Chapter 2

On convexity of weakly $k$-hyponormal region

2.1 Introduction

To express ourself clearly and systematically we begin by specifying the notations being followed. Let $W_\alpha$ be a weighted shift with weight sequence $\alpha = \{\alpha_n\}_{n=0}^{\infty}$. Here $\alpha_n$ is referred to as the $n^{th}$ weight. So $\alpha_0$ is the $0^{th}$ weight, $\alpha_1$ is the $1^{th}$ weight and so on. If for $y > 0$, and $j \geq 0$, $\alpha[j : y]$ denote the weight sequence $\alpha_0, \alpha_{j-1}, y, \alpha_{j+1}, \ldots$ then $W_\alpha[j : y]$ is called the perturbed shift where the $j^{th}$ weight $\alpha_j$ is perturbed to $y$. $W_\alpha[j : y]$ is a rank one perturbation of $W_\alpha$.

If the $i^{th}$ and $j^{th}$ weights of $\alpha$ are perturbed to $x$ and $y$ respectively, then the perturbed shift $W_\alpha[(i : x), (j : y)]$ is called a rank two perturbation of $W_\alpha$. Similarly, we can define any finite rank perturbation of $W_\alpha$.

The issue of perturbation of weights in a weighted shift operator, is intricately related to the question of convexity of the domain of perturbation. In [24, Theorem 6.5] it was shown that rank-one perturbations of $k$-hyponormal weighted shifts which preserve $k$-hyponormality form a convex set. That is, if $W_\alpha$ is $k$-hyponormal then $\Omega_\alpha(k, j) := \{x : W_\alpha[j : x] \text{ is } k\text{-hyponormal}\}$ is a convex set. However, it is not known whether a similar result holds for weakly $k$-hyponormal
weighted shift $W_\alpha$. In this chapter we show that, "if $W_\alpha$ is weakly $k$-hyponormal, then $\omega_\alpha(k, j) := \{x : W_{\alpha[j,x]} \text{ is weakly } k\text{-hyponormal}\}$ is a convex set."

In trying to ascertain this result, our first attempt is to come up with an example having this property. We try to achieve this for the case of a weak 2-hyponormal (i.e. quadratic hyponormal) operator using the characterization of quadratic hyponormality given in [65].

In examples 2.2.1 and 2.2.2 we construct two quadratically hyponormal weighted shifts $W_\alpha$ and $W_\beta$ where the weight sequences $\alpha$ and $\beta$ are as follows:

$$\alpha : \sqrt{\frac{1}{2}}, \sqrt{\frac{1}{2}}, \sqrt{\frac{43}{80}}, \sqrt{\frac{2}{3}}, \ldots$$

and

$$\beta : \sqrt{\frac{1}{2}}, \sqrt{\frac{1}{2}}, \sqrt{\frac{47}{80}}, \sqrt{\frac{2}{3}}, \ldots$$

Let $\gamma(x)$ denote the weight sequence $\sqrt{\frac{1}{2}}, \sqrt{\frac{1}{2}}, \sqrt{x}, \sqrt{\frac{2}{3}}, \ldots$ In Proposition 2.2.3 we show that $W_{\gamma(x)}$ is quadratically hyponormal for all $x \in \left[\frac{43}{80}, \frac{47}{80}\right]$. For this we make use of the Theorem 2.2.2 and Mathematica graphs. The insight gained from this example enables us to prove that for a weighted shift operator $W_\alpha$, if $W_{\alpha[j,x]}$ and $W_{\alpha[j,y]}$ are weakly $k$-hyponormal, then $W_{\alpha[j,z]}$ is weakly $k$-hyponormal for all $z$ between $x$ and $y$. In other words, $\omega_\alpha(k, j)$ is a convex set, and this is shown in Theorem 2.4.3.

2.2 Examples for quadratic hyponormality

We begin by recapitulation of definitions introduced in [12, 17, 65]. Let $\{e_n\}_{n=0}^{\infty}$ be the canonical orthonormal basis for $l^2(\mathbb{Z}_+)$ and let $\alpha := \{\alpha_n\}_{n=0}^{\infty}$ be a bounded sequence of positive numbers. Let $W_\alpha$ be the unilateral weighted shift defined by $W_\alpha e_n = \alpha_n e_{n+1}$ ($\forall n \geq 0$).

By definition, an operator $T$ is quadratically hyponormal (q.h.) if $T + sT^2$ is hyponormal for every $s \in \mathbb{C}$. 
Lemma 2.2.1. [25] $W_\alpha$ is quadratically hyponormal if and only if $W_\alpha + sW_\alpha^2$ is hyponormal for every $s \geq 0$.

