Chapter 1
PRELIMINARIES

1.1 Introduction

Topology is an indispensable object of study in Mathematics. The study of topological spaces, their continuous functions and general properties make up a branch of topology known as General topology. Open sets as well as closed sets being the most fundamental concepts in the study and investigation in general topological spaces, the concepts have been generalized and studied by many authors from different points of views. The present work is an elaborate study of a new type of generalized closed sets in topological spaces called $g\delta s$-closed sets, their respective continuous functions, closed functions, homeomorphisms, compactness, connectedness, regular spaces, normal spaces and also semi totally continuous functions.

In this chapter, the author recalls the recent developments in topology contributed by various authors. Section 1 begins with the discussion of stronger and weaker forms of open sets. Section 2 deals with the stronger and weaker forms of continuous functions, while section 3 is devoted to irresolute functions, closed and open functions. Some generalized homeomorphisms, new types of compact spaces, connected spaces, regular spaces and normal spaces are discussed in section 4, while section 5 outlines the studies carried out by the author on new generalizations of closed sets and continuous functions.

Throughout the thesis $(X, \tau)$, $(Y, \sigma)$ and $(Z, \eta)$ or simply $X$, $Y$ and $Z$ always mean topological spaces on which no separation axioms are assumed unless explicitly stated. For a subset $A$ of a space $X$, the closure and interior of $A$ with respect to $\tau$ are denoted by $Cl(A)$ and $Int(A)$ respectively. The
complement of A is denoted by $A^c$ or $X - A$.

1.2 Stronger and weaker forms of open sets

In 1937, Stone [75] introduced new sets, which are stronger forms of open sets and closed sets called regular open and regular closed sets respectively in topological spaces. The stronger forms of open sets called $\theta$-open sets and $\delta$-open sets respectively were introduced and studied by Tong [80] and Velicko [82]. Some weaker forms of closed sets such as semi preclosed ($\beta$-closed [1], semiclosed and preclosed sets were introduced by Andrijevic [3], Levine [39] and Mashhour [49] respectively.

The study of so called generalized closed (briefly, $g$-closed) sets, i.e. the sets whose closure belongs to every open superset was the initiation of Levine [38]. The concept of $g$-closed sets was modified and studied in last decade by using weaker forms of open sets such as $\alpha$-open sets [53], semiopen sets [39] preopen sets [49] and semi preopen sets [3]. The majority of the modifications are in fact weaker than $g$-closedness.

The complements of the various types of open (resp. closed) sets are called same type of closed (resp. open) sets. That is complement of the regular open set is called a regular closed set and the complement of the generalized closed set is called generalized open set and soon. The following are some definitions and results from various authors which are useful in the sequel of the thesis.

**Definition 1.2.1.** A subset $A$ of a topological space $X$ is called

(i) regular open [75] if $A = \text{Int}(\text{Cl}(A))$

(ii) $\alpha$-open [53] if $A \subset \text{Int}(\text{Cl}(\text{Int}(A)))$

(iii) preopen [49] if $A \subset \text{Int}(\text{Cl}(A))$

(iv) semiopen [39] if $A \subset \text{Cl}(\text{Int}(A))$

(v) semi preopen [3] ($=\beta$-open set [1]) if $A \subset \text{Cl}(\text{Int}(\text{Cl}(A)))$
(vi) $\delta$-closed [82] if $A = Cl_\delta(A)$, where $Cl_\delta(A) = \{ x \in X : \text{Int}(Cl(U)) \cap A \neq \emptyset \land U \in \tau \text{ and } x \in U \}$

(vii) $\delta$-preopen [65] if $A \subset \text{Int}(Cl_\delta(A))$

For a subset $A$ of $X$, the intersection of all semiclosed subsets of $X$ containing $A$ is called semi closure of $A$ and is denoted by $sCl(A)$ and semi interior of $A$ is the union of all semiopen sets of $X$ contained in $A$ and is denoted by $sInt(A)$. The $\alpha$-closure (resp. pre closure, semi preclosure) of a subset $A$ of a space $X$ is the intersection of all $\alpha$-closed (resp. pre closed, semi preclosed) sets that contain $A$ and is denoted by $\alpha-Cl(A)$ (resp. $pCl(A)$, $spCl(A)$). The $\alpha$-interior (resp. pre interior, semi pre interior) of a subset $A$ of a space $X$ is the union of all $\alpha$-closed (resp. preopen, semi preopen) sets contained in $A$ and is denoted by $\alpha-Int(A)$ (resp. $pInt(A)$, $spInt(A)$).

