_{k=0}^{r-1} (-1)^k \binom{r-1}{k} E(X_{r-k:n}) , \)

\[
= \sum_{k=0}^{r-1} (-1)^k \binom{r-1}{k} \int_0^\infty x(F(x))^{r-k-1} (1-F(x))^k f(x)dx , \tag{6.1}
\]

where \( X_{r:n} \) is the \( r^{th} \) order statistic in a sample of size \( n \) from \( F(x) \) and \( f(x) \) is the density function of \( X \). The truncated variable \( X_t = (X|X > t) \) has survival function \( \overline{F}(x) = \frac{F(x)}{F(t)} \) so that (6.1) becomes

\[
L_t(t) = \sum_{k=0}^{r-1} (-1)^k \binom{r-1}{k} \int_t^\infty x \left( \frac{\overline{F}(x) - \overline{F}(x)}{F(t)} \right)^{r-k-1} \left( \frac{\overline{F}(x)}{F(t)} \right)^k \frac{f(x)}{F(t)} dx . \tag{6.2}
\]

In particular, setting \( r = 1 \), we have

\[
L_t(t) = \frac{1}{F(t)} \int_t^\infty x f(x)dx = E(X|X > t) ,
\]

which is the vitality function discussed widely in reliability analysis. Since \( L_t(t) \) is widely discussed with references to its properties and applications (e.g. Kupka and Loo (1989)), we bestow our attention to the second moment. When \( r = 2 \)

\[
L_t(t) = \sum_{k=0}^{1} (-1)^k \frac{1}{k} \int_t^\infty x \left( \frac{\overline{F}(x)}{F(t)} \right)^{1-k} \left( \frac{\overline{F}(x)}{F(t)} \right)^k \frac{f(x)}{F(t)} dx
\]

\[
= \frac{1}{F^2(t)} \int_t^\infty (\overline{F}(x) - 2\overline{F}(x)) xf(x)dx
\]

\[
= \frac{1}{F(t)} \int_t^\infty xf(x)dx - \frac{2}{F^2(t)} \int_t^\infty x\overline{F}(x)f(x)dx
\]

\[
= L_t(t) - t - (\overline{F}(t))^{-2} \int_t^\infty \overline{F}^2(x)dx
\]

(applying integration by parts to the second integral)

\[
= m(t) - (\overline{F}(t))^{-2} \int_t^\infty \overline{F}^2(x)dx , \tag{6.3}
\]
where \( m(t) \) is the mean residual function. It follows that \( L_2(t) \leq m(t) \). However, the equality sign does not hold for any non-degenerate distribution. Thus \( L_2(t) \) is strictly less than the mean residual life function.

Differentiating (6.3), we have

\[
L'_2(t) = m'(t) - \frac{-\left(\bar{F}(t)\right)^4 + 2\left(\bar{F}(t)\right)^2 f(t) \int^\infty_t (\bar{F}(x))^2 \, dx}{(\bar{F}(t))^4}
\]

\[
= m'(t) + 1 - \frac{2h(t) \int^\infty_t (\bar{F}(x))^2 \, dx}{(\bar{F}(t))^4}
\]

\[
= m(t)h(t) - 2h(t)(m(t) + L_2(t)) \quad (\text{using (2.50)})
\]

\[
= h(t)(2L_2(t) - m(t)).
\] (6.4)

Setting \( F(x) = p \) and \( F(t) = u \) in (6.2), we get the expression for the \( r \)-th \( L \)-moment residual quantile function of \( X \) as

\[
\alpha_r(u) = L_r(Q(u)) = \sum_{k=0}^{r-1} (-1)^k \binom{r-1}{k} \int_u^1 \left( \frac{p-u}{1-u} \right)^{r-k-1} \left( 1 - \frac{1-p}{1-u} \right)^k Q(p) \, dp.
\] (6.5)

In particular, from (6.5)

\[
\alpha_1(u) = (1-u)^{-1} \int_u^1 Q(p) \, dp
\] (6.6)

and

\[
\alpha_2(u) = \frac{1}{2} \int_u^1 (2p-u-1)Q(p) \, dp.
\] (6.7)

Of the last two functions \( \alpha_1(u) \) is the quantile form of vitality function. Hence its properties are not pursued further. Note that \( \alpha_1(u) \) determines \( Q(u) \) through the formula
obtained from (6.6). The following theorem establishes the relationship between \( \alpha_1(u) \), \( \alpha_2(u) \) and \( M(u) \), and shows that these functions determine \( Q(u) \) uniquely.

**Theorem 6.1.** The function \( \alpha_1(u) \), \( \alpha_2(u) \) and \( M(u) \) determine each other and \( Q(u) \) uniquely.

**Proof:** We have

\[
M(u) = \frac{1}{(1-u)\alpha_1(u)-Q(u)} \int_u^1 (Q(p)-Q(u)) dp
\]

\[
= \alpha_1(u) - Q(u)
\]

\[
= \alpha_1(u) - (\alpha_1(u) - (1-u)\alpha_1'(u))
\]

\[
= (1-u)\alpha_1'(u).
\]

Differentiating (6.7), we have

\[
(1-u)^2\alpha_1'(u) - 2(1-u)\alpha_2(u) = -2uQ(u) - \left[-(u+1)Q(u) + \int_u^1 Q(p) dp\right]
\]

\[
= (1-u)Q(u) - \int_u^1 Q(p) dp
\]

\[
= (1-u)Q(u) - (1-u)(M(u) + Q(u))
\]

\[
= -(1-u)M(u)
\]

or

\[
M(u) = 2\alpha_2(u) - (1-u)\alpha_2'(u).
\]  

(6.10)

Again from (2.79)

\[
Q(u) = \mu - M(u) + \int_0^u (1-p)^{-1} M(p) dp.
\]

