Chapter 6

$g\delta s$-HOMEOMORPHISMS AND OTHER CONCEPTS

6.1 Introduction


The chapter contains five sections. In the section 2, the concept of a new class of homeomorphisms namely $g\delta s$-homeomorphisms, strongly $g\delta s$-homeomorphisms in topological spaces are introduced and some of their characterizations and properties are studied.

In the section 3, the class of $g\delta s$-quotient, strongly $g\delta s$-quotient and completely $g\delta s$-quotient functions in topological spaces are introduced and some of their properties are obtained.

In the section 4, the concepts of $g\delta s$-compact, countably $g\delta s$-compact and $g\delta s$-Lindelöf by using $g\delta s$-open sets in topological spaces are introduced and investigated some of their properties.

In the last section the concept of $g\delta s$-connectedness is introduced and discussed some of their properties.
6.2 $g\delta s$-homeomorphisms

**Definition 6.2.1.** A bijective function $f : X \to Y$ is said to be $g\delta s$-homeomorphism if $f$ is both $g\delta s$-continuous and $g\delta s$-open, equivalently, if $f$ and $f^{-1}$ both are $g\delta s$-continuous.

The family of all $g\delta s$-homeomorphism of space $X$ on to itself is denoted by $g\delta s-h(X)$.

**Example 6.2.2.** Let $X = Y = \{a, b, c\}$, $\tau = \{X, \emptyset, \{a\}, \{a, c\}\}$ and $\sigma = \{Y, \emptyset, \{a, b\}\}$ be topologies on $X$ and $Y$ respectively. Define a function $f : X \to Y$ by $f(a) = a$, $f(b) = c$ and $f(c) = b$, then $f$ is bijective, $g\delta s$-continuous and $g\delta s$-open. Therefore, $f$ is a $g\delta s$-homeomorphism.

**Remark 6.2.3.** Every homeomorphism is a $g\delta s$-homeomorphism. But converse need not be true in general.

**Example 6.2.4.** In Example 6.2.2, function $f$ is $g\delta s$-homeomorphism but not a homeomorphism. Because for an open set $\{a\}$ in $X$, $f(\{a\}) = \{a\}$ is not an open set in $Y$, implies $f$ is not open.

**Remark 6.2.5.** Every $g$-homeomorphism is $g\delta s$-homeomorphism. But converse need not be true in general.

**Example 6.2.6.** Let $X = Y = \{a, b, c\}$, $\sigma = \{X, \emptyset, \{a\}, \{a, c\}\}$ and $\sigma = \{Y, \emptyset, \{a\}, \{a, b\}\}$ be topologies on $X$ and $Y$ respectively. Then the identity function $f : X \to Y$ is $g\delta s$-homeomorphism but not $g$-homeomorphism, because for an open set $\{a, c\}$ in $X$, $f(\{a, c\}) = \{a, c\}$ is not $g$-open in $Y$, implies $f$ is not $g$-open.

**Theorem 6.2.7.** If $f : X \to Y$ is a bijective and $g\delta s$-continuous, then following statements are equivalent.

(i) $f$ is $g\delta s$-open

(ii) $f$ is $g\delta s$-homeomorphism

(iii) $f$ is $g\delta s$-closed
Proof: (i) $\iff$ (ii) Obvious from definition.

(i)$\iff$(iii) Suppose $f$ is a $g\delta s$-open function and $F$ is a closed set in $X$, then $X - F$ is an open set in $X$. By (i), $f(X - F) = Y - f(F)$ is $g\delta s$-open set in $Y$. This implies $f(F)$ is $g\delta s$-closed set in $Y$. Therefore, $f$ is $g\delta s$-closed function.

**Remark 6.2.8.** The composition of two $g\delta s$-homeomorphisms need not be a $g\delta s$-homeomorphism in general.

**Example 6.2.9.** Let $X = Y = Z = \{a, b, c\}$, $\tau = \{X, \phi, \{a\}, \{b\}, \{a, b\}\}$, $\sigma = \{Y, \phi, \{a\}\}$ and $\eta = \{Z, \phi, \{a\}, \{a, b\}\}$ be topologies on $X$, $Y$ and $Z$ respectively. Define a function $f : X \to Y$ by $f(a) = b$, $f(b) = a$ and $f(c) = c$ and $g : Y \to Z$ by $g(a) = b$, $g(b) = c$ and $g(c) = a$. Then $f$ and $g$ are both $g\delta s$-homeomorphisms but the composition $(g \circ f)$ is not a $g\delta s$-homeomorphism, because for an open set $\{a\}$ in $Z$, $(g \circ f)^{-1}(\{a\}) = f^{-1}(g^{-1}(\{a\})) = f^{-1}(\{c\}) = \{c\}$ is not a $g\delta s$-open in $X$.

**Theorem 6.2.10.** If $f : X \to Y$ and $g : Y \to Z$ be two $g\delta s$-homeomorphism functions and $Y$ is $Tg\delta s$-space, then $(g \circ f)$ is $g\delta s$-homeomorphism.

**Proof:** Suppose $U$ be an open set in $Z$, then $(g \circ f)^{-1}(U) = f^{-1}(g^{-1}(U)) = f^{-1}(V)$ where $g^{-1}(U) = V$. Since $g$ is $g\delta s$-continuous implies $V$ is $g\delta s$-open in $Y$ and since $Y$ is $Tg\delta s$-space, implies $V$ is open in $Y$. Again, $f$ is $g\delta s$-continuous implies $f^{-1}(V)$ is $g\delta s$-open in $X$. Thus $(g \circ f)$ is $g\delta s$-continuous.

Also, for an open set $G$ in $X$, $(g \circ f)(G) = g(f(G)) = g(W)$ where $W = f(G)$, since $f$ is $g\delta s$-open, implies $W$ is $g\delta s$-open in $Y$ and $Y$ is $Tg\delta s$-space, implies $W$ is open in $Y$. Therefore $g(W) = g(f(G)) = (g \circ f)(G)$ is $g\delta s$-open in $Z$, as $g$ is $g\delta s$-open. Therefore, $(g \circ f)$ is $g\delta s$-open. Hence $(g \circ f)$ is bijective, $g\delta s$-continuous and $g\delta s$-open, implies $(g \circ f)$ is $g\delta s$-homeomorphism.

**Definition 6.2.11.** A bijective function $f : X \to Y$ is called strongly $g\delta s$-homeomorphism if $f$ is both $g\delta s$-irresolute and strongly $g\delta s$-open.
Equivalently, if both $f$ and $f^{-1}$ are $g\delta s$-irresolute. The family of all strongly $g\delta s$-homeomorphism of space $X$ on to itself is denoted by $Sg\delta s-h(X)$

**Example 6.2.12.** Let $X = Y = \{a, b, c\}$, $\tau = \{X, \phi, \{a\}, \{a, b\}\}$ and $\sigma = \{Y, \phi, \{a\}\}$ be topologies on $X$ and $Y$ respectively. Define a function $f : X \to Y$ by $f(a) = b$, $f(b) = a$ and $f(c) = c$, then $f$ is both $g\delta s$-irresolute and strongly $g\delta s$-open. Therefore $f$ is strongly $g\delta s$-homeomorphism.

**Remark 6.2.13.** Every strongly $g\delta s$-homeomorphism is $g\delta s$-homeomorphism. But converse need not be true in general.

**Example 6.2.14.** Let $X = Y = \{a, b, c\}$, $\tau = \{X, \phi, \{a\}, \{b\}, \{a, b\}\}$ and $\sigma = \{Y, \phi, \{a\}\}$ be topologies on $X$ and $Y$ respectively. Define a function $f : X \to Y$ by $f(a) = a$, $f(b) = b$ and $f(c) = c$, then $f$ is $g\delta s$-homeomorphism. But not strongly $g\delta s$-homeomorphism, because for the $g\delta s$-open set $\{c\}$ in $Y$, $f^{-1}(\{c\}) = \{c\}$ is not $g\delta s$-open in $X$, implies $f$ is not $g\delta s$-irresolute.

**Remark 6.2.15.** Implication diagram of the above results is given as follows

![Implication Diagram](image)

**Theorem 6.2.16.** If $f : X \to Y$ and $g : Y \to Z$ be two strongly $g\delta s$-homeomorphism functions, then $(g \circ f) : X \to Z$ is also strongly $g\delta s$-homeomorphism.