**Definition 1.2.2.** [10] The intersection of all $\delta$-open subsets of $X$ containing $A$ is called $\delta$-kernel of $A$ and is denoted by $ker_\delta(A)$.

**Definition 1.2.3.** [66] The intersection of all semiopen subsets of $X$ containing $A$ is called semi kernel of $A$ and is denoted by $sker(A)$.

**Definition 1.2.4.** [15] A point $x \in X$, is said to be a semi limit point of subset $A$ of $X$ if and only if every semiopen set $U$ containing $x$, $U \cap (A - \{x\}) \neq \emptyset$. The set of all semi limit points of $A$ is called semi derived set of $A$ and is denoted by $D_\delta(A)$.

**Definition 1.2.5.** A subset $A$ of a topological space $X$ is called

(i) a generalized closed (briefly $g$-closed) [38] if $Cl(A) \subset U$ whenever $A \subset U$ and $U$ is open in $X$.

(ii) a semi generalized closed (briefly $sg$-closed) [8] if $sCl(A) \subset U$ whenever $A \subset U$ and $U$ is semiopen in $X$.

(iii) a generalized semiclosed (briefly $gs$-closed) [5] if $sCl(A) \subset U$ whenever $A \subset U$ and $U$ is open in $X$. 

3
(iv) an $\alpha$-generalized closed (briefly $\alpha g$-closed) \cite{44} if $\alpha Cl(A) \subset U$ whenever $A \subset U$ and $U$ is open in $X$.

(v) generalized preclosed (briefly $gp$-closed) \cite{47} if $pCl(A) \subset U$ whenever $A \subset U$ and $U$ is open in $X$.

(vi) $\omega$-closed \cite{76} if $Cl(A) \subset U$ whenever $A \subset U$ and $U$ is semiopen in $X$.

**Definition 1.2.6.** A topological space $X$ is said to be

(i) $\alpha$-space \cite{53} if every $\alpha$-closed subset of $X$ is closed in $X$.

(ii) $T_{1/2}$-space \cite{38} if every $g$-closed subset of $X$ is closed in $X$.

(iii) semi $T_{1/2}$-space \cite{8} if every $sg$-closed subset of $X$ is semiclosed in $X$.

(iv) $T_{3/4}$ \cite{23} if every $\delta g$-closed subset of $X$ is $\delta$-closed in $X$.

**Lemma 1.2.7.** \cite{15} Let $X$ be a topological space and $A \subset X$, then

(i) $A \subset sCl(A)$ and $sCl(A) = A \cup D_s(A)$

(ii) $A$ is semiclosed if and only if $A = sCl(A)$

(iii) $sInt(A) \subset A$

(iv) $A$ is semiopen if and only if $A = sInt(A)$

(v) $sCl(A) \subset sCl(B)$ and $sInt(A) \subset sInt(B)$, whenever $A \subset B$.

(vi) $sCl(sCl(A)) = sCl(A)$ and $sInt(sInt(A)) = sInt(A)$

(vii) $sCl(X - A) = X - sInt(A)$

(viii) $sInt(X - A) = X - sCl(A)$

### 1.3 Stronger and weaker forms of continuous functions

The class of continuous functions plays an important role in general topological space. The stronger and weaker forms of continuity were introduced and studied by several topologist. This section presents an overview of strong and weak forms of continuous functions contributed by various topologist. Levine \cite{40}, Noiri \cite{60} and Jain \cite{34} introduced and studied strong continuous functions, perfectly continuous functions and totally continuous
functions respectively, which are strong forms of continuous functions.

