CHAPTER - 5

WEAKLY DISTRIBUTIVE *SEMILATTICE

§0 Introduction

T.P. speed[23] has introduced the concept of distributive * - lattice, the same condition was applied on a distributive lattice to become a * - lattice was placed on any meet semilattice with and such semilattice are called as * - semilattice and showed that almost all the results of speed can be extended to a more general class of distributive * - semilattices by Dr. M. Krishna Murthy[16].

In pseudocomplemented semilattices and distributive semilattice, the set of annihilators of an element is an ideal in the sense of Greatzer[6]. But it is not so in general and thus a weakly distributive semilattice $S$ is defined. Also we study the necessary and sufficient condition for modular * semilattice to be weakly distributive semilattice.

The concept of * - semilattice has been extended to semilattice by Krishna Murthy [16] and that of pseudocomplement by Katrinak. T. [15], Sankappanavar H.P.[22]. In this chapter we extend the concept of * - semilattice to each section $[o,a]$ in a semilattice. We also established its connections with pseudocomplemented semilattices.
§ 1 Preliminaries

1.1 Definition: A meet semilattice \((S, \wedge)\) is a set with an idempotent, communicative and associative binary operations on \(S\).

1.2 Definition: A partial order may be defined on \(S\) by \(a \geq b\) if and only if

\[ a \wedge b = b \quad \text{where } a, b \in S. \]

1.3 Definition: A meet semilattice \(S\) is said to be directed above if and only if for \(x, y\) in \(S\) there exists \(a\) in \(S\) such that \(a \geq x, a \geq y\).

1.4 Definition: A non empty subset \(I\) of a meet semilattice \(S\) is called an Ideal if

(i) for \(x, y\) in \(I, \ x \wedge y\) is in \(I\)

(ii) for \(x\) in \(I, t\) in \(S\) such that \(x \leq t\), implies \(t\) is in \(I\).

1.5 Definition: A non empty subset \(D\) of a meet semilattice \(S\) is called an filter of \(S\) if and only if

(i) \(x\) in \(D\) and \(a \geq x\) , implies \(a\) is in \(D\)

(ii) \(x, y\) in \(D\) implies there exists \(z\) in \(D\), such that \(z \geq x\) and \(z \geq y\).

1.6 Definition: A meet semilattice \(S\) with \(0\) is called a weakly distributive semilattice, if \((a)^*\) is an ideal for every \(a \in S\), where

\[(a)^* = \{ x \in S/ x \wedge a = 0, \forall a \in S \}. \]

1.7 Definition: A meet semilattice \(S\) with \(0\) is said to be a \(*\)-semilattice if and only if for any \(a\) in \(S\), there exists \(a^1\) in \(S\), such that \((a)^* = (a^1)^*\), where \(a \wedge a^1 = 0\)

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1.8 Definition: If S is a meet semilattice with 0 then for any subset A of S,

$A^*$ stands for \{x in S/x \land a = 0, for all a in A\}.

If A = \{a\} then $A^* = (a)^*$

1.9 Definition: An element a in S is called a dense element of S if and only if

$(a)^* = \{0\}$

1.10 Remark: If D denotes the set of all dense elements of S, then D is a filter.

Proof: Let $D = \{x/x$ is dense element of S\}

= \{x/(x)^* = \{0\}, for x \in S\}

(i) Let $x \in D$ and $a \in S$ such that $a \geq x$

$(a)^* \subseteq (x)^* = \{0\}$

Also \{0\} $\subseteq (a)^*$ as 0 is least element of S.

Therefore $(a)^* = \{0\}$.

Hence $a \in D$.

(i) Let $x, y \in D$. Then x and y are dense elements.

Thus we have $(x)^* = \{0\}; (y)^* = \{0\}$.

Let $z \in S$ such that $z \geq x, z \geq y$.

Then $(z)^* \subseteq (x)^*$ and $(z)^* \subseteq (y)^*$

$\Rightarrow (z)^* \subseteq \{0\}$;

Also \{0\} $\subseteq (z)^*$ as 0 is least element.

Therefore $(z)^* = \{0\}$.

Hence D is a filter.
Now we produce a condition for a * semilattice with modularity property to be a weakly distributive semilattice

**1.11 Theorem:** In a weakly distributive semilattice $S$, the following are equivalent.

(i) $S$ is a * - semilattice

(ii) For any $x \in S$, there exists $x^1 \in S$ such that $x \land x^1 = 0$ and

$[x) \cap [x^1) \subseteq D$, where $D$ is a filter.