(6.11)

Thus \( M(u) \) determines \( Q(u) \), and \( \alpha_1(u) \) and \( \alpha_2(u) \) determine \( M(u) \). Also we have from (6.9)

\[
\alpha_1(u) = \int_0^u \frac{M(p)}{1-p} dp
\]

(6.12)
Equation (6.12) and (6.13) determines $\alpha_1(u)$ in terms of $M(u)$ and $\alpha_2(u)$, and (6.8) recovers $Q(u)$ from $\alpha_1(u)$. We also have

$$
\frac{d}{du}(1-u)^2 \alpha_2(u) = -(1-u)M(u).
$$

Integrating

$$(1-u)^2 \alpha_2(u) = -\int_u^1 (1-p)M(p)dp
$$
or

$$
\alpha_2(u) = (1-u)^{-2} \int_u^1 (1-p)M(p)dp
= (1-u)^{-2} \int_u^1 (1-p)^2 \alpha_1(p)dp,
$$
determining $\alpha_2(u)$ from $M(u)$ and $\alpha_1(u)$. Given $\alpha_2(u)$, $M(u)$ can be determined from (6.10) and hence $Q(u)$ from (6.11). Hence the proof.

**Remark 6.1.** Equation (6.3) is important in deducting the conditions for the monotonic behaviour of $L_2(t)$ when $F(x)$ is used instead of $Q(u)$.

**Remark 6.2** Recall the definition of Gini’s mean difference given in (2.9) through (2.11). Gini’s mean difference of the residual random variable $X$, is

$$
G(t) = 2 \int_t^\infty F_j(x) \overline{F}_j(x)dx.
$$
In terms of the quantile functions, this becomes

$$
\Delta(u) = G(Q(u)) = 2 \int_u^1 \frac{(1-p)(p-u)}{(1-u)^2} Q'(p)dp
= \frac{2}{(1-u)^2} \int_u^1 (p - p^2 - u + up)Q'(p)dp.
$$
Integrating by parts, we have...
Further $\alpha_2(0)$ is half the mean difference of $X$, which is extensively used as a measure of spread and the latter is an accepted measure of dispersion in the analysis of income and poverty in theoretical and applied economics.

The second L-moment of the conditional distribution of $X \mid X > t$ is half the mean difference of $X \mid X > t$. Since the mean difference is location invariant the second L-moment of $X_t$ is same as that of $X_t = X - t \mid X > t$. Thus we can treat $\alpha_2(u)$ as the second L-moment of residual life, a measure of variation and alternative to variance residual quantile function.

Remark 6.3. Theorem 6.1 shows that the dispersion of the residual life in the sense of mean difference is specified in terms of the mean, by means of equation (6.15).

To derive more reliability implications of $\alpha_2(u)$, we have connected it with some other important reliability functions. Firstly consider the total time on test transform (TTT) defined in (2.99)

$$T(u) = \int_0^u (1 - p)Q(p)dp.$$  

Using its relationship with $M(u)$ (equation (2.101))

$$T(u) = \mu - (1 - u)M(u)$$

and (6.15), we can easily write
\[ \alpha_2(u) = (1-u)^{-2} \int_u^1 (\mu - T(p)) dp. \quad (6.17) \]

Since \( \alpha_2(u) \) is conceived as a measure of dispersion its relationship with the variance residual function is of interest. We have

\[
V(u) = (1-u)^{-1} \int_u^1 M^2(p) dp \\
= (1-u)^{-1} \int_u^1 (2\alpha_2(p) - (1-p)\alpha'_2(p))^2 dp.
\]

A comparison between \( V(u) \) and \( \alpha_2(u) \) seems to be in order as they are competing measures of variability in the residual life. The functional form of \( \alpha_2(u) \) or equivalently that of the mean difference quantile function (or its reversed form) characterizes the life distribution, and therefore it can be used to identify the distribution. The variance of residual life also characterizes the associated distribution, but unlike \( \alpha_2(u) \), there is no simple expression relating \( Q(u) \) in terms of \( V(u) \) or between \( \overline{F}(t) \) and \( \sigma^2(t) \). As mentioned in Section 2.1.6, Yitzhaki (2003) has compared the relative merits of variance and mean difference as measures of variability, which is also valid for \( V(u) \) and \( \alpha_2(u) \). He points out that

(a) the mean difference is more informative than the variance in deriving properties of distributions that depart from normality

(b) mean difference can be used to form necessary conditions for second degree stochastic dominance while variance cannot.

We notice that most of the reliability models are non-normal and second order stochastic dominance is used in defining ageing concepts. In these contexts \( \alpha_2(u) \) seems to have preference over variance residual life.
The two functions $V(u)$ and $\alpha_2(u)$ may not exhibit same kind of monotonic behaviour. Even when $V(u)$ increases for larger $u$, $\alpha_2(u)$ can show a decreasing trend. As an example, consider the distribution with quantile function

$$Q(u) = 4u^3 - 3u^5, \quad 0 \leq u \leq 1,$$

which is a particular case of Govindarajulu (1977) model discussed in Chapter 3. In this case, using the expression of $M(u)$ given in (3.19) with $\beta = 3$, we have

$$V(u) = \frac{1}{1-u} \int_u^1 M^2(p) dp$$

$$= \frac{1}{175} \left(22 - 6u - 34u^2 - 62u^3 + 50u^4 + 78u^5 + 106u^6 + 9u^7 - 38u^8\right)$$

giving

$$\frac{dV(u)}{du} = \frac{1}{175} \left(-6 - 68u - 186u^2 + 200u^3 + 390u^4 + 636u^5 + 63u^6 - 324u^7\right),$$

which initially decreases in $(0, u_0)$ and then increases in $(u_0, 1)$ with a unique change point at $u_0 = 0.554449$. On the other hand

$$\alpha_2(u) = \frac{(1-u^2)^2}{5}$$

and

$$\alpha'_2(u) = -\frac{4}{5}u(1-u^2) < 0,$$

showing that $\alpha_2(u)$ is decreasing for all $u$ in $(0,1)$.