**Proof:** Suppose $U$ is a $g\delta s$-open set in $Z$, then $(g \circ f)^{-1}(U) = f^{-1}(g^{-1}(U)) = f^{-1}(V)$ where $g^{-1}(U) = V$. Since $g$ is $g\delta s$-irresolute, implies $V$ is $g\delta s$-open in $Y$ and again $f$ is $g\delta s$-irresolute, implies $f^{-1}(V)$ is $g\delta s$-open in $X$. Therefore, $(g \circ f)$ is $g\delta s$-irresolute. Also, for a $g\delta s$-open set $G$ in $X$, $(g \circ f)(G) =$
\( g(f(G)) = g(W) \) where \( W = f(G) \), since \( f \) is strongly \( g\delta s \)-open, implies \( W = f(G) \) is \( \delta s \)-open in \( Y \) and again \( g \) is strongly \( g\delta s \)-open, implies, \( g(W) = g(f(G)) \) is \( g\delta s \)-open in \( Z \). This implies \((g \circ f)\) is strongly \( g\delta s \)-open. Therefore, \((g \circ f)\) is strongly \( g\delta s \)-homeomorphism.

**Theorem 6.2.17.** If \( f : X \rightarrow Y \) is strongly \( g\delta s \)-homeomorphism, then \( g\delta s \text{-Cl}(f^{-1}(A)) = f^{-1}(g\delta s \text{-Cl}(A)) \), for every subset \( A \) of \( Y \).

**Proof:** Suppose \( f : X \rightarrow Y \) is a strongly \( g\delta s \)-homeomorphism, then \( f \) is both \( g\delta s \)- irresolute and strongly \( g\delta s \)-open. Since \( g\delta s \text{-Cl}(A) \) is a \( g\delta s \)-closed set in \( Y \), implies \( f^{-1}(g\delta s \text{-Cl}(A)) \) is \( g\delta s \)-closed in \( X \). Since \( f^{-1}(A) \subseteq f^{-1}(g\delta s \text{-Cl}(A)) \), implies \( g\delta s \text{-Cl}(f^{-1}(A)) \subseteq g\delta s \text{-Cl}(f^{-1}(g\delta s \text{-Cl}(A))) = f^{-1}(g\delta s \text{-Cl}(A)) \). This implies \( g\delta s \text{-Cl}(f^{-1}(A)) \subseteq f^{-1}(g\delta s \text{-Cl}(A)) \). ...(i)

Again, since \( g\delta s \text{-Cl}(f^{-1}(A)) \) is a \( g\delta s \)-closed set in \( X \) and \( f \) is strongly \( g\delta s \)-open, implies \( f(g\delta s \text{-Cl}(f^{-1}(A))) \) is \( g\delta s \)-closed in \( Y \). Since \( f^{-1}(A) \subseteq g\delta s \text{-Cl}(f^{-1}(A)) \), implies \( A \subseteq f(g\delta s \text{-Cl}(f^{-1}(A))) \), therefore \( g\delta s \text{-Cl}(A) \subseteq f(g\delta s \text{-Cl}(f^{-1}(A))) \). This implies \( f^{-1}(g\delta s \text{-Cl}(A)) \subseteq g\delta s \text{-Cl}(f^{-1}(A)) \). ...(ii) Thus, from (i) and (ii), \( g\delta s \text{-Cl}(f^{-1}(A)) = f^{-1}(g\delta s \text{-Cl}(A)) \), for every subset \( A \) of \( Y \).

**Corollary 6.2.18.** If \( f : X \rightarrow Y \) is strongly \( g\delta s \)-homeomorphism then \( g\delta s \text{-Cl}(f(A)) = f(g\delta s \text{-Cl}(A)) \), for every subset \( A \) of \( X \).

**Proof:** Since \( f : X \rightarrow Y \) is a strongly \( g\delta s \)-homeomorphism, \( f^{-1} : Y \rightarrow X \) is also strongly \( g\delta s \)-homeomorphism. By theorem 6.2.17, \( g\delta s \text{-Cl}((f^{-1})^{-1}(A)) = (f^{-1})^{-1}(g\delta s \text{-Cl}(A)) \), for every subset \( A \) of \( X \). Hence \( g\delta s \text{-Cl}(f(A)) = f(g\delta s \text{-Cl}(A)) \), for every subset \( A \) of \( X \).

**Corollary 6.2.19.** If \( f : X \rightarrow Y \) is strongly \( g\delta s \)-homeomorphism then \( f(g\delta s \text{-Int}(A)) = g\delta s \text{-Int}(f(A)) \), for every subset \( A \) of \( X \).

**Proof:** For any subset \( A \) of \( X \), \( g\delta s \text{-Int}(A) = X - g\delta s \text{-Cl}(X - A) \), using theorem 2.5.16. Therefore, \( f(g\delta s \text{-Int}(A)) = f(X - g\delta s \text{-Cl}(X - A)) = \)
\[ Y - f(g_{\delta s}-\text{Cl}(X - A)) = Y - g_{\delta s}-\text{Cl}(f(X - A)), \] using corollary 6.2.18.
\[ = Y - g_{\delta s}-\text{Cl}(Y - f(A)) = g_{\delta s}-\text{Int}(f(A)), \] again using theorem 2.5.16.

**Corollary 6.2.20.** If \( f : X \to Y \) is strongly \( g_{\delta s}\)-homeomorphism, then \( f^{-1}(g_{\delta s}-\text{Int}(A)) = g_{\delta s}-\text{Int}(f^{-1}(A)), \) for every subset \( A \) of \( Y \).

**Proof:** If \( f : X \to Y \) is a strongly \( g_{\delta s}\)-homeomorphism, then \( f^{-1} : Y \to X \) is also strongly \( g_{\delta s}\)-homeomorphism. Therefore proof follows from 6.2.19.

### 6.3 \( g_{\delta s}\)-quotient Functions

**Definition 6.3.1.** A surjective function \( f : X \to Y \) is said to be \( g_{\delta s}\)-quotient if \( f \) is \( g_{\delta s}\)-continuous and \( f^{-1}(V) \) is open in \( X \) implies \( V \) is \( g_{\delta s}\)-open in \( Y \).

**Theorem 6.3.2.** If a function \( f : X \to Y \) is surjective, \( g_{\delta s}\)-continuous and \( g_{\delta s}\)-open, then \( f \) is \( g_{\delta s}\)-quotient function.

**Proof:** Since \( f : X \to Y \) is \( g_{\delta s}\)-continuous, it is enough to prove \( f^{-1}(V) \) is open in \( X \) implies \( V \) is \( g_{\delta s}\)-open in \( Y \). Let \( f^{-1}(V) \) is an open set in \( X \). Since \( f \) is \( g_{\delta s}\)-open, surjective implies \( f(f^{-1}(V)) = V \) is a \( g_{\delta s}\)-open in \( Y \). Therefore \( f \) is \( g_{\delta s}\)-quotient function.

**Remark 6.3.3.** Every homeomorphism is \( g_{\delta s}\)-quotient function. But converse need not be true in general.

**Example 6.3.4.** Let \( X = Y = \{a, b, c\}, \tau = \{X, \phi, \{a\}, \{a, c\}\} \) and \( \sigma = \{Y, \phi, \{a, b\}\} \) be topologies on \( X \) and \( Y \) respectively. Define a function \( f : X \to Y \) by \( f(a) = a, f(b) = c \) and \( f(c) = b \). Then \( f \) is \( g_{\delta s}\)-continuous and \( f^{-1}(V) \) is open in \( X \) implies \( V \) is \( g_{\delta s}\)-open in \( Y \), therefore \( f \) is \( g_{\delta s}\)-quotient. But for an open set \( \{a\} \) in \( X \), \( f(\{a\}) = \{a\} \) is not an open set in \( Y \), implies \( f \) is not an open function. Therefore \( f \) is not homeomorphism.
Theorem 6.3.5. If \( f : X \rightarrow Y \) is an open surjective, \( g\delta s \)-irresolute and \( g : Y \rightarrow Z \) is a \( g\delta s \)-quotient function, then \((g \circ f) : X \rightarrow Z\) is \( g\delta s \)-quotient function.