In 1963, Levine [39] introduced and studied the weak forms of continuity namely, semi continuity. The weaker forms of continuous functions such as $g$-continuity, $\alpha$-continuity and pre-continuity, $\beta$-continuity, $sg$-continuity, $gpr$-continuity, $gs$-continuity and $\alpha g$-continuity, $gsp$-continuity, $gp$-continuity, $\omega$-continuity and $\alpha gs$-continuity were the contributions of Balachandran et al [6], Mashhour et al [49, 50], Abd El-Monsef et al [1], Biswas [9], Gnanambal [33], Devi et al [17, 19], Dontchev [24], Maki et al [45], Sundaram and Sheik John [76], and Rajamani and Viswanathan [64] respectively.

**Definition 1.3.1.** A function $f : X \rightarrow Y$ is called

(i) $\delta$-continuous [58] $f^{-1}(A)$ is $\delta$-closed in $X$ for every $\delta$-closed set $A$ of $Y$.
(ii) semi continuous [39] if $f^{-1}(A)$ is semiclosed in $X$ for every closed set $A$ of $Y$.
(iii) pre continuous [49] if $f^{-1}(A)$ is pre closed in $X$ for every closed set $A$ of $Y$.
(iv) $\alpha$-continuous [50] if $f^{-1}(A)$ is $\alpha$-closed in $X$ for every closed set $A$ of $Y$.
(v) $\beta$-continuous [1] if $f^{-1}(A)$ is $\beta$-closed in $X$ for every closed set $A$ of $Y$.
(vi) semi pre continuous [3] if $f^{-1}(A)$ is semi preopen set in $X$ for every open set $A$ of $Y$.
(vii) $g$-continuous [6] if $f^{-1}(A)$ is $g$-closed set in $X$ for every closed set $A$ of $Y$.
(viii) $sg$-continuous [9] if $f^{-1}(A)$ is $sg$-closed set in $X$ for every closed set $A$ of $Y$.
(ix) $gs$-continuous [78] if $f^{-1}(A)$ is $gs$-closed set in $X$ for every closed set $A$ of $Y$.
(x) $\alpha g$-continuous [17] if $f^{-1}(A)$ is $\alpha g$-closed set in $X$ for every closed set $A$ of $Y$. 
(xi) $gsp$-continuous [24] if $f^{-1}(A)$ is $gsp$-closed set in $X$ for every closed set $A$ of $Y$.

(xii) $gp$-continuous [47] if $f^{-1}(A)$ is $gp$-closed set in $X$ for every closed set $A$ of $Y$.

(xiii) $gpr$-continuous [33] if $f^{-1}(A)$ is $gpr$-closed set in $X$ for every closed set $A$ of $Y$.

(xiv) $\omega$-continuous [66] if $f^{-1}(A)$ is $\omega$-closed set in $X$ for every closed set $A$ of $Y$.

(xv) $agr$-continuous [81] if $f^{-1}(A)$ is $agr$-closed set in $X$ for every closed set $A$ of $Y$.

**Definition 1.3.2.** A function $f : X \to Y$ is called
(i) strongly continuous [40] if $f^{-1}(G)$ is both open and closed in $X$ for each subset $G$ of $Y$.

(ii) perfectly continuous [60] if $f^{-1}(G)$ is both open and closed set in $X$ for each open set $G$ of $Y$.

(iii) totally semi continuous function [61] if $f^{-1}(G)$ is both semiopen and semiclosed set in $X$ for each open set $G$ of $Y$.

(iv) perfectly $g$-continuous [77] if $f^{-1}(G)$ is both open and closed set in $X$ for each $g$-open set $G$ of $Y$.

(v) completely continuous [4] if $f^{-1}(G)$ is regular open set in $X$ for each open set $G$ of $Y$.

(vi) totally continuous [34] if $f^{-1}(G)$ is clopen set in $X$ for each open set $G$ of $Y$.

(vii) strongly semi continuous [61] if $f^{-1}(G)$ is semi clopen in $X$ for each subset $G$ of $Y$.

(viii) $R$-map [12] if $f^{-1}(G)$ is regular open set in $X$ for each regular open set $G$ of $Y$.
1.4 Irresolute, closed and open functions

Crossely and Hildebrand [13] introduced the concepts of irresolute functions, which are independent of continuous functions and stronger than semi continuous functions. Sundaram [78] introduced and investigated $g$-irresolute functions, while Devi et.al [16], Sheik John [66] and Rajamani and Viswanathan [64] introduced and studied $\omega$-irresolute, $\omega$-irresolute and $\omega g$-irresolute functions respectively.