**Proof:** Let $S$ be a weakly distributive semilattice, then for every $a \in S$,

(a)* = \{x \in S / x \land a = 0\} is an ideal.

(i) $\Rightarrow$ (ii)

Let $S$ be a * - semilattice, then for every $x$ in $S$, there exists $x^1$ in $S$, such that

$(x)^* = (x^1)^{**}$ where $x \land x^1 = 0$.

Implies $x^1 \in (x)^* = (x^1)^{**}$

implies $x^1 \in (x^1)^{**}$.

Let $t \in S$ such that $t \land x = 0 = t \land x^1$ implies $t \in (x)^*$,

Since $x \in S$ and as $S$ is a weakly distributive semilattice, we have $(x)^*$ is an ideal for $x \in S$ such that $x \leq t$, then $t \in [x)$ also $t \in (x^1)^*$.

And since $(x^1)^*$ is an ideal for $x^1 \in S$, such that $x^1 \leq t$ implies $t \in [x^1)$.

Thus $t \in [x) \cap [x^1)$ and as $D$ is filter of $S$ for $x$, $x^1$ in $D$ there exists $t$ in $D$

such that $t \geq x$ and $t \geq x^1$.

Hence $[x) \cap [x^1) \subseteq D$. 
Assume that for any \( x \in S \), there exists \( x^1 \in S \) such that \( x \land x^1 = 0 \) and \([x) \cap [x^1) \subseteq D\).

Now to prove that \( S \) is a * semilattice.

Since \( x \land x^1 = 0 \) implies \( x^1 \in (x)^* \).

Let \( s \in (x^1)^* \), then \( s \land (x^1)^* = 0 \) which implies \( s \land x = 0 \) for \( x \in (x)^* \), so \( s \in (x)^* \).

Thus \( (x^1)^* \subseteq (x)^* \) --- (i).

Let \( t \in S \) be such that \( t \land x = 0 \), implies \( t \in (x)^* \).

Let \( r \in S \) be such that \( r \land x^1 = 0 \), implies \( r \in (x^1)^* \) then \( t \land x \land r = 0 \land r = 0 \)

which implies \( x \in (t \land r)^* \).

Similarly \( t \land r \land x^1 = t \land 0 = 0 \), then \( x^1 \in (t \land r)^* \).

Since for \( t, r \in S \), we have \( t \land r \in S \) and \((t \land r)^* \) is an ideal of \( S \) (since \( S \) is a weakly distributive).

It follows that there exists \( z \in S \) such that \( z \geq x, x^1 \) as \( z \in (t \land r)^* \)

which implies \( z \land t \land r = 0 \).

Since \( z \geq x, x^1 \), we have \( z \in [x) \) and \( z \in [x^1) \)

which implies \( z \in [x) \cap [x^1) \subseteq D \). Thus \( z \in D \).

Since \( z \in (t \land r)^* \) and \( z \in D \) we have \((t \land r)^* \subseteq D \), as \( D \) is a filter and set of all elements of \( D \) are dense, we have \((t \land r)^* = \{0\} \).

Implies \( t \land r = 0 \), implies \( t \in (r)^* \).

As \( r \in (x^1)^* \), we have \( t \in (x^1)^** \), which shows \( (x)^* \subseteq (x^1)^** \) --- (ii)
Therefore from (i) & (ii), we have \((x)^* = (x^1)^**\) for \(x \wedge x^1 = 0\).

Hence S is a * semilattice.

1.12 Note: For any filter F of a semilattice S, \[
\theta_F = \{ (x,y) \in S \times S / x \wedge f = y \wedge f \text{ for some } f \in F \} \] is a congruence on S.

1.13 Definition: A meet semilattice S is called modular if and only if \(a \wedge b \leq w \leq a\).

Implies there exists \(y\) in S such that \(y \geq b\) and \(w = a \wedge y\).

1.14 Theorem: In a modular * semilattice S, we have \(\theta_D = R\), where \[
R = \{ (x, y) \in S \times S / (x)^* = (y)^* \} .
\]

Proof: Let S be a modular * semilattice.

For any filter D of S, \(\theta_D = \{ (x,y) \in S \times S / x \wedge d = y \wedge d \text{ for } d \in D \} \) is a congruence on S.

Given \(R = \{ (x,y) \in S \times S / (x)^* = (y)^* \} \).

Let \((x,y) \in \theta_D\), then \(x \wedge d = y \wedge d\) for \(d \in D\),

which implies \(x = y\) then \((x)^* = (y)^*\). Thus \((x, y) \in R\).