Proof: Let \( U \) be an open set in \( Z \). Since \( g \) is a \( g\delta s \)-quotient, implies \( g^{-1}(U) \) is a \( g\delta s \)-open in \( Y \). Also, since \( f \) is \( g\delta s \)-irresolute, \( f^{-1}(g^{-1}(U)) = (g \circ f)^{-1}(U) \) is \( g\delta s \)-open in \( X \). Therefore \((g \circ f)\) is \( g\delta s \)-continuous. Assume \( (g \circ f)^{-1}(U) = f^{-1}(g^{-1}(U)) \) is open in \( X \) for some subset \( U \) in \( Z \). Since \( f \) is an open and surjective, implies \( f(f^{-1}(g^{-1}(U))) = g^{-1}(U) \) is open in \( Y \) and since \( g \) is a \( g\delta s \)-quotient function, implies \( U \) is a \( g\delta s \)-open set in \( Y \). This shows that, \((g \circ f)\) is \( g\delta s \)-quotient function.

Definition 6.3.6. A surjective function \( f : X \rightarrow Y \) is said to be strongly \( g\delta s \)-quotient if \( f \) is \( g\delta s \)-continuous and \( f^{-1}(V) \) is \( g\delta s \)-open in \( X \) implies \( V \) is \( g\delta s \)-open in \( Y \).

Remark 6.3.7. Every strongly \( g\delta s \)-quotient function is \( g\delta s \)-quotient. But converse need not be true in general.

Example 6.3.8. Let \( X = Y = \{a, b, c\} \), \( \tau = \{X, \phi, \{a\}\} \) and \( \sigma = \{Y, \phi, \{a\}, \{b\}, \{a, b\}\} \) be topologies on \( X \) and \( Y \) respectively. Define a function \( f : X \rightarrow Y \) by \( f(a) = b, f(b) = a \) and \( f(c) = c \). Then \( f \) is \( g\delta s \)-quotient but not strongly \( g\delta s \)-quotient. Because for set \( \{c\} \) in \( Y \), \( f^{-1}(\{c\}) \) is \( g\delta s \)-open in \( X \) and \( \{c\} \) is not \( g\delta s \)-open in \( Y \).

Definition 6.3.9. A surjective function \( f : X \rightarrow Y \) is said to be completely \( g\delta s \)-quotient if \( f \) is \( g\delta s \)-irresolute and \( f^{-1}(V) \) is \( g\delta s \)-open set in \( X \) implies \( V \) is an open set in \( Y \).

Theorem 6.3.10. If \( f : X \rightarrow Y \) is surjective, strongly \( g\delta s \)-open and \( g\delta s \)-irresolute and \( g : Y \rightarrow Z \) is a completely \( g\delta s \)-quotient, then \((g \circ f) : X \rightarrow Z\) is completely \( g\delta s \)-quotient.
Proof: Since by hypothesis both $f$ and $g$ are $g\delta$s-irresolute, implies $(g \circ f)$ is a $g\delta$s-irresolute as composition of two $g\delta$s-irresolute functions is again a $g\delta$s-irresolute. Suppose that $(g \circ f)^{-1}(U) = f^{-1}(g^{-1}(U))$ is $g\delta$s-open in $X$ for some subset $U$ in $Z$. Since $f$ is surjective and strongly $g\delta$s-open, implies $f(f^{-1}(g^{-1}(U))) = g^{-1}(U)$ is $g\delta$s-open in $Y$. Also since $g$ is completely $g\delta$s-quotient, implies that $U$ is open in $Z$. This proves, $(g \circ f) : X \rightarrow Z$ is completely $g\delta$s-quotient function.

Remark 6.3.11. Every completely $g\delta$s-quotient function is a strongly $g\delta$s-quotient function. But converse need not be true in general.

Example 6.3.12. Let $X = Y = \{a, b, c\}$, $\tau = \{X, \phi, \{a\}, \{b\}, \{a, b\}\}$ and $\sigma = \{Y, \phi, \{a\}\}$ be topologies on $X$ and $Y$ respectively. Define a function $f : X \rightarrow Y$ by $f(a) = f(c) = b, f(b) = a$. Then $f$ is strongly $g\delta$s-quotient but not completely $g\delta$s-quotient. Since for set $\{b\}$ in $Y$, $f^{-1}(\{b\})$ is $g\delta$s-open in $X$, but $\{b\}$ is not open in $Y$.

Theorem 6.3.13. Let $X$, $Y$ both are $Tg\delta$s-spaces and $f : X \rightarrow Y$ be surjective function. Then following are equivalent

(i) $f$ is a completely $g\delta$s-quotient function.

(ii) $f$ is a strongly $g\delta$s-quotient function.

(iii) $f$ is a $g\delta$s-quotient function.

Proof: (i)$\Rightarrow$(ii) Suppose (i) holds. Clearly $f$ is $g\delta$s-continuous, because every $g\delta$s-irresolute function is $g\delta$s-continuous. Let $f^{-1}(V)$ is $g\delta$s-open, by (i) $V$ is open set. Since every open set is $g\delta$s-open, implies $V$ is $g\delta$s-open.

Therefore (ii) holds.

(ii)$\Rightarrow$(iii) Suppose (ii) holds. Therefore $f$ is $g\delta$s-continuous. Let $f^{-1}(V)$ is open, and hence it is $f^{-1}(V)$ is $g\delta$s-open. By (ii) $V$ is $g\delta$s-open set. Therefore (iii) holds.

(iii)$\Rightarrow$ (i) Suppose (iii) holds. Let $V$ be a $g\delta$s-open set in $Y$ and $Y$ is $Tg\delta$s-space, implies $V$ is an open set in $Y$. Since $f$ is $g\delta$s-continuous, implies
\(f^{-1}(V)\) is \(g\delta s\)-open in \(X\). This implies \(f\) is \(g\delta s\)-irresolute. Suppose \(f^{-1}(V)\) is \(g\delta s\)-open in \(X\). Since \(X\) is \(Tg\delta s\)-space, \(f^{-1}(V)\) is open in \(X\). By (iii) \(V\) is \(g\delta s\)-open in \(Y\). Since \(Y\) is a \(Tg\delta s\)-space, \(V\) is open in \(Y\). Hence (i) hold.

6.4 \(g\delta s\)-compactness

**Definition 6.4.1.** A collection \(\{A_i : i \in I\}\) of \(g\delta s\)-open sets in a topological space \(X\) is called \(g\delta s\)-open cover of a subset \(A\) in \(X\) if \(A \subseteq \bigcup_{i \in I} A_i\).

**Definition 6.4.2.** A topological space \(X\) is called \(g\delta s\)-compact if every \(g\delta s\)-open cover of \(X\) has a finite subcover.

**Definition 6.4.3.** A subset \(A\) of a space \(X\) is called \(g\delta s\)-compact relative to \(X\) if for every collection \(\{A_i : i \in I\}\) of \(g\delta s\)-open subsets of \(X\) such that \(A \subseteq \bigcup_{i \in I} A_i\) there exists a finite subset \(I_0\) of \(I\) such that \(A \subseteq \bigcup_{i \in I_0} A_i\).

**Definition 6.4.4.** A subset \(A\) of a topological space \(X\) is called \(g\delta s\)-compact if \(A\) is \(g\delta s\)-compact as a subspace of \(X\).

**Theorem 6.4.5.** A \(g\delta s\)-closed subset of \(g\delta s\)-compact space is \(g\delta s\)-compact relative to \(X\).

**Proof:** Let \(X\) be a \(g\delta s\)-compact space and \(A\) is a \(g\delta s\)-closed subset of \(X\). Then \(X - A\) is \(g\delta s\)-open in \(X\). Let \(S = \{A_i : i \in I\}\) be a \(g\delta s\)-open cover of \(A\) by \(g\delta s\)-open subsets in \(X\). Then \(S^* = S \cup (X - A)\) is a \(g\delta s\)-open cover of \(X\). That is \(X = (\bigcup \{A_i : i \in I\}) \cup (X - A)\). By hypothesis \(X\) is \(g\delta s\)-compact and hence \(S^*\) is reducible to a finite subcover of \(X\) say \(X = A_{i_1} \cup A_{i_2} \cup \ldots \cup A_{i_n} \cup (X - A), A_{i_k} \in S^*\). But \(A\) and \(X - A\) are disjoint. Hence \(A \subseteq A_{i_1} \cup A_{i_2} \cup \ldots \cup A_{i_n}\). Thus a \(g\delta s\)-open cover \(S\) of \(A\) contains a finite subcover. Therefore \(A\) is \(g\delta s\)-compact relative to \(X\).