There are situations when $\alpha_2(u)$ promises to give better results than residual variance quantile function. We give two such examples that bring out the comparison.
Example 6.1 Let $X$ be distributed as exponential with parameter $\lambda$. Then $V(u) = \lambda^{-2}$ and $\alpha_2(u) = (2\lambda)^{-1}$. Five hundred samples were generated from the distribution for each of the values $\lambda = 0.5$, $\lambda = 1$ and $\lambda = 5$. The parameter $\lambda$ was estimated by equating the sample and population values of $V(u)$ and $\alpha_2(u)$. We found that $\alpha_2(u)$ gives a better approximation to the model (equivalently estimates of $\lambda$ with less bias). Also the variance of the estimates of $\lambda$ is considerably less when we use $\alpha_2(u)$. Table 6.1 contains the number of cases in which each of the functions gave better model, reveal that $\alpha_2(u)$ perform better.

<table>
<thead>
<tr>
<th>Function</th>
<th>Number of cases of better approximation</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\lambda = 0.5$</td>
</tr>
<tr>
<td>$\alpha_2(u)$</td>
<td>360</td>
</tr>
<tr>
<td>$V(u)$</td>
<td>140</td>
</tr>
</tbody>
</table>

Example 6.2 Mudholkar and Hutson (1996) analyzed the data on annual flood discharge rates of the Floyd river at James, Iowa using the exponentiated Weibull distribution. In an attempt of modelling the data using power-Pareto distribution

$$Q(u) = Cu^\lambda (1-u)^{-\lambda_2}, \ C, \lambda_1, \lambda_2 > 0,$$

discussed in Chapter 3, we have classified the 39 observations into 5 classes and estimated the parameters by equating the sample and population L-moments (This method has been discussed in Chapter 3). The estimates thus obtained were

$$\hat{C} = 3495.2, \ \hat{\lambda}_1 = 0.6226, \ \hat{\lambda}_2 = 0.5946.$$
The $\chi^2$ value of 2.375 as against the tabulated value 3.84 for one degree of freedom does not reject the power-Pareto model for the data.

In studying the variation, among the two measures $V(u)$ and $\alpha_2(u)$, only $\alpha_2(u)$ can be utilized as the variance residual quantile function (given in Chapter 3) does not exist for the above parameter values since $\lambda_2 > 0.5$. The above discussions reveal some reasonable grounds on which further properties of the second L-moment of residual life can be pursued in the following sections.

Now we consider the implications between mean residual quantile function $M(u)$ and $\alpha_2(u)$. We show with the following examples that $M(u)$ and $\alpha_2(u)$ may or may not possess same monotonicity.

**Example 6.3.** Consider the modified Tukey-Lambda distribution of Freimer et al. (1988) discussed in Chapter 3. The distribution has

$$M(u) = \frac{1}{\lambda_2} \left[ \frac{(1-u)^{\lambda_4}}{\lambda_3 + 1} - \frac{u^{\lambda_3}}{\lambda_3} + \frac{1-u^{\lambda_3+1}}{(1+\lambda_3)(1-u)} \right]$$

and

$$\alpha_2(u) = \frac{1-u}{\lambda_2 \lambda_3} - \frac{2(1-u^{\lambda_3+2})}{\lambda_2 \lambda_5 (\lambda_3 + 1)(\lambda_3 + 2)(1-u)^2} + \frac{1-u^{\lambda_4}}{\lambda_4 (1+\lambda_4)(2+\lambda_4)} + \frac{(1-u)(1+u^{\lambda_3+1})}{\lambda_2 \lambda_5 (\lambda_5 + 1)(1-u)^2}.$$  

After taking location and scale parameters $\lambda_1 = 0$, $\lambda_2 = 1$, we have,

$$M(u) = \frac{(1-u)^{\lambda_4}}{\lambda_4} - \frac{u^{\lambda_3}}{\lambda_3 (1+\lambda_3)(1-u)} + \frac{1-u^{\lambda_3+1}}{\lambda_3 (1+\lambda_3)(1-u)}$$

and

$$\alpha_2(u) = \frac{2(1-u^{\lambda_3+2})}{\lambda_3 (\lambda_3 + 2)(1-u)^2} + \frac{(1-u)^{\lambda_4}}{(1+\lambda_4)(2+\lambda_4)} - \frac{(1+u)(1-u^{\lambda_3+1})}{\lambda_3 (\lambda_5 + 1)(1-u)^2}. $$
In this case, both $M(u)$ and $\alpha_2(u)$ can be decreasing (e.g. $\lambda_3 = 2, \lambda_4 = 1$), linear ($\lambda_3 = 1, \lambda_4 = 1$) or decreasing first and then increasing ($\lambda_3 = 10, \lambda_4 = 5$). However the behaviour of $M(u)$ and $\alpha_2(u)$ need not be similar as in the case of $\lambda_3 = 1, \lambda_4 = -5$ in which case the former is decreasing and increasing, while the latter is decreasing.

In Chapter 3, we fitted the distribution to the aluminium coupon data with parameter values

$\hat{\lambda}_3 = 1382.18, \hat{\lambda}_2 = 0.0033, \hat{\lambda}_4 = 0.2706$ and $\hat{\lambda}_4 = 0.2211$.

The graphs of $M(u)$ and $\alpha_2(u)$ of the fitted distribution given in Figure 6.1 shows that both are decreasing functions of $u$.