Proof. By the definition of quadratic hyponormality it is trivial that $W_\alpha$ is quadratically hyponormal implies $W_\alpha + sW_\alpha^2$ is hyponormal for every $s \geq 0$.

Conversely, suppose $W_\alpha + sW_\alpha^2$ is hyponormal for every $s \geq 0$. We need to show that $W_\alpha + sW_\alpha^2$ is hyponormal for all $s \in \mathbb{C}$.

Let $s \in \mathbb{C}$ and $s = re^{i\theta}$ for $r > 0$. Define $u_\theta : \ell^2 \to \ell^2$ as $u_\theta e_n = e^{-i\theta}e_n$. Then $u_\theta^*e_n = e^{i\theta}e_n$, and so, $u_\theta$ is unitary.

Also, $u_\theta W_\alpha u_\theta^*e_n = e^{-i\theta}\alpha_n e_{n+1} = e^{-i\theta}W_\alpha e_n$ that is, $u_\theta W_\alpha u_\theta^* = e^{-i\theta}W_\alpha$.


t_\alpha(W_\alpha + sW_\alpha^2)u_\theta^* = u_\theta W_\alpha u_\theta^* + su_\theta W_\alpha^2 u_\theta^*

\begin{align*}
&= u_\theta W_\alpha u_\theta^* + s(u_\theta W_\alpha u_\theta^*)^2 \\
&= e^{-i\theta}W_\alpha + re^{i\theta}e^{-2i\theta}W_\alpha^2 \\
&= e^{-i\theta}(W_\alpha + rW_\alpha^2)
\end{align*}

Since $W_\alpha + rW_\alpha^2$ is hyponormal, therefore $W_\alpha + sW_\alpha^2$ is hyponormal ($\forall s \in \mathbb{C}$). □

For a hyponormal weighted shift $W_\alpha$ and $s \geq 0$, let $D(s) := [(W_\alpha + sW_\alpha^2)^*, (W_\alpha + sW_\alpha^2)]$. Then we have,

\begin{align*}
D(s) &= [(W_\alpha + sW_\alpha^2)^*, (W_\alpha + sW_\alpha^2)] \\
&= (W_\alpha + sW_\alpha^2)^*(W_\alpha + sW_\alpha^2) - (W_\alpha + sW_\alpha^2)(W_\alpha + sW_\alpha^2)^* \\
\end{align*}

It can be easily shown that

\begin{align*}
[W_\alpha^*, W_\alpha]e_n &= (\alpha_n^2 - \alpha_{n-1}^2)e_n \ (\forall n \geq 0) \\
[W_\alpha^2, W_\alpha]e_n &= \alpha_n(\alpha_n^2 - \alpha_{n-1}^2)e_{n+1} \ (\forall n \geq 0)
\end{align*}
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\[ [W^{*2}_\alpha, W_\alpha]e_n = \begin{cases} 0 & \text{if } n = 0 \\ \alpha_{n-1}(\alpha_n^2 - \alpha_{n-2}^2)e_{n-1} & \text{if } n \geq 1 \end{cases} \]

\[ [W^{*2}_\alpha, W^2_\alpha]e_n = (\alpha_n^2\alpha_{n+1}^2 - \alpha_{n-1}^2\alpha_{n-2}^2)e_n \text{ (}\forall n \geq 0\text{)} \]

Let \( P_n \) be the projection of \( l^2(\mathbb{Z}) \) onto \( \bigvee_{i=0}^n \{e_i\} \) and for \( (n \geq 0) \), let \( D_n := D_n(s) = P_n D(s) P_n \). Then

\[
D_n = \begin{pmatrix}
q_0 & r_0 & 0 & \ldots & 0 & 0 \\
r_0 & q_1 & r_1 & \ldots & 0 & 0 \\
0 & r_1 & q_2 & \ldots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \ldots & q_{n-1} & r_{n-1} \\
0 & 0 & 0 & \ldots & r_{n-1} & q_n
\end{pmatrix},
\]

where

\[
q_k := u_k + s^2v_k
\]

\[
r_k := s\sqrt{w_k}
\]

\[
u_k := \alpha_k^2 - \alpha_{k-1}^2
\]

\[
v_k := \alpha_k^2\alpha_{k+1}^2 - \alpha_{k-1}^2\alpha_{k-2}^2
\]

\[
w_k := \alpha_k^2(\alpha_{k+1}^2 - \alpha_{k-1}^2)^2
\]

for \( k \geq 0 \) and \( \alpha_{-1} = \alpha_{-2} := 0 \).