**Theorem 6.4.6.** If \(f : X \to Y\) is surjective, \(g\delta s\)-continuous (resp. semi-\(g\delta s\)-continuous) function and \(X\) is \(g\delta s\)-compact, then \(Y\) is compact (resp. semi compact).
Proof: Let $f: X \to Y$ be surjective, $g\delta s$-continuous (resp. semi-$g\delta s$-continuous) function from $g\delta s$-compact space $X$ to space $Y$. Let $\{A_i : i \in I\}$ be an open (resp. semiopen) cover of $Y$. Since $f$ is $g\delta s$-continuous (resp. semi-$g\delta s$-continuous), implies $\{f^{-1}(A_i) : i \in I\}$ is $g\delta s$-open cover of $X$. Since $X$ is $g\delta s$-compact, implies $g\delta s$-open cover $\{f^{-1}(A_i) : i \in I\}$ has a finite subcover say $\{f^{-1}(A_i) : i = 1...n\}$. Therefore $X = \bigcup_{i=1}^{n} f^{-1}(A_i)$, which implies $f(X) = \bigcup_{i=1}^{n} A_i$. That is, $Y = \bigcup_{i=1}^{n} A_i$ as $f$ is surjective. Thus $\{A_1, A_2, ... A_n\}$ is a finite subcover of $\{A_i : i \in I\}$ for $Y$. Hence $Y$ is compact (resp. semi compact).

Theorem 6.4.7. If $f: X \to Y$ is $g\delta s$-irresolute and $B \subset X$ is $g\delta s$-compact relative to $X$, then the image $f(B)$ is $g\delta s$-compact relative to $Y$.

Proof: Let $\{A_i : i \in I\}$ be any collection of $g\delta s$-open sets in $Y$ such that $f(B) \subset \bigcup_{i \in I} A_i$. Then $B \subset \bigcup_{i \in I} f^{-1}(A_i)$, where $\{f^{-1}(A_i) : i \in I\}$ is family of $g\delta s$-open sets in $X$. Since $B$ is $g\delta s$-compact relative to $X$, the open cover $\{f^{-1}(A_i) : i \in I\}$ of $X$ has a finite subcover say $\{f^{-1}(A_i) : i = 1, 2...n\}$ such that $B \subset \bigcup_{i=1}^{n} f^{-1}(A_i)$. Therefore $f(B) \subset \bigcup_{i=1}^{n} A_i$. Hence $f(B)$ is $g\delta s$-compact relative to $Y$.

Remark 6.4.8. Every strongly $g\delta s$-homeomorphic image of a $g\delta s$-compact space is $g\delta s$-compact because if $f$ is a strongly $g\delta s$-homeomorphism then it is certainly $g\delta s$-irresolute.

Theorem 6.4.9. If $f: X \to Y$ is strongly $g\delta s$-continuous function from a compact space $X$ onto a topological space $Y$, then $Y$ is $g\delta s$-compact.

Proof: Let $\{A_i : i \in I\}$ be a $g\delta s$-open cover of $Y$. Since $f$ is strongly $g\delta s$-continuous, $\{f^{-1}(A_i) : i \in I\}$ is an open cover of $X$. Again, since $X$ is compact space, the open cover $\{f^{-1}(A_i) : i \in I\}$ of $X$ has a finite subcover say $\{f^{-1}(A_i) : i = 1...n\}$. Therefore $X = \bigcup_{i=1}^{n} f^{-1}(A_i)$ which implies $f(X) = \bigcup_{i=1}^{n} A_i$ so that $Y = \bigcup_{i=1}^{n} A_i$. That is $\{A_1, A_2...A_n\}$ is a finite subcover of $\{A_i : i \in I\}$ for $Y$. Hence $Y$ is $g\delta s$-compact.
Theorem 6.4.10. Let $f : X \to Y$ be a semi-$g\delta s$-continuous function from a $g\delta s$-compact space $X$ onto a topological space $Y$. If $Y$ is $g\delta sT_{1/2}$ space, then $Y$ is $g\delta s$-compact.

**Proof:** Let $\{A_i : i \in I\}$ be a $g\delta s$-open cover of $Y$. As $Y$ is $g\delta sT_{1/2}$ space, $\{A_i : i \in I\}$ is a semiopen cover of $Y$. Since $f$ is semi-$g\delta s$-continuous, $\{f^{-1}(A_i) : i \in I\}$ is a $g\delta s$-open cover of $X$ and $X$ is $g\delta s$-compact, implies that the $g\delta s$-open cover $\{f^{-1}(A_i) : i \in I\}$ of $X$ has a finite subcover say $\{f^{-1}(A_i) : i = 1 \ldots n\}$. Therefore $X = \bigcup_{i=1}^{n} f^{-1}(A_i)$ which implies $f(X) = Y = \bigcup_{i=1}^{n} A_i$, that is $\{A_i : i = 1 \ldots n\}$ is a finite subcover of $\{A_i : i \in I\}$ for $Y$. Hence $Y$ is $g\delta s$-compact.

Corollary 6.4.11. Let $f : X \to Y$ be a $g\delta s$-continuous function from a $g\delta s$-compact space $X$ onto a space $Y$. If $Y$ is $Tg\delta s$-space, then $Y$ is $g\delta s$-compact.

**Proof:** Since every semi-$g\delta s$-continuous function is $g\delta s$-continuous and proof follows from theorem 6.4.10.

Theorem 6.4.12. Let $f : X \to Y$ be a perfectly $g\delta s$-continuous surjection. If $X$ is mildly compact, then $Y$ is $g\delta s$-compact.

**Proof:** Let $f : X \to Y$ be a perfectly $g\delta s$-continuous function and let $\{A_i : i \in I\}$ be a $g\delta s$-open cover of $Y$. Since $f$ is perfectly $g\delta s$-continuous, $\{f^{-1}(A_i) : i \in I\}$ is clopen cover of $X$. Again since $X$ is mildly compact space, the clopen cover $\{f^{-1}(A_i) : i \in I\}$ of $X$ has a finite subcover say $\{f^{-1}(A_i) : i = 1 \ldots n\}$. Therefore $X = \bigcup_{i=1}^{n} f^{-1}(A_i)$ which implies $f(X) = Y = \bigcup_{i=1}^{n} A_i$. That is $\{A_1, A_2 \ldots A_n\}$ is a finite subcover of $\{A_i : i \in I\}$ for $Y$. Hence $Y$ is $g\delta s$-compact.

Theorem 6.4.13. Let $f : X \to Y$ be a completely $g\delta s$-continuous surjection. If $X$ is nearly compact, then $Y$ is $g\delta s$-compact.
Proof: Let \( f : X \to Y \) be a completely \( g\delta s \)-continuous function and let \( \{A_i : i \in I\} \) be a \( g\delta s \)-open cover of \( Y \). Since \( f \) is completely \( g\delta s \)-continuous, \( \{f^{-1}(A_i) : i \in I\} \) is regular open cover of \( X \). Again since \( X \) is nearly compact space, the regular open cover \( \{f^{-1}(A_i) : i \in I\} \) of \( X \) has a finite subcover say \( \{f^{-1}(A_i) : i = 1...n\} \). Therefore \( X = \bigcup_{i=1}^{n} f^{-1}(A_i) \) which implies \( f(X) = Y = \bigcup_{i=1}^{n} A_i \). That is \( \{A_1, A_2,...A_n\} \) is a finite subcover of \( \{A_i : i \in I\} \) for \( Y \). Hence \( Y \) is \( g\delta s \)-compact.

**Theorem 6.4.14.** Every \( g\delta s \)-compact space is compact.

**Proof:** Let \( X \) be a \( g\delta s \)-compact space and \( \{A_i : i \in I\} \) be an open cover of \( X \). Then \( \{A_i : i \in I\} \) is a \( g\delta s \)-open cover of \( X \) as every open set is \( g\delta s \)-open set. Since \( X \) is \( g\delta s \)-compact, the \( g\delta s \)-open cover \( \{A_i : i \in I\} \) of \( X \) has a finite subcover say \( \{A_i : i = 1...n\} \) for \( X \). This shows that every open cover \( \{A_i : i \in I\} \) of \( X \) has a finite subcover. Therefore \( X \) is compact.