![Graph showing $M(u)$ and $\alpha_2(u)$](image)

**Figure 6.1**  $M(u)$ and $\alpha_2(u)$ of the generalised lambda distribution

**Example 6.4** The Govindarajulu distribution has (see Section 3.5.5),

$$M(u) = \sigma \left( \frac{2 - (\beta + 1)(\beta + 2)u^\beta + 2\beta(\beta + 2)u^{\beta+1} - \beta(\beta + 1)u^{\beta+2}}{(\beta + 2)(1-u)} \right)$$

and

$$\alpha_2(u) = \sigma \left( \frac{2\beta - 2(\beta + 3)u + (\beta + 2)(\beta + 3)u^{\beta+1} - 2\beta(\beta + 3)u^{\beta+2} + \beta(\beta + 1)u^{\beta+3}}{(\beta + 2)(\beta + 3)(1-u)^2} \right)$$
we find that $M(u)$ and $\alpha_2(u)$ decrease for $\beta < 1$ and when $\beta > 1$ both functions either decrease or first increase and then decrease with the change point increasing as $\beta$ increases. In Govindarajulu (1977), the distribution is fitted to the data on failure times of a set of refrigerator motors, with the estimate of $\beta$ viz. $\hat{\beta} = 2.94$. Taking $\sigma = 1$, for this value of $\beta$, $M(u)$ initially increases and then decreases with approximate change point at $u = 0.2673$ and $\alpha_2(u)$ decreases for all $u$. See Figure 6.2.

\[u \rightarrow\]

**Figure 6.2** - $M(u)$ and $\alpha_2(u)$ of Govindarajulu distribution

**Example 6.5** In the case of power Pareto distribution described in Chapter 3,

$$M(u) = C(1-u)^{-1} \left( B_{1-u} (\lambda_1 + 1, 1 - \lambda_2) - u^{\lambda_1} (1-u)^{1-\lambda_2} \right)$$

and

$$\alpha_2(u) = C(1-u)^{-2} \left( 2B_{1-u} (\lambda_1 + 2, 1 - \lambda_2) - (u + 1)B_{1-u} (\lambda_1 + 1, 1 - \lambda_2) \right),$$

where

$$B_{1-u} (p, q) = \int_u^1 t^{p-1}(1-t)^{q-1} dt.$$  

In general $M(u)$ and $\alpha_2(u)$ possess different patterns of failures, through its functional behaviour, such as, both are increasing (e.g. $C = 1, \lambda_1 = 1, \lambda_2 = 0.5$), first increasing and then decreasing (e.g. $C = 1, \lambda_1 = 0.5, \lambda_2 = 5$), first decreasing and then increasing (e.g. $C = 1, \lambda_1 = 0.5, \lambda_2 = 1.5$), or decreasing (e.g. $C = 1, \lambda_1 = 0.5, \lambda_2 = 1$).
For the electric cart data given in Chapter 3, we have found the estimates as

$$\hat{\lambda}_1 = 0.234621, \hat{\lambda}_2 = 0.0966912, \hat{C} = 1530.53$$

The nature of $M(u)$ and $\alpha_2(u)$ functions for the data is presented in Figure 6.3.

![Figure 6.3: $M(u)$ and $\alpha_2(u)$ of power-Pareto distribution](image)

In the above discussions we compared the second L-moment of residual life with the mean and variance of the residual life. The coefficient of variation of residual life defined by

$$C(x) = \frac{\sigma(x)}{\mu(x)}.$$  \hfill (6.18)

Encouraging from the properties of coefficient of variation of residual life, here we define the L-coefficient of variation by

$$c(u) = \frac{\alpha_2(u)}{\alpha_1(u)}.$$  \hfill (6.19)

Gupta and Kirmani (2000) have shown that the coefficient of variation of residual life characterizes the life distribution. We now demonstrate that
a similar result exists for the L-coefficient of variation of the residual quantile function defined as \( c(u) = \frac{\alpha_{-2}(u)}{\alpha_{-1}(u)} \).

**Theorem 6.2.** If \( c(u) \) is differentiable, then

\[
Q(u) = g(u) \exp \left[ - \int g(u) \, du \right],
\]

where

\[
g(u) = \frac{(1-u)c'(u) - c(u) + 1}{(1-u)(1+c(u))}.
\]

**Proof:** From the definition of \( c(u) \), (6.6) and (6.7),

\[
\int_{u}^{1} (2p - u - 1)Q(p)\, dp = (1-u)c(u)\int_{u}^{1} Q(p)\, dp.
\]

Differentiating and simplifying

\[
(1-u)(1+c(u))Q(u) = [(1-u)c'(u) - c(u) + 1] \int_{u}^{1} Q(p)\, dp,
\]

rearranging the terms

\[
\frac{Q(u)}{\int_{u}^{1} Q(p)\, dp} = \frac{(1-u)c'(u) - c(u) + 1}{(1-u)(1+c(u))}.
\]

Integrating the above, we get

\[
-\log \int_{u}^{1} Q(p)\, dp = \int \frac{(1-u)c'(u) - c(u) + 1}{(1-u)(1+c(u))},
\]

from which (6.20) follows.

**Example 6.6** As a simple example, if

\[
c(u) = \frac{(1-u)}{3(1+u)}
\]

then

\[
\frac{(1-u)c'(u) - c(u) + 1}{(1-u)(1+c(u))} = \frac{2u}{1-u^2}
\]
Applying Theorem 6.20, we have
\[ Q(u) = 2u, \]
shows that \( c(u) \) determine the quantile function of uniform distribution with a change of scale.

In the next section we give the definition and the properties of L-moments of reversed residual life.