By the definition of quadratically hyponormal operator, we immediately see that \( W_\alpha \) is q.h. if and only if \( D_n(s) \geq 0 \) for every \( s \geq 0 \) and every \( n \geq 0 \).

For \( x_0, x_1, \ldots, x_n \) and \( s \) in \( \mathbb{R}_+ \), we define the following:

\[
F_n := F_n(x_0, x_1, \ldots, x_n, s)
\]

\[
= \sum_{i=0}^n q_i x_i^2 - 2 \sum_{i=0}^{n-1} r_i x_i x_{i+1}
\]
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\[ n = \sum_{i=0}^{n} u_i x_i^2 - 2s \sum_{i=0}^{n-1} \sqrt{u_i x_i x_{i+1}} + s^2 \sum_{i=0}^{n} u_i x_i^2 \]

and recall, for further use, the following result:

**Theorem 2.2.2.** [65]: Let \( W_\alpha \) be a weighted shift with a weight sequence \( \alpha \). Then the followings are equivalent:

(i) \( W_\alpha \) is quadratically hyponormal;

(ii) \( F_n(x_0, x_1, \ldots, x_n, s) \geq 0 \) for any \( x_0, x_1, \ldots, x_n, s \in \mathbb{R}_+ (n \geq 2) \);

(iii) There exists a positive integer \( N \) such that \( F_n(x_0, x_1, \ldots, x_n, s) \geq 0 \) for any \( x_0, x_1, \ldots, x_n, s \in \mathbb{R}_+ (n \geq N) \).

**Example 2.2.1.** Let \( \alpha \) be the positive weight sequence given by \( \sqrt{\frac{1}{2}}, \sqrt{\frac{1}{2}}, \sqrt{\frac{43}{80}}, \sqrt{\frac{3}{4}}, \ldots \). We will show that the weighted shift operator \( W_\alpha \) is quadratically hyponormal.

For this, let \( \{\alpha_n\}_{n=0}^{\infty} \) denote the sequence \( \alpha \) so that \( \alpha_{n+1} = \frac{n}{n+1} \forall n \geq 2 \).

In view of Theorem 2.2.2, it is sufficient to show that \( F_n \geq 0 \) \( \forall n \geq 5 \).

For \( x_0, x_1, \ldots, x_5, s \) reals, we denote a function \( G_5 = G_5(x_0, x_1, \ldots, x_5, s) \) by

\[ G_5 = F_5 - v_5 t x_5^2 \text{ where } s^2 = t \]

Then,

\[ F_6 = \sum_{i=0}^{6} (u_i + t v_i) x_i^2 - 2 \sum_{i=0}^{5} \sqrt{u_i t x_i x_{i+1}} \]

\[ = G_5 + t v_5 x_5^2 + (u_6 + t v_6) x_6^2 - 2 \sqrt{u_5 t x_5 x_6} \]

\[ = G_5 + \left( v_5 t - \frac{w_5 t}{u_6 + t v_6} \right) x_5^2 + \left( \frac{\sqrt{w_5 t} x_5 - \sqrt{u_6 + t v_6 x_6}}{u_6 + t v_6} \right)^2 \]
Suppose \( F_6(x_0, \ldots, x_6, s) \geq 0 \) for any \( x_0, \ldots, x_6, s \in \mathbb{R}_+ \). Then since \( x_6 \) is arbitrary non-negative real, we will take \( x_6 = \frac{\sqrt{w_5 t}}{u_6 + t v_6} x_5 \) so that
\[
F_6 \geq 0 \Rightarrow G_5 + \left( v_5 t - \frac{w_5 t}{u_6 + t v_6} \right) x_5^2 \geq 0
\]

Conversely, if \( G_5 + (v_5 t - \frac{w_5 t}{u_6 + t v_6}) x_5^2 \geq 0 \) then
\[
F_6 = G_5 + \left( v_5 t - \frac{w_5 t}{u_6 + t v_6} \right) x_5^2 + \left( \frac{\sqrt{w_5 t}}{u_6 + t v_6} x_5 - \sqrt{u_6 + t v_6} x_6 \right)^2 \geq 0
\]

Hence,
\[
F_6(x_0, \ldots, x_6, s) \geq 0 \quad \text{for any} \quad x_0, \ldots, x_6, s \in \mathbb{R}_+
\]
\[
\Leftrightarrow G_5(x_0, \ldots, x_5, s) + \left( v_5 t - \frac{w_5 t}{u_6 + t v_6} \right) x_5^2 \geq 0 \quad \text{for any} \quad x_0, \ldots, x_5, s \in \mathbb{R}_+
\]
\[
\Leftrightarrow G_5(x_0, \ldots, x_5, s) + \frac{x_6 t}{1 + x_6 t} v_5 t x_5^2 \geq 0 \quad \text{for any} \quad x_0, \ldots, x_5, s \in \mathbb{R}_+
\]
\[
\left( \text{using} \quad w_n = u_{n+1} v_n \quad \forall \ n \geq 5 \quad \text{and} \quad z_n = \frac{v_n}{u_n} \right)
\]