**Theorem 6.4.15.** If \( X \) is compact and \( Tg\delta s \)-space, then \( X \) is \( g\delta s \)-compact.

**Proof:** Let \( \{A_i : i \in I\} \) be a \( g\delta s \)-open cover of \( X \). As \( X \) is \( Tg\delta s \)-space, \( \{A_i : i \in I\} \) is an open cover of \( X \). Since \( X \) is compact, the open cover \( \{A_i : i \in I\} \) of \( X \) has a finite subcover say \( \{A_i : i = 1,...,n\} \). This shows that every \( g\delta s \)-open cover \( \{A_i : i \in I\} \) of \( X \) has a finite subcover. Therefore \( X \) is \( g\delta s \)-compact.

**Theorem 6.4.16.** If \( X \) is semi compact space and \( g\delta s T_{1/2} \) space, then \( X \) is \( g\delta s \)-compact.

**Proof:** Let \( X \) be a semi compact space and \( \{A_i : i \in I\} \) be a \( g\delta s \)-open cover of \( X \). As \( X \) is \( g\delta s T_{1/2} \) space \( \{A_i : i \in I\} \) is a semiopen cover of \( X \). Since \( X \) is semi compact, the semiopen cover \( \{A_i : i \in I\} \) of \( X \) has a finite subcover say \( \{A_i : i \in I\} \) of \( X \). This shows that every \( g\delta s \)-open cover \( \{A_i : i \in I\} \) of \( X \) has a finite subcover. Therefore \( X \) is \( g\delta s \)-compact.
Theorem 6.4.17. A topological space $X$ is $g\delta s$-compact if and only if every family of $g\delta s$-closed sets of $X$ having finite intersection property has a nonempty intersection.

**Proof:** Suppose $X$ is $g\delta s$-compact. Let $\{A_i : i \in I\}$ be a family of $g\delta s$-closed sets with finite intersection property. To prove, $\cap_{i \in I} A_i \neq \emptyset$. Suppose $\cap_{i \in I} A_i = \emptyset$. Then, $X - \cap_{i \in I} A_i = X$. This implies, $\cup_{i \in I} (X - A_i) = X$. Thus the cover $\{X - A_i : i \in I\}$ is a $g\delta s$-open cover of $X$. Since $X$ is $g\delta s$-compact, the $g\delta s$-open cover $\{X - A_i : i \in I\}$ has a finite subcover say $\{X - A_i : i = 1 \ldots n\}$. This implies $X = \cup_{i=1}^n (X - A_i)$ which implies that $X = X - \cap_{i=1}^n A_i$ which implies $X - X = X - (X - \cap_{i=1}^n A_i)$ implies that $\cap_{i=1}^n A_i = \emptyset$. This contradicts the hypothesis. Therefore, $\cap_{i \in I} A_i \neq \emptyset$.

Conversely, suppose every family of $g\delta s$-closed sets of $X$ with finite intersection property has a nonempty intersection and if possible, let $X$ be not compact, then there exists a $g\delta s$-open cover of $X$ say $\{G_i : i \in I\}$ having no finite subcover. This implies for any finite sub family $\{G_i : i = 1 \ldots n\}$ of $\{G_i : i \in I\}$, $\cup_{i=1}^n G_i \neq X$ which implies that $X - \cup_{i=1}^n G_i \neq X - X$, this implies $\cap_{i=1}^n (X - G_i) \neq \emptyset$. Then the family $\{X - G_i : i \in I\}$ of $g\delta s$-closed sets has a finite intersection property. Therefore $\cap_{i \in I} (X - G_i) \neq \emptyset$, which implies, $\cap_{i \in I} (X - G_i)$ is an infinite collection of $g\delta s$-closed sets with f.i.p. Also, by hypothesis $\{G_i : i \in I\}$ being a $g\delta s$-open covering of $X$. Therefore $X = \cup_{i \in I} G_i$. Taking complements, $\emptyset = X - \cup_{i \in I} G_i = \cap_{i \in I} (X - G_i)$, which is an infinite collection of $g\delta s$-closed subsets of $X$ having f.i.p with empty intersection. This is a contradiction due to the fact that $X$ is not compact. Hence $X$ is $g\delta s$-compact.

**Definition 6.4.18.** A topological space $X$ is said to be countably $g\delta s$-compact if every countable $g\delta s$-open cover of $X$ has a finite subcover.

**Definition 6.4.19.** A subset $A$ of a space $X$ is called countably $g\delta s$-compact relative to $X$ if for every countable collection $\{A_i : i \in I\}$ of $g\delta s$-open
subsets of $X$ such that $A \subseteq \cup_{i \in I} A_i$ there exists a finite subset $I_0$ of $I$ such that $A \subseteq \cup_{i \in I_0} A_i$

**Theorem 6.4.20.** A $g\delta$s-closed subset of countably $g\delta$s-compact space is countably $g\delta$s-compact relative to $X$.

**Proof:** Similar to proof of theorem 6.4.5.

**Theorem 6.4.21.** If $X$ is a countably $g\delta$s-compact space, then $f$ is countably compact.

**Proof:** Let $\{A_i : i \in I\}$ be a countable open cover of $X$. Then $\{A_i : i \in I\}$ is countable $g\delta$s-open cover of $X$. Since $X$ is countably $g\delta$s-compact, the countable $g\delta$s-open cover of $X$ has a finite subcover say $\{A_i : i = 1...n\}$. Hence $X$ is countably compact.

**Theorem 6.4.22.** If $X$ is countably compact and $Tg\delta$s-space, then $X$ is countably $g\delta$s-compact.

**Proof:** Let $\{A_i : i \in I\}$ be a countable $g\delta$s-open cover of $X$ by $g\delta$s-open sets. As $X$ is $Tg\delta$s-space, $\{A_i : i \in I\}$ is countable open cover of $X$. Since $X$ is countably compact, the open cover $\{A_i : i \in I\}$ of $X$ has a finite subcover $\{A_i : i = 1...n\}$. Hence $X$ is countably $g\delta$s-compact.

**Theorem 6.4.23.** If $X$ is semi countably compact and $g\delta sT_{1/2}$ space, then $X$ is countably $g\delta$s-compact.

**Proof:** Let $\{A_i : i \in I\}$ be a countable $g\delta$s-open cover of $X$ by $g\delta$s-open sets. As $X$ is $g\delta sT_{1/2}$ space, $\{A_i : i \in I\}$ is countable semiopen cover of $X$. Since $X$ is semi countably compact, the semiopen cover $\{A_i : i \in I\}$ of $X$ has a finite subcover $\{A_i : i = 1...n\}$. Hence $X$ is countably $g\delta$s-compact.
Theorem 6.4.24. Every $g\delta s$-compact space is countably $g\delta s$-compact.

Proof: Let $X$ be a $g\delta s$-compact space. Let $\{A_i : i \in I\}$ be a countable $g\delta s$-open cover of $X$ containing $g\delta s$-open sets. Then $\{A_i : i \in I\}$ is a $g\delta s$-open cover of $X$, which has a finite subcover $\{A_i : i = 1\ldots n\}$. Hence $X$ is countably $g\delta s$-compact.

Theorem 6.4.25. If $f : X \to Y$ is $g\delta s$-continuous (resp. semi-$g\delta s$-continuous) function from a countably $g\delta s$-compact space $X$ onto a topological space $Y$, then $Y$ is countably compact (resp. semi countably compact).

Proof: Let $\{A_i : i \in I\}$ be a countable open (resp. countable semiopen) cover of $Y$. Since $f$ is $g\delta s$-continuous (resp. semi-$g\delta s$-continuous), then $\{f^{-1}(A_i) : i \in I\}$ is countable $g\delta s$-open cover of $X$. Again since $X$ is countably $g\delta s$-compact, the countable $g\delta s$-open cover $\{f^{-1}(A_i) : i \in I\}$ of $X$ has a finite subcover say $\{f^{-1}(A_i) : i = 1\ldots n\}$. Therefore $X = \bigcup_{i=1}^{n} f^{-1}(A_i)$ which implies $f(X) = Y = \bigcup_{i=1}^{n} A_i$. That is $\{A_1, A_2\ldots A_n\}$ is a finite subcover of $\{A_i : i \in I\}$ for $Y$. Hence $Y$ is countably compact (resp. semi countably compact).