### 6.3 L-moments of reversed residual life

On almost similar lines we can treat the functions related to reversed residual life by considering \( X = (X | X \leq t) \) whose distribution is \( F(x) = \frac{F(x)}{F(t)}, \ 0 < x \leq t \). Using (6.1) the \( r^{th} \) L-moment of \( X \) has the expression
\[
B_r(t) = \sum_{k=0}^{r-1} (-1)^k \binom{r-1}{k} \int_0^t \left( \frac{F(x)}{F(t)} \right)^{r-k-1} \left( 1 - \frac{F(x)}{F(t)} \right)^k f(x) \frac{dx}{F(t)}. 
\]
(6.22)

In particular
\[
B_1(t) = \int_0^t \frac{xf(x)dx}{F(t)} = E(X | X \leq x) \]
(6.23)
and
\[
B_2(t) = \frac{1}{F(t)^2} \int_0^t (2F(x) - F(t))xf(x)dx. 
\]
(6.24)

Setting \( u = F(t) \) and \( p = F(x) \), we have
\[
\beta_i(u) = u^{-1} \int_0^u Q(p)dp 
\]
(6.25)
\[
\beta_1(u) = u^{-2} \int_{0}^{u} (2p - u)Q(p)\,dp. \quad (6.26)
\]

From (6.25), we have
\[
Q(u) = u\beta'_1(u) + \beta_1(u). \quad (6.27)
\]

Further from (6.26),
\[
\frac{d}{du} u^2 \beta_2(u) = u \int_{0}^{u} (2p - u)Q(p)\,dp
\]

\[
\Rightarrow u \beta'_2(u) + 2 \beta_2(u) = Q(u) - \frac{1}{u} \int_{0}^{u} Q(p)\,dp
\]

\[
= \frac{1}{u} \int_{0}^{u} (Q(u) - Q(p))\,dp
\]

\[
= R(u). \quad (6.28)
\]

Again from (6.28), we have
\[
\frac{d}{du} u^2 \beta_2(u) = uR(u).
\]

Integrating, we have
\[
\beta_2(u) = u^{-2} \int_{0}^{u} pR(p)\,dp. \quad (6.29)
\]

From (2.91) we have
\[
Q(u) = R(u) + \int_{0}^{u} p^{-1}R(p)\,dp. \quad (6.30)
\]

Hence using similar arguments in Theorem 6.1, we can conclude that each of \(Q(u), \ R(u), \ \beta_1(u)\) and \(\beta_2(u)\) determine the others uniquely. We have from (2.97), the reversed variance residual quantile function
\[
D(u) = u^{-1} \int_{0}^{u} R^2(p)\,dp
\]

\[
= u^{-1} \int_{0}^{u} \left( p\beta'_1(p) + 2\beta_2(p) \right)^2 \,dp. \quad (6.31)
\]

Similar to (6.19), we define the L-coefficient of variation in reversed time as
\[ \theta(u) = \frac{\beta_2(u)}{\beta_1(u)}. \]  
(6.32)

Following the steps of the proof of Theorem 6.2, we can show that \( \theta(u) \) determines the distribution up to a change of scale as

\[ Q(u) = \frac{u \theta'(u) + \theta(u) + 1}{u(1 - \theta(u))} \exp \left[ \int \frac{u \theta'(u) + \theta(u) + 1}{u(1 - \theta(u))} \, du \right]. \]  
(6.33)

As an example we can easily show that the power distribution is characterized by a constant value for \( \theta(u) \).

### 6.4 Characterizations

In this section we present some characterization theorems employing the reliability concepts discussed above that can help the identification of the underlying lifetime distribution. Our first result concerns the generalized Pareto distribution with quantile function

\[ Q(u) = \frac{b}{a} \left( (1 - u)^{-\frac{a}{a+1}} - 1 \right), \quad a > -1, \ b > 0, \]  
(6.34)

which is a family consisting of the exponential distribution \( (a \to 0) \), rescaled beta \( (-1 < a < 0) \) and the Lomax distribution \( (a > 0) \). The family is characterized by a linear mean residual life (reciprocal linear hazard rate) function in the conventional reliability analysis.

**Theorem 6.3.** Let \( X \) be a nonnegative continuous random variable with \( E(X) < \infty \). Then \( X \) follows the generalized Pareto distribution (6.34) if and only if any one of the following conditions is satisfied for all \( u \) in \( (0, 1) \).

1. \( \alpha_2(u) = CM(u), \quad 0 < C < 1 \)
2. \( \alpha_2(u) = a_1 \alpha_1(u) + a_2, \quad a_1 > -1, \ a_2 > 0 \)
3. \( \alpha_1(u) = AM(u) + B. \)
Proof: For the model (6.34), we have
\[ \alpha_1(u) = ba^{-1}\left( (a+1)(1-u)^{-\frac{a}{a+1}} - 1 \right), \]
\[ \alpha_2(u) = b(a+1)(a+2)^{-1}(1-u)^{-\frac{a}{(a+1)}} \]
and
\[ M(u) = b(1-u)^{-\frac{a}{(a+1)}}. \]
Then we have
\[ \alpha_2(u) = \frac{a+1}{a+2} M(u), \tag{6.35} \]
\[ \alpha_2(u) = \frac{1}{a+2} \left[ a\alpha_1(u) + \frac{b}{a(a+2)} \right], \tag{6.36} \]
and
\[ \alpha_1(u) = \frac{a+1}{a} M(u) - \frac{b}{a}. \tag{6.37} \]
Equations (6.35), (6.36) and (6.37) verify the conditions (i), (ii) and (iii).
To prove the only if part of (i), we assume (i), then from (6.15)
\[ C(1-u)^2 M(u) = \int_u^1 (1-p)M(p)dp. \]
On differentiation,
\[ C(1-u)M'(u) = (2C-1)M(u). \]
Rearranging,
\[ \frac{M'(u)}{M(u)} = \frac{2C-1}{C} \frac{1}{1-u}. \]
On integration this leads to
\[ M(u) = k(1-u)^{\frac{1-2C}{C}}, \quad k = M(0) = \mu. \]
Since $0 < C < 1$, we can write $C = \frac{a + 1}{a + 2}$ for $a > -1$ and obtain the generalized Pareto distribution. Notice that the exponential (rescaled beta, Lomax) distribution is characterized by $C = \frac{1}{2} \left( \frac{1}{2}, \frac{1}{2} \right)$.