Hence \(\theta_D \subseteq R\).

Let \((x)^* = (y)^*\) for \(x, y \in S\) then \((x \wedge x)^* = (x \wedge y)^*\)

which implies \((x \wedge y)^* = (x)^*\).

Since S is a modular * semilattice, we have \((x)^* = (x^1)^**\) for \(x^1 \in S\), where \(x \wedge x^1 = 0\).

Then \((x \wedge y)^* = (x^1)^**\).

Since \(x \wedge x^1 = 0, x^1 \in (x)^* = (y)^*\), thus \(y \wedge x^1 = 0 \leq y \wedge x \leq y\).

Now, by using the modularity of S, there exists \(x_2 \geq x^1\) such that

\(y \wedge x = y \wedge x_2\), since \(x_2 \geq x^1\) and \(x_2 \geq (y \wedge x)\) implies \(x_2 \geq y \wedge x \wedge x^1\),

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implies $x^2 \geq 0$, implies $(x^2)^* \subseteq \{0\}$, but $\{0\} \subseteq (x^2)^*$

Thus $(x^2)^* = 0$, which implies $x^2$ is dense.

Similarly, $x \land x^1 = 0 \leq y \land x \leq x$, by modularity of $S$,

there exists $y_2 \geq x^1$ such that $y \land x = y_2 \land x$.

Since $y_2 \geq x^1$ and $y_2 \geq y \land x$, implies $y_2 \geq y \land x \land x^1 = 0$

Thus $(y_2)^* \subseteq \{0\}$, but $\{0\} \subseteq (y_2)^*$, hence $(y_2)^* = 0$

Hence $y_2$ is dense element.

Hence $x^2 \land y_2$ is dense.

Now $x \land x^2 \land y_2 = x \land x^2 \land y = y \land x^2 \land x = y \land x_2 \land y_2$

Therefore $x \land x^2 \land y_2 = y \land x^2 \land y_2$ for $x_2 \land y_2 \in D$.

Thus $R \subseteq \theta_D$. Hence $\theta_D = R$.

In the following theorem, it is provided a necessary and sufficient condition for a modular * semilattice to be weakly distributive.

1.15 Theorem: If $S$ is a * - semilattice which is directed above such that $\theta_D = R$, then $S$

is a weakly distributive semilattice if and only if for all $x$, $y$ in $S$

$([x) \land [y) \lor D = ([x) \lor D) \land ([y) \lor D)$

Proof: Let $S$ be * semilattice which is directed above such that $\theta_D = R$.

Assume $S$ is weakly distributive.

Now to prove the condition $([x) \land [y) \lor D = ([x) \land D) \cap ([y) \lor D)$ holds.

Let $t \in S$ be such that $t \in [([x) \lor D) \land ([y) \lor D)]$ which implies that there exists

$d_1, d_2 \in D$ such that $x \land d_1 \leq t$ and $y \land d_2 \leq t$. 

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Put \( d = d_1 \land d_2 \in D \).

It follows that, there exists \( d \in D \) such that \( x \land d \leq t \) and \( y \land d \leq t \).

Since \( S \) is a * semilattice, there exists \( t^1 \in S \) such that \( (t^*) = (t^1)^* \) where \( t \land t^1 = 0 \).

Implies \( x \land d \land t \land t^1 = x \land d \land 0 = 0 \), implies \( x \land d \land t \land t^1 = 0 \).

Which implies \( x \land d \land t^1 = 0 \) as \( x \land d \leq t \).

Similarly we can prove \( y \land d \land t^1 = 0 \).

Since \( d \in D \) is a dense element and \( S \) is a weakly distributive, there exists \( z \in S \) such that \( z \geq x, y \) and \( z \land t^1 = 0 \).

Since \( z \land t^1 = 0 \), we have \( (z)^* = (t^1)^* = (t)^* \) which implies \( (z)^* = (t)^* \) or \( (z \land t)^* = (z)^* \), since \( \theta_D = R \), there exists \( e \in D \) such that \( z \land e = z \land t \land e \) which implies that \( z \land e \leq t \).

Hence there exists \( z \in [x) \cap [y) \) and \( e \in D \) such that \( z \land e \leq t \).

Thus we have \( t \in ([x) \cap [y)) \lor D \).