Theorem 6.4.26. Let $f : X \to Y$ be $g\delta s$-continuous function from a countably $g\delta s$-compact space $X$ onto a topological space $Y$. If $Y$ is $Tg\delta s$-space, then $Y$ is countably $g\delta s$-compact.

Proof: Let $\{A_i : i \in I\}$ be a countable $g\delta s$-open cover of $Y$ by $g\delta s$-open sets in $Y$. Since $Y$ is $Tg\delta s$-space, $\{A_i : i \in I\}$ is a countable open cover of $Y$. Then $\{f^{-1}(A_i) : i \in I\}$ is a countable $g\delta s$-open cover of $X$ as $f$ is $g\delta s$-continuous. Again since $X$ is countably $g\delta s$-compact, the countable $g\delta s$-open cover $\{f^{-1}(A_i) : i \in I\}$ of $X$ has a finite subcover say $\{f^{-1}(A_i) : i = 1,\ldots n\}$. Therefore $X = \bigcup_{i=1}^{n} f^{-1}(A_i)$ which implies $f(X) = Y = \bigcup_{i=1}^{n} A_i$. That is $\{A_1, A_2\ldots A_n\}$ is a finite subcover of $\{A_i : i \in I\}$ for $Y$. Hence $Y$ is countably $g\delta s$-compact.
Theorem 6.4.27. Let \( f : X \rightarrow Y \) be semi-\( g\delta s \)-continuous function from a countably \( g\delta s \)-compact space \( X \) onto a topological space \( Y \). If \( Y \) is \( g\delta s T_{1/2} \) space, then \( Y \) is countably \( g\delta s \)-compact.

Proof: Let \( \{ A_i : i \in I \} \) be a countable \( g\delta s \)-open cover of \( Y \) by \( g\delta s \)-open sets in \( Y \). Since \( Y \) is \( g\delta s T_{1/2} \) space, \( \{ A_i : i \in I \} \) is a countable semiopen cover of \( Y \). Then \( \{ f^{-1}(A_i) : i \in I \} \) is a countable \( g\delta s \)-open cover of \( X \) as \( f \) is semi-\( g\delta s \)-continuous. Again since \( X \) is countably \( g\delta s \)-compact, the countable \( g\delta s \)-open cover \( \{ f^{-1}(A_i) : i \in I \} \) of \( X \) has a finite subcover say \( \{ f^{-1}(A_i) : i = 1,...,n \} \). Therefore \( X = \bigcup_{i=1}^{n} f^{-1}(A_i) \) which implies \( f(X) = Y = \bigcup_{i=1}^{n} A_i \). That is \( \{ A_1, A_2...A_n \} \) is a finite subcover of \( \{ A_i : i \in I \} \) for \( Y \). Hence \( Y \) is countably \( g\delta s \)-compact.

Theorem 6.4.28. Let \( f : X \rightarrow Y \) be strongly \( g\delta s \)-continuous function from a countably compact space \( X \) onto a space \( Y \), then \( Y \) is countably \( g\delta s \)-compact.

Proof: Let \( \{ A_i : i \in I \} \) be a countable \( g\delta s \)-open cover of \( Y \) by \( g\delta s \)-open sets in \( Y \). Then \( \{ f^{-1}(A_i) : i \in I \} \) is a countable open cover of \( X \) as \( f \) is strongly \( g\delta s \)-continuous function. Since \( X \) is countably compact, the countable open cover \( \{ f^{-1}(A_i) : i \in I \} \) of \( X \) has a finite subcover say \( \{ f^{-1}(A_i) : i = 1,...,n \} \). Therefore \( X = \bigcup_{i=1}^{n} f^{-1}(A_i) \) which implies \( f(X) = Y = \bigcup_{i=1}^{n} A_i \). That is \( \{ A_1, A_2...A_n \} \) is a finite subcover of \( \{ A_i : i \in I \} \) for \( Y \). Hence \( Y \) is countably \( g\delta s \)-compact.

Theorem 6.4.29. The image of a countably \( g\delta s \)-compact space under \( g\delta s \)-irresolute function is countably \( g\delta s \)-compact.

Proof: Let \( f : X \rightarrow Y \) be a \( g\delta s \)-irresolute function from a countably \( g\delta s \)-compact space \( X \) onto a topological space \( Y \). Let \( \{ A_i : i \in I \} \) be a countable \( g\delta s \)-open cover of \( Y \). Then \( \{ f^{-1}(A_i) : i \in I \} \) is a countable \( g\delta s \)-open cover of \( X \) as \( f \) is \( g\delta s \)-irresolute. Since \( X \) is countably \( g\delta s \)-compact, the countable
gδs-open cover \( \{ f^{-1}(A_i) : i \in I \} \) of \( X \) has a finite subcover say \( \{ f^{-1}(A_i) : i = 1...n \} \). Therefore \( X = \bigcup_{i=1}^{n} f^{-1}(A_i) \) which implies \( f(X) = Y = \bigcup_{i=1}^{n} A_i \). That is \( \{ A_1, A_2...A_n \} \) is a finite subcover of \( \{ A_i : i \in I \} \) for \( Y \). Hence \( Y \) is countably gδs-compact.

**Theorem 6.4.30.** A space \( X \) is countably gδs-compact if and only if every countable family of gδs-closed sets of \( X \) having finite intersection property (f. i. p.) has a non empty intersection.

**Proof:** Similar to the proof of theorem 6.4.17.

**Definition 6.4.31.** A topological space \( X \) is said to be gδs-Lindelöf if every gδs-open cover of \( X \) has a countable subcover.

**Theorem 6.4.32.** Every gδs-Lindelöf space is Lindelöf.

**Proof:** Let \( X \) be a gδs-Lindelöf space. Let \( \{ A_i : i \in I \} \) be an open cover of \( X \). Then \( \{ A_i : i \in I \} \) is gδs-open cover of \( X \) as every open set is gδs-open set in \( X \). Since \( X \) is gδs-Lindelöf space, the gδs-open cover \( \{ A_i : i \in I \} \) of \( X \) has countable subcover. Hence \( X \) is Lindelöf space.

**Theorem 6.4.33.** If \( X \) is Lindelöf and \( Tgδs \)-space, then \( X \) is gδs-Lindelöf space.

**Proof:** Let \( \{ A_i : i \in I \} \) be a gδs-open cover of \( X \). Since \( X \) is \( Tgδs \)-space implies \( \{ A_i : i \in I \} \) is an open cover of \( X \). Since \( X \) is Lindelöf space, the open cover \( \{ A_i : i \in I \} \) of \( X \) has a countable subcover. Hence \( X \) is gδs-Lindelöf space.

**Theorem 6.4.34.** Every gδs-compact space is gδs-Lindelöf space.

**Proof:** Let \( X \) be a gδs-compact space. Let \( \{ A_i : i \in I \} \) be gδs-open cover of \( X \). Then \( \{ A_i : i \in I \} \) has a finite subcover say \( \{ A_i : i = 1...n \} \) as \( X \) is gδs-compact. Since every finite subcover is always a countable subcover. Therefore \( \{ A_i : i = 1...n \} \) is a countable subcover of \( \{ A_i : i \in I \} \) for \( X \). Hence \( X \) is gδs-Lindelöf space.
Theorem 6.4.35. If $f : X \rightarrow Y$ is $g\delta s$-continuous function from a $g\delta s$-Lindelöf space $X$ onto a space $Y$, then $Y$ is Lindelöf.

Proof: Let $\{A_i : i \in I\}$ be an open cover of $Y$. Since $f$ is $g\delta s$-continuous, $\{f^{-1}(A_i) : i \in I\}$ is $g\delta s$-open cover of $X$. Since $X$ is $g\delta s$-Lindelöf, the $g\delta s$-open cover $\{f^{-1}(A_i) : i \in I\}$ has a countable subcover say $\{f^{-1}(A_{i_n}) : n \in N\}$. Therefore $X = \bigcup_{n \in N} f^{-1}(A_{i_n})$ which implies $f(X) = Y = \bigcup_{n \in N} A_{i_n}$, that is $\{A_{i_n} : n \in N\}$ is a countable subcover of $\{A_i : i \in I\}$ for $Y$. Hence $Y$ is Lindelöf space.