In 1982, Malghan [48] introduced and studied the concept of generalized closed functions and obtained the preservation theorem of normality and regularity. After that several topologist like R. Devi, H. Maki and K. Balachandran [18] introduced the concept of $sg$ and $gs$ closed maps and studied some of their basic properties. Sundaram [78], Noiri [55], Biswas [9], Mashhour et al [49, 50], Maki et al [47], Devi et al [16], Gnanambal [33], Sheik John[66], Veera Kumar [81] and Rajamani and Viswanathan [64] introduced and studied generalized open functions, semiclosed functions, semiopen functions, $\alpha$-open functions, $g$-closed functions, $\alpha g$-closed and $\alpha g s$-closed functions, $g pr$-closed functions, $\omega$-closed and $\omega$-open functions, $\alpha g r$-closed functions and $\alpha g s$-closed and $\alpha g s$-open functions respectively. The concept of quasi-$\alpha$-closed and strongly $\alpha$-closed maps in topological spaces was introduced by G. B. Navalagi [52].

**Definition 1.4.1.** A function $f : X \to Y$ is called

(i) irresolute [13] if $f^{-1}(G)$ is semiopen set in $X$ for each semiopen set $G$ of $Y$.

(ii) $g$-irresolute [78] if $f^{-1}(G)$ is $g$-closed set in $X$ for $g$-closed set $G$ of $Y$.

(iii) $\alpha g$-irresolute [17] if $f^{-1}(G)$ is $\alpha g$-closed set in $X$ for every $\alpha g$-closed set $G$ of $Y$.

(iv) $gs$-irresolute [78] if $f^{-1}(G)$ is $gs$-closed set in $X$ for every $gs$-closed set $G$ of $Y$. 
(v) $\omega$- irresolute [66] if $f^{-1}(G)$ is $\omega$-closed set in $X$ for $\omega$-closed set $G$ of $Y$.
(vi) $\alpha_{gs}$- irresolute [64] if $f^{-1}(G)$ is $\alpha_{gs}$-closed set in $X$ for every $\alpha_{gs}$-closed set $G$ of $Y$.

**Definition 1.4.2.** A function $f : X \rightarrow Y$ is called
(i) $\delta$-closed [56] if $f(G)$ is $\delta$-closed in $Y$ for every $\delta$-closed set $G$ in $X$.
(ii) preclosed [59] if $f(G)$ is closed in $Y$ for every semiclosed set $G$ in $X$.
(iii) semi closed [71] if $f(G)$ is semiclosed in $Y$ for every semiclosed set $G$ in $X$.
(iv) $g$-closed [48] (resp. $g$-open [78]) if $f(G)$ is $g$-closed (resp. $g$-open) in $Y$ for every closed (resp. open) set $G$ in $X$.
(v) semi closed [55] if $f(G)$ is semi closed in $Y$ for every closed set $G$ in $X$.
(vi) $sg$-open [18] if $f(G)$ is $sg$-open in $Y$ for every open set $G$ in $X$.
(vii) $\alpha_{gs}$-closed [16] (resp. $\alpha_{gs}$-open [16]) if $f(G)$ is $\alpha_{gs}$-closed (resp. $\alpha_{gs}$-open) in $Y$ for every closed (resp. open) set $G$ in $X$.
(viii) $gp$-closed [57] if $f(G)$ is $gp$-closed in $Y$ for every closed set $G$ in $X$.
(ix) preclosed [30] if $f(G)$ is preclosed in $Y$ for every closed set $G$ in $X$.

**Definition 1.4.3.** A function $f : X \rightarrow Y$ is called
(i) quasi-$\alpha$-closed [52] if the image of each $\alpha$-closed set in $X$ is closed in $Y$.
(ii) strongly $\alpha$-closed [52] if the image of each $\alpha$-closed set in $X$ is $\alpha$-closed in $Y$.
(iii) quasi-$\alpha$-open [36] if the image of each $\alpha$-open set in $X$ is open in $Y$.
(iv) strongly $\alpha$-open [36] if the image of each $\alpha$-open set in $X$ is $\alpha$-open in $Y$.