Thus \( ([x) \lor D) \land ([y) \lor D) \subseteq ([x) \land [y)) \lor D \). \( ...............(i) \)

Let \( t \in ([x) \cap [y)) \lor D \), implies \( t \in ([x) \cap [y)) \) or \( t \in D \), implies \( t \in [x) \) and \( t \in [y), t \in D \).

Thus \( t \geq x \) and \( t \geq y \) and \( t = d \) for \( d = d_1 \land d_2 \in D \).

Implies \( t \geq x \land d \) and \( t \geq y \land d \), which shows that \( t \in ([x) \lor D) \land ([y) \lor D) \).

Hence \( ([x) \land [y)) \lor D \subseteq ([x) \lor D) \land ([y) \lor D) \). \( ...............(ii) \)

Therefore from (i) and (ii) we have

\( ([x) \land [y)) \lor D = ([x) \lor D) \land ([y) \lor D) \).
Conversely, suppose that the condition \((x \land y) \lor D = (x) \lor (y) \lor D\) holds. Now to show that \(S\) is a weakly distributive semilattice.

Let \(x, y \in (a)^*\), to prove that \((a)^*\) is an ideal for every \(a \in S\).

Thus \(x \land a = y \land a = 0\).

Since \(S\) is a * semilattice for \(a \in S\) there exists \(a^1\) in \(S\) such that \((a)^* = (a^1)^{**}\), where \(a \land a^1 = 0\).

Let \(t \in (x \land a^1)^*\) then \(t \land x \land a^1 = 0\) which implies \(t \land x = 0\), \(t \in (x)^*\).

Thus \((x \land a^1)^* \subseteq (x)^*\).

Similarly, we obtain \((x)^* \subseteq (x \land a^1)^*\).

Therefore \((x \land a^1)^* = (x)^*\).

Similarly \((y \land a^1)^* = (y)^*\) where \(a^1\) is an element of \(S\) satisfying the condition \((a)^* = (a^1)^{**}\).

Therefore \(x \land a^1 \in (x) \lor D\) and \(y \land a^1 \in (y) \lor D\), since \(\theta_D = R\), we get that \(a^1 \in (((x) \lor D) \land (y) \lor D)\).

By assumption \(a^1 \in (((x) \land (y)) \lor D)\), it follows that there exists \(e \in D\) such that \(z \land e \leq a^1\) for since \(z \geq x, y\).

Now \(z \land e \land a \leq a^1 \land a = 0\), we have \(z \land e \land a = 0\), hence we have \(z \land a = 0\) thus \(z \in (a)^*\) for \(e \in D\).

Hence \((a)^*\) becomes an ideal.

Hence \(S\) is a weakly distributive.
Now we define a *-semilattice as sectionally and we give a relation with pseudocomplemented and sectionally pseudo complemented semilattices.

1.16 **Definition:** A meet semilattice S with 0 is said to be a sectionally *-semilattice if and only if for every a in S, the interval [0, a] is a *-semilattice.

1.17 **Theorem:** Every *-semilattice is a sectionally *-semilattice.

**Proof:** Let (S, ∧) be a semilattice and a in S be arbitrary.

Since S is a *-semilattice, for x in S there exists a in S such that (x)* = (a)** where $x \land a = 0$.

Claim: (S, ∧) is sectionally *-semilattice.

Consider [0, a] in S and Put $X = [0, a]$

To show that (X, ∧) is a *-semilattice.

Clearly 0 is in X, let x be in X.

Since $X \subseteq S$ and x is in X, x is in S.

Thus there exists $a^1$ in S such that (x)* = $(a^1)**$ where $x \land a^1 = 0$.

Put $x^1 = x \land a^1 = a \land a^1 = 0$ (take $x = a$)

⇒ $x^1 = 0$ is in X

Therefore $x \land x^1 = x \land (a \land a^1) = x \land 0 = 0$

Therefore $x \land (a \land a^1) = 0$;

Hence x is in $(a \land a^1)^*$.