Theorem 6.4.36. The image of $g\delta s$-Lindelöf space under $g\delta s$-irresolute function is $g\delta s$-Lindelöf.

Proof: Let $f : X \rightarrow Y$ be a $g\delta s$-irresolute form a $g\delta s$-Lindelöf space $X$ onto a space $Y$. Let $\{A_i : i \in I\}$ be a $g\delta s$-open cover of $Y$. Then $\{f^{-1}(A_i) : i \in I\}$ is $g\delta s$-open cover of $X$ as $f$ is $g\delta s$-irresolute. Since $X$ is $g\delta s$-Lindelöf, the $g\delta s$-open cover $\{f^{-1}(A_i) : i \in I\}$ of $X$ has a countable subcover say $\{f^{-1}(A_{i_n}) : n \in N\}$. Therefore $X = \bigcup_{n \in N} f^{-1}(A_{i_n})$ which implies $f(X) = Y = \bigcup_{n \in N} A_{i_n}$, that is $\{A_{i_n} : n \in N\}$ is a countable subfamily of $\{A_i : i \in I\}$ for $Y$. Hence $Y$ is Lindelöf.

Theorem 6.4.37. If $f : X \rightarrow Y$ is strongly $g\delta s$-continuous function from a Lindelöf space $X$ onto a space $Y$, then $Y$ is $g\delta s$-Lindelöf.

Proof: Let $\{A_i : i \in I\}$ be a $g\delta s$-open cover of $Y$. Since $f$ is strongly $g\delta s$-continuous, $\{f^{-1}(A_i) : i \in I\}$ is open cover of $X$. Again, since $X$ is Lindelöf, the open cover $\{f^{-1}(A_{i_n}) : i \in I\}$ of $X$ has a countable subcover say $\{f^{-1}(A_{i_n}) : i \in I\}$. Therefore $X = \bigcup_{n \in N} f^{-1}(A_{i_n})$ which implies $f(X) = Y = \bigcup_{n \in N} A_{i_n}$. So $\{A_{i_n} : n \in N\}$ is a countable subcover of $\{A_i : i \in I\}$ for $Y$. Hence $Y$ is $g\delta s$-Lindelöf.

Theorem 6.4.38. Let $f : X \rightarrow Y$ be a $g\delta s$-continuous function from a $g\delta s$-Lindelöf space $X$ onto a space $Y$ and $Y$ is $Tg\delta s$ space, then $Y$ is $g\delta s$-Lindelöf.
Proof: Let \( \{ A_i : i \in I \} \) be a \( g\delta s \)-open cover of \( Y \). Then \( \{ A_i : i \in I \} \) is open cover of \( Y \) as \( Y \) is \( Tg\delta s \) space. Since \( f \) is \( g\delta s \)-continuous, \( \{ f^{-1}(A_i) : i \in I \} \) is \( g\delta s \)-open cover of \( X \). Again since \( X \) is \( g\delta s \)-Lindelöf, the \( g\delta s \)-open cover \( \{ f^{-1}(A_i) : i \in I \} \) of \( X \) has a countable subcover say \( \{ f^{-1}(A_{i_n}) : n \in N \} \). Therefore \( X = \bigcup_{n \in N} f^{-1}(A_{i_n}) \) which implies \( f(X) = Y = \bigcup_{n \in N} A_{i_n} \), that is \( \{ A_{i_n} : n \in N \} \) is a countable subcover of \( \{ A_i : i \in I \} \) for \( Y \). Hence \( Y \) is \( g\delta s \)-Lindelöf.

**Theorem 6.4.39.** Let \( f : X \to Y \) be a perfectly \( g\delta s \)-continuous surjection and \( X \) is mildly Lindelöf, then \( Y \) is \( g\delta s \)-Lindelöf.

Proof: Let \( f : X \to Y \) be a perfectly \( g\delta s \)-continuous function and let \( \{ A_i : i \in I \} \) be a \( g\delta s \)-open cover of \( Y \). Since \( f \) is perfectly \( g\delta s \)-continuous, \( \{ f^{-1}(A_i) : i \in I \} \) is clopen cover of \( X \). Again since \( X \) is mildly Lindelöf space, the clopen cover \( \{ f^{-1}(A_i) : i \in I \} \) of \( X \) has a countable subcover say \( \{ f^{-1}(A_{i_n}) : n \in N \} \). Therefore \( X = \bigcup_{n \in N} f^{-1}(A_{i_n}) \) which implies \( f(X) = Y = \bigcup_{n \in N} A_{i_n} \), that is \( \{ A_{i_n} : n \in N \} \) is a countable subcover of \( \{ A_i : i \in I \} \) for \( Y \). Hence \( Y \) is \( g\delta s \)-Lindelöf.

**Theorem 6.4.40.** Let \( f : X \to Y \) be a completely \( g\delta s \)-continuous surjection and \( X \) is nearly Lindelöf, then \( Y \) is \( g\delta s \)-Lindelöf.

Proof: Let \( f : X \to Y \) be a completely \( g\delta s \)-continuous function and let \( \{ A_i : i \in I \} \) be a \( g\delta s \)-open cover of \( Y \). Since \( f \) is completely \( g\delta s \)-continuous, \( \{ f^{-1}(A_i) : i \in I \} \) is regular open cover of \( X \). Again since \( X \) is nearly compact space, the regular open cover \( \{ f^{-1}(A_i) : i \in I \} \) of \( X \) has a countable subcover say \( \{ f^{-1}(A_{i_n}) : n \in N \} \). Therefore \( X = \bigcup_{n \in N} f^{-1}(A_{i_n}) \) which implies \( f(X) = Y = \bigcup_{n \in N} A_{i_n} \) is a countable subcover of \( \{ A_i : i \in I \} \) for \( Y \). Hence \( Y \) is \( g\delta s \)-Lindelöf.

**Theorem 6.4.41.** If \( X \) is \( g\delta s \)-Lindelöf and countably \( g\delta s \)-compact space, then \( X \) is \( g\delta s \)-compact.
Proof: Suppose $X$ is countably $g\delta s$-compact and $g\delta s$-Lindelöf space. Let 
\{A_i : i \in I\} be a $g\delta s$-open cover of $X$. Since $X$ is $g\delta s$-Lindelöf, 
\{A_i : i \in I\} has a countable subcover say \{A_{i_n} : n \in N\}. Therefore \{A_{i_n} : n \in N\} is a

countable subcover of $X$ and \{A_{i_n} : n \in N\} is a subfamily of \{A_i : i \in I\}

and so \{A_{i_n} : n \in N\} is a countably $g\delta s$-open cover of $X$. Again since $X$

is countably $g\delta s$-compact, \{A_{i_n} : n \in N\} has a finite subcover say \{A_{i_n} : 

n \in N\} \subset \{A_i : i \in I\}. Therefore \{A_{i_n} : n \in N\} is a finite subcover of

\{A_i : i \in I\} for $X$. Hence $X$ is $g\delta s$-compact space.

6.5 $g\delta s$-connectedness

Definition 6.5.1. A topological space $X$ is said to be $g\delta s$-connected if $X$
cannot be written as the disjoint union of two non empty $g\delta s$-open sets.

Remark 6.5.2. Every $g\delta s$-connected space is connected. But converse need

not be true in general.

Example 6.5.3. Let $X = \{a, b, c\}$ and $\tau = \{X, \phi, \{a\}\}$ be topology on

$X$. Then $X$ is connected space, but not $g\delta s$-connected space, because $X = 

\{b\} \cup \{a, c\}$ where $\{b\}$ and $\{a, c\}$ are $g\delta s$-open sets.

Theorem 6.5.4. For a topological space $X$, the following are equivalent

(i) $X$ is $g\delta s$-connected.

(ii) The only subsets of $X$ which are both $g\delta s$-open and $g\delta s$-closed are the

empty set $\phi$ and $X$.