In the case of (ii), it implies
\[
(1-u)^{-2} \int_u^1 (2p-u-1)Q(p)dp = (1-u)^{-1} a_i \int_u^1 Q(p)dp + a_2
\]
or
\[
\int_u^1 (2p-u-1)Q(p)dp = a_i (1-u) \int_u^1 Q(p)dp + a_2 (1-u)^2.
\]
Differentiating and simplifying
\[
(1+a_i) (1-u) Q(u) + 2a_2 (1-u) = (1-a_i) \int_u^1 Q(p)dp.
\]
Again differentiating, we have
\[
Q'(u) - \frac{2a_i Q(u)}{(1+a_i)(1-u)} = \frac{2a_2}{(1+a_i)(1-u)}.
\]
This is a linear differential equation with integrating factor $(1-u)^{-\frac{2a_i}{1+a_i}}$ and hence the solution is
\[
(1-u)^{-\frac{2a_i}{1+a_i}} Q(u) = -\frac{a_2}{a_i} (1-u)^{-\frac{2a_i}{1+a_i}} + C.
\]
Setting $u = 0$, $C = \frac{a_2}{a_i}$ and therefore,
\[
Q(u) = \frac{a_2}{a_i} \left( (1-u)^{-\frac{2a_i}{1+a_i}} - 1 \right),
\]
which is a generalized Pareto form (the form (6.34) results from the reparametrisation $a_i = \frac{a}{a + 2}, \ a_2 = \frac{b}{a + 2}$). The result (iii) follows from (i) and (ii) and the proof is completed.
Remark 6.3 The relationship between variance residual life and $\alpha_2(u)$ as measures of dispersion is of interest. It is seen from direct calculations that for the generalized Pareto distribution, 

$$V(u) = \frac{1 + a}{1 - a} b^2 (1 - u)^{-\frac{2a}{\alpha+1}}$$

$$= K\alpha_2^2(u), \quad K = \frac{(a + 2)^2}{1 - a^2}. \quad (6.38)$$

Now we examine whether (6.38) is a characteristic property. Equation (6.38) means that

$$K\alpha_2^2(u) = (1 - u)^{-1} \int_u^1 Q^2(p) dp - (M(u) + Q(u))^2.$$  

Differentiating and simplifying the resulting expression,

$$2K(1 - u)\alpha_2(u)\alpha_2'(u) - K\alpha_2'(u) = -Q^2(u) - 2(M(u) + Q(u))(M'(u) + Q'(u))(1 - u)$$

$$+ (M(u) + Q(u))^2.$$  

$$\quad (6.39)$$

Since

$$(1 - u)(M(u) + Q(u)) = \int_u^1 Q(p) dp,$$

$$(1 - u)(M'(u) + Q'(u)) - (M(u) + Q(u)) = -Q(u)$$

or

$$(1 - u)(M'(u) + Q'(u)) = M(u).$$

Substituting in (6.39),

$$K\alpha_2^2(u) - 2K(1 - u)\alpha_2(u)\alpha_2'(u) = M^2(u).$$

But using (6.10), we have

$$-3K\alpha_2^2(u) + 2K\alpha_2(u)M(u) = M^2(u),$$

which can be written by taking $y = \frac{\alpha_2(u)}{M(u)}$ as

$$3Ky^2 - 2Ky + 1 = 0,$$  

$$\quad (6.40)$$
The solutions of (6.40) are

\[ y = \frac{1}{3} \left( 1 \pm \left( \frac{K - \frac{3}{3}}{K} \right)^{\frac{1}{2}} \right). \]

The first solution leads to

\[ \alpha_2(u) = \frac{1}{3} \left( 1 + \left( \frac{K - \frac{3}{3}}{K} \right)^{\frac{1}{2}} \right) M(u) \]

and as such by (i) of Theorem 6.3, \( X \) is distributed as generalized Pareto distribution and with exponential (rescaled beta; Lomax) when \( K = 4 \) \((3 \leq K < 4, K > 4)\). However the second solution gives

\[ \alpha_2(u) = \frac{1}{3} \left( 1 - \left( \frac{K - \frac{3}{3}}{K} \right)^{\frac{1}{2}} \right) M(u) < \frac{1}{2} M(u), \]

for all \( K \geq 3 \) and therefore \( X \) is distributed as rescaled beta. As an example when \( X \) is exponential \((a = 0)\) or rescaled beta with \( a = -\frac{4}{5} \) gives \( \sigma^2(u) = 4\alpha^2_2(u) \). Thus (6.38) is not a characteristic property of the generalized Pareto distribution.

**Remark 6.4.** \( V(u) = \frac{1 + a}{1 - a} M^2(u), \ a > -1 \) characterizes the generalized Pareto distribution. To see this, use the above relationship in

\[ V(u) = (1 - u)^{-1} \int_u^1 M^2(p) dp, \]

\[ \frac{1 + a}{1 - a} M^2(u) = (1 - u)^{-1} \int_u^1 M^2(p) dp. \]

Differentiating

\[ \frac{1 + a}{1 - a} [2(1 - u)M(u)M'(u) - M^2(u)] = -M^2(u), \]

which leads to
\[ M'(u) = \frac{2a}{M(u)} \]

Integrating and simplifying

\[ M(u) = K(1-u)^{-2a}, \]

the expression for \( M(u) \) of the generalized Pareto distribution. This result is proved earlier in Gupta and Kirmani (2004) using the distribution function approach.