To show that $(x)^*_{[0, a]} = (a \land a^1)^*_{[0, a]}$

Let y be in $(a \land a^1)^*_{[0, a]}$

⇒ $y \land (a \land a^1)^* = 0$
⇒ y ∧ x = 0 for x is in (a ∧ a^1)^*

⇒ y is in (x)^*[0,a]

⇒ (a ∧ a^1)^*[0,a] ⊆ (x)^*[0,a]

Let y is in (x)^*[0,a] be arbitrary, then y ∧ x = 0

⇒ y is in (x)^* for all x in [0,a] ⊆ S, y is in (x)^* for x is in S

⇒ y is (a^1)^**

⇒ y ∧ (a^1)^* = 0

⇒ y ∧ z = 0 for all z in (a^1)^* = \{z ∈ S/ z ∧ a^1 = 0\}

⇒ y ∧ z = 0, for every z in [0,a] with z ∧ a ∧ a^1 = 0

[since z ∈ (a^1)^* ⇒ z ∧ a^1 = 0 ⇒ z ∧ a^1 ∧ a = 0 ∧ a = 0]

⇒ z ∈ (a ∧ a^1)^*[0,a]

⇒ y ∧ z = 0 for every z in (a ∧ a^1)^*[0,a]

⇒ y is in (a ∧ a^1)^*[0,a]

Therefore (x)^*[0,a] ⊆ (a ∧ a^1)^*[0,a]

Therefore (x)^*[0,a] = (a ∧ a^1)^*[0,a]

Therefore (X, ∧) is a *-semilattice

Therefore [0,a] is a *-semilattice.

Hence (S, ∧) is a sectionally *-semilattice.

Hence, every *-semilattice is a sectionally *-semilattice.
1.18 **Theorem:** Every sectionally * - semi lattice is not necessarily a * - semilattice.

**Proof** Let S be a sectionally * - semilattice. Then for every a in S, the interval [o,a] is a * - semilattice.

We prove S need not be a * - semilattice by taking an example.

Let S = {o, a, x, d} be a sectionally * - semilattice

![Diagram](image)

Clearly the intervals [o,a], [o, d] and [o,x] are * - semilattices, but for any b in S,

\[(0)^* \neq (b)^**\]

Hence, S is not a * - semilattice.

1.19 **Theorem:** A directed above semilattice S is a * - semilattice if and only if it is a sectionally * - semilattice and S contains a dense element.

**Proof:** Let S be directed above and a * - semilattice.

To prove that S contains a dense element.

Since S is a * - semilattice, then S is a sectionally * - semilattice, and for every x in S there exists x\(^1\) in S such that \((x)^* = (x\(^1\))^**\), where \(x \land x\(^1\) = 0\).

Since S is directed above semilattice there exists an element d in S such that \(d \geq x, x\(^1\)\).

\[\Rightarrow d \geq x \land x\(^1\) = 0\]

\[\Rightarrow (d)^* \subseteq (0)^* = S\]

Now for any a in \((d)^*\), we have \(a \land d = 0\)

\[\Rightarrow a \land d \land x = 0\]
\[ \Rightarrow a \land x = 0 \text{ for } d \geq x \]

also \[ a \land d \land x^1 = 0 \Rightarrow a \land x^1 = 0 \text{ for } d \geq x^1 \]

Thus a is \((x)^*\) and a is in \((x^1)^*\)

\[ \Rightarrow a \text{ is in } (x)^* \cap (x^1)^* \]

\[ \Rightarrow a \text{ is in } (x^1)^* \cap (x^1)^* \]

\[ \Rightarrow a \text{ is in } \{0\} \]

\[ \Rightarrow a = 0 \]

Therefore \((d)^* = \{0\}\).

Therefore S contains a dense element.

Conversely, Let S be a directed above sectionally * - semilattice and it contains a dense element d.

To prove that S is a * - semilattice.

Let x is in S and choose a such that \[ a \geq x, d \]

Then a is a dense element because \((a)^* \subseteq (d)^* = \{0\}\).

Since S is a sectionally * - semilattice, we have for every x in the interval \([0,a] \) there exists \(x^1\) in the interval \([0,a] \) such that

\[ (x)^*_{[0,a]} = (x^1)^*_{[0,a]}, \text{ where } x \land x^1 = 0. \]

Now, it is enough to show that \((x)^* = (x^1)^*\).

Since \[ x \land x^1 = 0 \text{ in } [0, a] \subseteq S. \]

We have \[ x \text{ is in } (x^1)^* \Rightarrow (x^1)^* \subseteq (x)^* \] ----------- (i)