(iii) Each $g\delta s$-continuous function of $X$ into a discrete space $Y$ with at least

two points is a constant function.

Proof: (i)$\Rightarrow$(ii) Suppose (i) holds and $F$ is a proper subset of $X$, which

is both $g\delta s$-open and $g\delta s$-closed. Then $X - F$ is also both $g\delta s$-open and

$g\delta s$-closed. Therefore $X = F \cup (X - F)$ is a disjoint union of two non empty

$g\delta s$-open sets. This contradicts the fact that $X$ is $g\delta s$-connected. Hence
\( F = \phi \) or \( X \).

(ii) \( \Rightarrow \) (i) Suppose (ii) holds. If possible \( X \) is not \( g\delta s \)-connected, then \( X = A \cup B \), where \( A \) and \( B \) are disjoint non empty \( g\delta s \)-open sets in \( X \). Since \( A = X - B \), implies \( A \) is \( g\delta s \)-closed set. But by assumption, \( A = \phi \) or \( X \), which is contradiction. Hence (i) hold.

(ii) \( \Rightarrow \) (iii) Let \( f : X \to Y \) be a \( g\delta s \)-continuous function, where \( Y \) is a discrete space with at least two points. Then \( f^{-1}(\{y\}) \) is both \( g\delta s \)-open and \( g\delta s \)-closed for each \( y \in Y \) and \( X = \cup \{ f^{-1}(\{y\}) : y \in Y \} \). By assumption, \( f^{-1}(\{y\}) = X \) or \( \phi \). If \( f^{-1}(\{y\}) = \phi \) for all \( y \in Y \), then \( f \) will not be function. Also there cannot exist more than one point \( y \in Y \) such that \( f^{-1}(\{y\}) = X \). Hence there exists only one \( y \in Y \) such that \( f^{-1}(\{y\}) = X \) and \( f^{-1}(\{y_1\}) = \phi \) where \( y \neq y_1 \in Y \). This shows that \( f \) is constant function.

(iii) \( \Rightarrow \) (ii) Let \( F \) be both \( g\delta s \)-open and \( g\delta s \)-closed in \( X \). Suppose \( F \neq \phi \) and \( f : X \to Y \) be a \( g\delta s \)-continuous function defined by \( f(F) = \{a\} \) and \( f(X - F) = \{b\} \) for some distinct points \( a \) and \( b \) in \( Y \). By assumption, \( f \) is constant function. Therefore \( F = X \).

**Theorem 6.5.5.** If \( f : X \to Y \) is a \( g\delta s \)-continuous (resp. semi-\( g\delta s \)-continuous) surjection and \( X \) is \( g\delta s \)-connected, then \( Y \) is connected (resp. semi connected).

**Proof:** Suppose \( Y \) is not connected (resp. semi connected). Then \( Y = A \cup B \) where \( A \) and \( B \) are disjoint nonempty open (resp. semiopen) sets in \( Y \). Since \( f \) is a \( g\delta s \)-continuous (resp. semi-\( g\delta s \)-continuous) surjection, \( X = f^{-1}(A) \cup f^{-1}(B) \) are disjoint non empty \( g\delta s \)-open subsets of \( X \), implies \( X \) is not \( g\delta s \)-connected space. This is contradiction to the hypothesis. Therefore \( Y \) is connected (resp. semi connected).

**Theorem 6.5.6.** If \( X \) is \( Tg\delta s \)-space and connected, then \( X \) is \( g\delta s \)-connected.

**Proof:** Suppose \( X \) is not \( g\delta s \)-connected. Then \( X = A \cup B \) where \( A \) and
B are disjoint nonempty gδs-open sets in X. Since X is Tgδs-space, implies A and B are disjoint non empty open sets in X, implies X is not connected space. This is contradiction to the hypothesis. Therefore X is gδs-connected.

**Corollary 6.5.7.** If X is gδsT_{1/2} space and semi connected, then X is gδs-connected.

**Proof:** Since every Tgδs-space is gδsT_{1/2} space and by theorem 6.5.6.

**Theorem 6.5.8.** If f : X → Y is a gδs-irresolute, surjection and X is gδs-connected, then Y is gδs-connected.

**Proof:** Suppose Y is not gδs-connected. Then Y = A ∪ B where A and B are disjoint nonempty gδs-open sets in Y. Since f is a gδs-irresolute, surjection, X = f^{-1}(A) ∪ f^{-1}(B) are disjoint non empty gδs-open subsets of X, implies X is not gδs-connected space. This is contradiction to the hypothesis. Therefore Y is gδs-connected.

**Theorem 6.5.9.** If f : X → Y is a strongly gδs-continuous surjection and X is connected, then Y is gδs-connected.

**Proof:** Suppose Y is not gδs-connected. Then Y = A ∪ B where A and B are disjoint nonempty gδs-open sets in Y. Since f is a strongly gδs-continuous surjection, X = f^{-1}(A) ∪ f^{-1}(B) are disjoint non empty open subsets of X, implies X is not connected space. This is contradiction to the hypothesis. Therefore Y is gδs-connected.

**Theorem 6.5.10.** If f : X → Y is a bijective quasi gδs-open function and Y is connected then X is gδs-connected.

**Proof:** Suppose X is not gδs-connected. Then X = A ∪ B where A and B are disjoint nonempty open sets in X. Since f is bijective, quasi gδs-open function f(A) and f(B) are open sets in Y. Moreover f(A) ∩ f(B) = ∅ and
$Y = f(A) \cup f(B)$, implies $Y$ is not connected. This is contradiction to the hypothesis. Therefore $X$ is $g\delta s$-connected.

**Corollary 6.5.11.** If $f : X \to Y$ is a bijective strongly $g\delta s$-open function from a space $X$ onto $g\delta s$-connected space $Y$, then $X$ is $g\delta s$-connected.

**Proof:** Since every strongly $g\delta s$-open function is quasi $g\delta s$-open and by theorem 6.5.10.

**Corollary 6.5.12.** If $f : X \to Y$ is a bijective $g\delta s$-open function from a space $X$ onto $g\delta s$-connected space $Y$, then $X$ is connected.

**Proof:** Since every strongly $g\delta s$-open function is $g\delta s$-open and by corollary 6.5.11.

**Theorem 6.5.13.** If $f : X \to Y$ is a completely $g\delta s$-continuous surjection and $X$ is connected, then $Y$ is $g\delta s$-connected.

**Proof:** Suppose $Y$ is not $g\delta s$-connected. Then $Y = A \cup B$ where $A$ and $B$ are disjoint nonempty $g\delta s$-open sets in $Y$. Since $f$ is a completely $g\delta s$-continuous surjection, $X = f^{-1}(A) \cup f^{-1}(B)$ are disjoint non empty regular open and so open subsets of $X$, implies $X$ is not connected space. This is contradiction to the hypothesis. Therefore $Y$ is $g\delta s$-connected.

**Theorem 6.5.14.** If $f : X \to Y$ is a perfectly $g\delta s$-continuous surjection and $X$ is connected, then $Y$ is $g\delta s$-connected.

**Proof:** Suppose $Y$ is not $g\delta s$-connected. Then $Y = A \cup B$ where $A$ and $B$ are disjoint nonempty $g\delta s$-open sets in $Y$. Since $f$ is a perfectly $g\delta s$-continuous surjection, $X = f^{-1}(A) \cup f^{-1}(B)$ are disjoint non empty clopen and so open subsets of $X$, implies $X$ is not connected space. This is contradiction to the hypothesis. Therefore $Y$ is $g\delta s$-connected.

**Theorem 6.5.15.** If $X$ is a topological space with at least two points and if $SO(X) = SC(X)$, then $X$ is not $g\delta s$-connected space.
**Proof:** If $SO(X) = SC(X)$, then by theorem 2.2.18, $P(X) = G\delta SC(X)$. Therefore there exists a proper subset of $X$ which is both $g\delta s$-open and $g\delta s$-closed. Thus by the theorem 6.5.4, $X$ is not $g\delta s$-connected space.

Thus a new class of homeomorphisms namely $g\delta s$-homeomorphisms, strongly $g\delta s$-homeomorphisms, $g\delta s$-compact, countably $g\delta s$-compact, $g\delta s$-Lindelöf and $g\delta s$-connectedness are introduced and discussed some of their characterizations and properties.