The next theorem states the distribution corresponding to the sum of two second L-moment of residual lives (mean residual lives).

**Theorem 6.4.** If \( \alpha_{21}(u)(M_1(u)) \) and \( \alpha_{22}(u)(M_2(u)) \) are second L-moment (mean residual) quantile function of two random variable \( X \) and \( Y \), then \( \alpha_{21}(u)+\alpha_{22}(u)(M_1(u)+M_2(u)) \) is the second L-moment residual (mean residual) quantile function of the distribution with \( Q(u)=Q_1(u)+Q_2(u) \).

**Proof:** Follows directly from the definitions of \( \alpha_2(u) \) and \( M(u) \).

Parallel characterizations hold in the case of reversed L-moment quantile functions, where the role of the generalized Pareto distribution is taken by the power distribution. The proof follows the same pattern as in the previous cases and therefore they are omitted.

**Theorem 6.5** Let \( X \) be distributed as the power distribution with quantile function

\[ Q(u)=au^{\frac{1}{b}}, \quad a,b>0, \quad 0 \leq u \leq 1 \] (6.41)

Then for all \( u \),

1. \( \beta_1(u)=C_1Q(u), \quad 0<C_1<1 \)
2. \( \beta_2(u)=C_2\beta_1(u), \quad 0<C_2<1 \)
(iii) \( \beta_2(U) = C_3R(u) \),

and conversely.

**Remark 6.5** \( D(u) = K\beta_2^2(u) \) for the power distribution, reduces to the quadratic equations

\[
3Kz^2 - 2Kz + 1 = 0,
\]

where \( z = \frac{\beta_2(u)}{R(u)} \). As before the solutions are

\[
\beta_2(u) = \frac{1}{3} \left( 1 \pm \left( \frac{K-3}{K} \right)^{\frac{1}{3}} \right) R(u).
\]

Since \( \frac{\beta_2(u)}{R(u)} < \frac{1}{2} \) one should have \( 3 < K < 4 \) for the first solution and the second solution is valid for all \( K > 3 \). Hence there is a characterization of the power solution for all \( K \geq 4 \) and two power distributions result as solutions whenever \( 3 < K < 4 \).

**Theorem 6.6** The identity

\[
\beta_2(u) = \frac{a-bu}{c-au} \beta_1(u)
\]  

(6.42)

is satisfied for all \( u \) and constants \( a, b, c \) satisfying

\[
a > b, \ c > a, \text{ and } \frac{c-a}{c+a} = \frac{2b}{a-b}
\]  

(6.43)

if and only if \( X \) follows Govindarajulu distribution.

**Proof:** When \( X \) has the distribution stated in the theorem

\[
\beta_1(u) = \sigma u^\beta \left[ 1 - \frac{\beta}{\beta + 2} u \right]
\]

and
showing that form (6.42) and condition (6.43) are met with. Conversely from (6.42) and definitions of the function involved,

\[ \int_0^a (2p-u)Q(p)dp = \frac{u(a-bu)}{c-au} \int_0^a Q(p). \]  

(6.44)

Differentiating (6.44) with respect to \( u \) and simplifying

\[ \frac{Q(u)}{\int_0^a Q(p)} = \frac{(c-du)(a-2bu) + au(a-bu) + (c-au)^2}{u(c-au)(c-a+u(b-a))} \]

\[ = \frac{c+a}{(c-a)u} - \frac{a}{c-au} + \frac{\left( \frac{c+a}{c-a} \right)(a-b) - 2b}{(c-a+(b-a)u)}. \]

The last factor vanishes by virtue of (6.43) and hence on integration,

\[ \int_0^a Q(p)dp = Ku^{c-a}(c-au) \]

or

\[ Q(u) = Kc \left( \frac{c+a}{c-a} u^{c-a-1} - \left( \frac{c+a}{c-a} \right) u^{c-a} \right), \]

which is the quantile function of Govindarajulu distribution with parameters \( \sigma = Kc \) and \( \beta = \frac{c+a}{c-a} - 1 \). This completes the proof.

**Remark 6.6** Condition (6.43) can be modified to derive a more general family of distributions satisfying (6.42), but the resulting four-parameter family provides much complicated forms of properties which are not easy for practical use.
6.5 Applications

We have indicated some applications of $\alpha_2(u)$ and $\beta_2(u)$ in modelling lifetime data in the previous sections. In this section we point out some more applications in reliability analysis and also in economics. A detailed study has to be taken up separately.

6.5.1. Reliability:

When conceived as a reliability function the L-moment $\alpha_2(u)$ can also be employed in distinguishing life distributions based on its monotonic behaviour. Since $\alpha_2(u)$ and $\beta_2(u)$ are twice the mean difference, the monotonic behaviour of $\alpha_2(u)$ and $\beta_2(u)$ are those of the corresponding mean differences. Thus we have the following definition of the ageing class based on mean difference in terms of $\alpha_2(u)$ and $\beta_2(u)$.

**Definition 6.1:** The random variable $X$ is said to be increasing (decreasing) mean difference quantile function – IMDQ (DMDQ) according as $\alpha_2(u)$ is increasing (decreasing). Similarly increasing reversed mean difference quantile function (IRMDQ) and decreasing reversed mean difference quantile function (RDMDQ) are defined with respect to $\beta_2(u)$. Further the mean difference quantile function is first increasing (decreasing) and then decreasing (increasing) with change point at $u = u_0$ will be denoted by IDMDQ (DIMDQ).