Let y is in \((x)^*\) and z is in \((x^1)^*\)

\[ \Rightarrow y \land x = 0 \text{ and } z \land x^1 = 0 \]

\[ \Rightarrow y \land x \land z = 0 \text{ and } y \land z \land x^1 = 0 \]
\[
\Rightarrow y \land z \land a \land x = 0 \text{ and } y \land z \land a \land x^1 = 0
\]

\[
\Rightarrow y \land z \land a \text{ is in } (x)^*_{[o,a]} \text{ and } y \land z \land a \text{ is in } (x^1)^*_{[o,a]}
\]

\[
\Rightarrow y \land z \land a \text{ is in } (x)^*_{[o,a]} \cap (x^1)^*_{[o,a]} = \{0\}
\]

\[
\Rightarrow y \land z \land a = 0 \Rightarrow y \land z \text{ is in } (a)^* = \{0\} \text{, since } a \text{ is dense element}
\]

\[
\Rightarrow y \land z = 0
\]

\[
\Rightarrow y \text{ is in } (z)^* \text{ for } z \text{ is in } (x^1)^*
\]

\[
\Rightarrow y \text{ is in } (x^1)^**.
\]

Therefore \((x)^* \subseteq (x^1)^**\) \hspace{1cm} \text{(ii)}

Therefore from (i) and (ii)

we have \((x)^* = (x^1)^**\) where \(x \land x^1 = 0\).

Hence S is a *-semilattice.

1.20 Definition : A meet semilattice S with 0 is said to be pseudo complemented if and only if for a in S, \(a^1\) is a pseudo complement of a in S, that is \(x \land a = 0\) in S if and only if \(x \leq a^1\).

1.21 Definition : A meet semilattice S with 0 is called a sectionally Pseudo complemented semilattice if and only if for every a in S, the interval [0,a] is a pseudo complemented semilattice.

1.22 Remark: Every Pseudo complemented semilattice is a sectionally pseudo complemented semilattice.

Proof: Let S be the pseudo complemented semilattice, then for \(a \in S\), \(a^*\) is a pseudo complement of a in S that is \(x \land a = 0\) in S if and only if \(x \leq a^*\).

Since \(a^*\) is a pseudo complement of a, then \(a \land a^* = 0\).
Let $y \in \{0, a\}$

$\Rightarrow 0 \leq y \leq a$

$\Rightarrow y \land a = y$ also $y \land 0 = 0$

$\Rightarrow y \land a = 0$

$y \land a = y \land 0 = y \land a \land a^*$

$\Rightarrow y \land a = y \land a^* \land a$

$\Rightarrow y = y \land a^*$

$\Rightarrow y \leq a^*$

Let $y \leq a^*$, then $y \land a^* = y$

$\Rightarrow y \land a^* \land a = y \land a$

$\Rightarrow y \land 0 = y \land a$

$\Rightarrow 0 = y \land a$

Therefore for $y$ in $\{0, a\}$ there exist $a^*$ in $\{0, a\}$ such that $y \leq a^*$.

Therefore $\{0, a\}$ is pseudo complemented semilattice.

Hence every pseudo complemented semilattice is sectionally pseudo complemented semilattice.
1.23 Theorem: Every sectionally pseudo complemented semilattice is sectionally \* semilattice

Proof: Let S be a sectionally pseudo complemented semilattice.

To prove that for every a in S, the interval [0,a] is a \* - semilattice.

Let x in the interval [0,a] which is a pseudo complemented semilattice, then there exists a pseudo complement x\(^1\) of x in the interval [0,a] that is \(y \wedge x = 0\) in the interval [0,a] if and only if \(y \leq x\(^1\)\).

Then \((x)^* = (x\(^1\))\)

\(\Rightarrow (x)^{**} = ((x\(^1\))^*) = (x\(^1\))^*\)

\(\Rightarrow (x)^* = (x\(^1\))^{**}\) where \(x\(^1\)\) is a pseudo complement of x in the interval [0,a].

Therefore the interval [0,a] is a \* - semilattice.

Hence S is a sectionally \* - semilattice.

1.24 Theorem: Every sectionally \* - semilattice is not necessarily a sectionally pseudo complemented semilattice.

Proof: Let S be a sectionally \* - semilattice, then for every a in S, the interval [0,a] is a \* - semilattice.

[Diagram:]

For example S = \{0, a, x, d\} be a meet semilattice with 0, which is a sectionally \* - semilattice.

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But the interval \([0,d]\) is not a pseudo complemented semi lattice, because there does not exist pseudo complement \(x^1\) of \(x\) in the interval \([0,d]\) such that \(y \land x = 0\) in the interval \([0,d]\) if and only if \(y \leq x^1\).

Therefore \(S\) is not sectionally pseudo complemented semilattice.

Conclusion:

In this chapter we provided the necessary and sufficient condition for a modular * semilattice to be a weakly distributive semilattice, we extend the concept of * semilattice to each section \([0,a]\) in a meet semilattice and established its connections with pseudo complemented semilattice.