**Definition 1.4.4.** [37] A surjective function $f : X \rightarrow Y$ is called
(i) an $\alpha$-quotient function if $f$ is $\alpha$-continuous and $f^{-1}(V)$ is open in $X$ implies that $V$ is $\alpha$-open in $Y$.
(ii) a semi quotient function if $f$ is semi continuous and $f^{-1}(V)$ is open in $X$ implies that $V$ is semiopen in $Y$. 

8
(iii) a pre-quotient function if $f$ is pre-continuous and $f^{-1}(V)$ is open in $X$ implies that $V$ is pre-open in $Y$.

1.5 Generalized homeomorphisms and other related concepts

Many researchers have generalized the notion of homeomorphisms in topological spaces. Maki et al [46] introduced $g$-homeomorphisms in topological spaces.

In 1987, the study of a new class of compact spaces called $s$-closed spaces using semiopen sets was taken up by Di Maio and Noiri [20]. Balachandran et al [6] introduced the concept of $GO$-compactness using $g$-open sets followed by Devi [19] who introduced the notions of $\alpha GO$-compactness and $\alpha GO$-connectedness by using $\alpha g$-open sets. The recent additions to the field were by Gnanamandal et al [32] and Sheik John [66]. They introduced and investigated generalized pre-regular compact spaces (briefly $gpr$-compact) and $\omega$-compactness in topological spaces using $gpr$-open and $\omega$-open sets respectively. Sundaram [78] introduced the concepts of $GO$-compact spaces and $GO$-connected spaces using $g$-open sets. In 1994, semi pre connectedness and $\beta$-connectedness, which are equivalent notions were defined and investigated independently by Aho and Nieminan [2] and Popa and Noiri [63] respectively.

Definition 1.5.1. A bijective function $f : X \rightarrow Y$ is called
(i) generalized homeomorphism (g-homeomorphism) [46] if $f$ is both $g$-continuous and $g$-open.
(ii) gc-homeomorphism [46] if both $f$ and $f^{-1}$ are gc-irresolute functions.
(iii) $\alpha$-generalized homeomorphism (ag-homeomorphism) [19] if $f$ is both $ag$-continuous and $ag$-open.

Definition 1.5.2. A topological space $X$ is said to be
(i) semi compact [12] if every semiopen cover of $X$ has a finite subcover.
(ii) GO-compact [6] if every $g$-open cover of $X$ has a finite subcover.
(iii) $\alpha$GO-compact [19] if every $ag$-open cover of $X$ has a finite subcover.
(iv) GPR-compact [32] if every GPR-open cover of $X$ has a finite subcover.
(v) mildly compact [73] if every clopen cover of $X$ has a finite subcover.
(vi) nearly compact [70] if every regular open cover of $X$ has a finite subcover.
(vii) semi countably compact [11] if every countable cover of $X$ by semiopen sets has a finite subcover.
(viii) nearly countably compact [69] if every countable cover of $X$ by regular open sets has a finite subcover.
(ix) nearly Lindelöf [29] if every regular open cover of $X$ has a countable subcover.
(x) $S$-Lindelöf [29] if every cover of $X$ by regular closed sets has a countable subcover.
(xi) mildly Lindelöf [73] if every cover of $X$ by clopen sets has a countable subcover.
(xii) countably $S$-closed [21] if every countable cover of $X$ by regular closed sets has a finite subcover.
(xiii) $S$-closed [79] if every regular closed cover of $X$ has a finite subcover.
(xiv) Strongly $S$-closed [23] if every closed cover of $X$ has a finite subcover.
Definition 1.5.3. [14] Let \( X \) be a topological space and \( A \) and \( B \) be two non void subsets of \( X \). Then \( A \) and \( B \) are said to be semi separated if \( A \cap sCl(B) = sCl(A) \cap B = \emptyset \).

Definition 1.5.4. [61] Let \( X \) be a topological space. Then the set of all points \( y \) in \( X \) such that \( x \) and \( y \) cannot be separated by semi-separation of \( X \) is said to be the quasi semi component of \( x \).

A quasi semi component of point \( x \) in a space \( X \) means the intersection of all semi clopen sets containing \( x \).