**Example 6.7** From the expressions of $\alpha_2(u)$ given in the proof of Theorem 6.3, it is clear that the Lomax distribution is IMDQ and the beta distribution is DMDQ. In the Govindarajulu distribution, as it is
mentioned earlier that \( \alpha_2(u) \) is first increasing and then decreasing for \( \beta > 1 \), the Govindarajulu distribution is IDMDQ.

The analytic condition for \( X \) to be IMDQ or DMDQ is derived from
\[
M(u) = 2\alpha_2(u) - (1-u)\alpha'_2(u)
\]
as
\[
\alpha'_2(u) > 0 \Rightarrow \alpha_2(u) > \frac{1}{2} M(u).
\]
Thus \( X \) is IMDQ (DMDQ) according as \( \alpha_2(u) \geq (\leq) \frac{1}{2} M(u) \). In the case of ID (DI) MDQ, the change point \( u_0 \) is obtained from \( \alpha_2(u_0) = \frac{1}{2} M(u_0) \).

Obviously, the exponential distribution, as in the case of other ageing classes, separates the increasing and decreasing MDQ classes with constant mean difference.

On the other hand, the behaviour of \( \beta_2(u) \) results with
\[
\beta'_2(u) = u^{-1}(R(u) - 2\beta_2(u))
\]
and hence the turning point of \( \beta_2(u) \), if any will be the solution of \( R(u) = 2\beta_2(u) \). Looking at a more general equation, \( R(u) = C\beta_2(u) \), \( C > 0 \), we find
\[
u^2 R(u) = C \int_0^u p R(p) dp , \quad \text{using (6.29)}
\]
Differentiating and simplifying
\[
\frac{R'(u)}{R(u)} = \frac{c-2}{u}.
\]
On integration
\[
R(u) = Ku^{c-2}.
\]
Applying on (6.45)
\[ Q(u) = K \frac{C-1}{C-2} u^{c-2}, \]
which provides a proper distribution on the positive real line only if \( C > 2 \). For \( C > 2 \), \( \beta_2'(u) > 0 \) implies that for a nonnegative random variable there is no change point for \( \beta_2(u) \) and it is an increasing function for all \( u \). Unlike \( \alpha_2(u) \), there is limited use for \( \beta_2(u) \) in classifying life distributions on the basis of its monotonicity.

6.5.2 Economics

In this section we point out the scope of L-moment of reversed residual life in the field of economics. Let \( X \) be the random variable representing the personal incomes in a population. Modelling the distribution of incomes is a traditional problem in which the use of lambda distributions is of recent interest (Tarsitano (2004), Haritha et al. (2008)). One important application of income distributions is the analysis of poverty in a population. This is often accomplished by the choice of a criterion that decides whether an individual is poor and an index which summarizes the amount of poverty in the population under consideration. Taking the poverty line as \( X = t \), so that an individual whose income is below \( t \) is considered as poor, the well known index, proposed by Sen (1976) is often used in this context. The Sen Index is defined as

\[ p(t) = F(t)(i(t) + (1-i(t))g(t)), \]

where \( F(t) \) is interpreted as the headcount ratio,

\[ i(t) = 1 - E \left( \frac{X}{t} \mid X \leq t \right), \]

is known as the income gap ratio for the poor and
is the Gini index for poor. In terms of quantile functions, 

\[ G(u) = g(Q(u)) = 1 - \frac{2}{\beta_1(u)} \left( \int_{Q(u)}^u Q(p) \frac{u - p}{u^2} dp \right). \]  

(6.49)

Note that

\[ \int_{Q(u)}^u Q(p) dp = u \beta_1(u) \]

and

\[ \int_{Q(u)}^u pQ(p) dp = \frac{u^2}{2} \left[ \frac{1}{u^2} \int_{Q(u)}^u (2p - u)Q(p) dp + \frac{1}{u} \int_{Q(u)}^u Q(p) dp \right] \]

\[ = \frac{u^2}{2} [\beta_1(u) + \beta_2(u)]. \]

Thus (6.49) become

\[ G(u) = \frac{\beta_2(u)}{\beta_1(u)}, \]

the L-coefficient of variation and

\[ I(u) = i(Q(u)) = 1 - \frac{\beta_1(u)}{Q(u)}. \]

Thus the Sen index (5.46) has the quantile analogue

\[ P(u) = u \left[ 1 - \frac{\beta_1(u)}{Q(u)} + \frac{\beta_2(u)}{Q(u)} \right] \]

\[ = \frac{u}{Q(u)} (R(u) + \beta_2(u)). \]

Using the above expression we can express \( P(u) \) in terms of \( \beta_1(u) \) and \( \beta_2(u) \) alone by noting \( Q(u) = u \beta_1'(u) + \beta_1(u) \) as

\[ P(u) = u \left[ \frac{u \beta_1'(u) + \beta_2(u)}{u \beta_1'(u) + \beta_1(u)} \right]. \]
The above formula becomes handy when quantile functions, whose distributions are not available in closed form, are employed for modelling income data (like the lambda distributions). A detailed discussion in this respect is available in Haritha et al. (2008). Further from (5.29) and (5.30) it is evident that, instead of the distribution, if $R(u)$ can be specified then $P(u)$ can be determined from it.

From the earlier discussion it also evident that there is one to one correspondence between the income gap ratio $I(u)$ (and also the Gini index $G(u)$) and the baseline income distribution of $X$. The same cannot be said about the correspondence between $Q(u)$ and $P(u)$.

We conclude the present study by noting that the second $L$-moment of residual life (or equivalently the mean difference) possesses properties similar to the variance residual life. It can be useful in modelling, characterizing and analyzing lifetime data, and the quantile-based approach adds to its applicability to empirical models where the distribution function cannot be expressed in simple analytical form.