Definition 1.5.5. A topological space \( X \) is said to be
(i) semi connected [27] if \( X \) cannot be expressed as the union of two non empty semiopen sets.
(ii) pre connected [22] if \( X \) cannot be expressed as the union of two non empty preopen sets.
(iii) \( \beta \)-connected [63] if \( X \) cannot be expressed as the union of two non empty \( \beta \)-open sets.
(iv) hyperconnected [74] if every open set is dense.

Definition 1.5.6. A topological space \( X \) is said to be
(i) semi \( T_1 \) [41] (resp. clopen \( T_1 \) [28]) if for each pair of distinct points \( x \) and \( y \) of \( X \), there exist semiopen (resp. clopen) sets \( U \) and \( V \) containing \( x \) and \( y \) respectively such that \( x \in U \), \( y \notin U \) and \( y \in V \), \( x \notin V \).
(ii) semi \( T_2 \) [41] (resp. ultra Hausdorff or \( UT_2 \) [73]) if every two distinct points of \( X \) can be separated by disjoint semiopen (resp. clopen) sets.
(iii) Urysohn [62] if for each pair of distinct points \( x \) and \( y \) of \( X \), there exist open sets \( U \) and \( V \) containing \( x \) and \( y \) respectively such that \( Cl(U) \cap Cl(V) = \emptyset \).
(iv) weakly Hausdorff [72] if each element of \( X \) is an intersection of regular closed sets.
(v) s-regular [43] if for each closed set $F$ of $X$ and each point $x \notin F$, there exist disjoint semiopen sets $U$ and $V$ such that $x \in U$ and $F \subset V$.
(vi) semi regular [26] if for each semiclosed set $F$ of $X$ and each point $x \notin F$, there exist disjoint semiopen sets $U$ and $V$ such that $x \in U$ and $F \subset V$.
(vii) clopen regular [28] if for each clopen set $F$ of $X$ and each point $x \notin F$, there exist disjoint open sets $U$ and $V$ such that $x \in U$ and $F \subset V$.
(viii) ultra regular [31] if for each closed set $F$ of $X$ and each point $x \notin F$, there exist disjoint clopen sets $U$ and $V$ such that $x \in U$ and $F \subset V$.
(ix) almost regular [68] if for each regular closed set $F$ of $X$ and each point $x \notin F$, there exist disjoint open sets $U$ and $V$ such that $x \in U$ and $F \subset V$.

Definition 1.5.7. A topological space $X$ is said to be
(i) s-normal [42] if for each pair of disjoint closed sets $A$ and $B$ in $X$, there exist disjoint semiopen sets $U$ and $V$ such that $A \subset U$ and $B \subset V$.
(ii) semi normal [25] if for each pair of disjoint semiclosed sets $A$ and $B$ in $X$, there exist disjoint semiopen sets $U$ and $V$ such that $A \subset U$ and $B \subset V$.
(iii) clopen normal [28] if for each pair of disjoint clopen sets $A$ and $B$ in $X$, there exist disjoint open sets $U$ and $V$ such that $A \subset U$ and $B \subset V$.
(iv) almost normal [67] if for each pair of disjoint regular open sets $A$ and $B$ in $X$, there exist disjoint open sets $U$ and $V$ such that $A \subset U$ and $B \subset V$.
(v) ultra normal [73] if for each pair of disjoint open sets $A$ and $B$ in $X$, there exist disjoint open sets $U$ and $V$ such that $A \subset U$ and $B \subset V$.

1.6 Generalizations of closed sets and continuous functions

In the light of the above work, the author has obtained several interesting generalizations of closed sets and continuous functions in topological spaces on the following topics.
(i) $g\delta s$-closed and $g\delta s$-open sets in topological spaces and its properties.
(ii) strongly $g\delta s$-continuous and other related functions in topological spaces.
(iii) $g\delta s$-open, $g\delta s$-closed, $pg\delta s$-closed, strongly $g\delta s$-closed and quasi $g\delta s$-closed functions in topological spaces.

(iv) $g\delta s$-regular, $g\delta s$-normal spaces and weak separation axioms via $g\delta s$-closed sets.

(v) $g\delta s$-homeomorphisms, $g\delta s$-quotient functions, $g\delta s$-compactness, $g\delta s$-connectedness and other related results.

(vi) Semi totally continuous functions in topological spaces. The rest of the thesis is the detailed study of the above topics.