Similarly,
\[
F_7 = G_5 + \left( v_5 t - \frac{w_5 t}{u_6 + t v_6} \right) x_5^2
\]
\[
+ \left( \frac{\sqrt{w_5 t}}{u_6 + t v_6} - \frac{w_5 t}{w_7 + t v_7} \right) x_5 - \sqrt{(u_6 + t v_6) - \left( \frac{w_6 t}{u_7 + t v_7} \right) x_6}
\]
\[
+ \left( \frac{\sqrt{w_6 t}}{u_7 + t v_7} x_6 - \sqrt{u_7 + t v_7} x_7 \right)^2
\]
and so

\[ F_7(x_0, \ldots, x_7, s) \geq 0 \text{ for any } x_0, \ldots, x_7, s \in \mathbb{R}_+ \]

\[ \iff G_5(x_0, \ldots, x_5, s) + \left( v_5 t - \frac{w_5 t}{(u_6 + tv_6) - \frac{w_6 t}{u_7 + tv_7}} \right) x_5^2 \geq 0 \]

for any \( x_0, \ldots, x_5, s \in \mathbb{R}_+ \)

\[ \iff G_5(x_0, \ldots, x_5, s) + \frac{z_7 z_6 t^2}{1 + z_7 t + z_7 z_6 t^2} v_5 t x_5^2 \geq 0 \text{ for any } x_0, \ldots, x_5, s \in \mathbb{R}_+ \]

So, by Mathematical induction, for \( n \geq 6 \) we have

\[ F_n \geq 0 \iff G_5 + \frac{(z_n z_{n-1} \ldots z_6 t^{n-5}) v_5 t x_5^2}{1 + z_n t + z_n z_{n-1} t^2 + \cdots + z_n z_{n-1} \ldots z_6 t^{n-5}} \geq 0 \]

\[ \iff G_5 + \frac{1}{1 + z_n t + z_n z_{n-1} t^2 + \cdots + z_n z_{n-1} \ldots z_6 t^{n-5}} v_5 t x_5^2 \geq 0 \]

(2.2.1)

Claim 1: \( G_5(x_0, \ldots, x_5, s) \geq 0 \) for \( 0 \leq s \leq \sqrt{0.299} \)

The corresponding symmetric matrix to the quadratic form \( G_5 \) is

\[
A(t) = \begin{pmatrix}
u_0 + tv_0 & -\sqrt{u_6 t} & 0 & 0 & 0 & 0 \\
-\sqrt{u_6 t} & u_1 + tv_1 & -\sqrt{u_1 t} & 0 & 0 & 0 \\
0 & -\sqrt{u_1 t} & u_2 + tv_2 & -\sqrt{u_2 t} & 0 & 0 \\
0 & 0 & -\sqrt{u_2 t} & u_3 + tv_3 & -\sqrt{u_3 t} & 0 \\
0 & 0 & 0 & -\sqrt{u_3 t} & u_4 + tv_4 & -\sqrt{u_4 t} \\
0 & 0 & 0 & 0 & -\sqrt{u_4 t} & u_5
\end{pmatrix}
\]

We discuss the positivity of \( A(t) \) by Nested Determinant Test. By direct Computation, we have

\[
d_0 = \frac{1}{2} + \frac{1}{4} t \\
d_1 = \frac{3t}{320} + \frac{43 t^2}{640} \\
d_2 = \frac{43 t^2}{12800} + \frac{559 t^3}{76800}
\]
\[
d_3 = \frac{301t^2}{1024000} + \frac{731t^3}{1024000} + \frac{20683t^4}{12288000}
\]
\[
d_4 = \frac{301t^2}{12288000} + \frac{301t^3}{10240000} + \frac{34529t^4}{368640000} + \frac{599807t^5}{1474560000}
\]
\[
d_5 = \frac{301t^2}{245760000} - \frac{301t^3}{122880000} - \frac{35647t^4}{7372800000} - \frac{20683t^5}{9830400000}
\]

If \(0 < t \leq 0.299\), then \(d_0, d_4 > 0\) and \(d_5 > 0\), which implies that \(A(t) \geq 0\) for \(0 < t \leq 0.299\) and \(G_5(x_0, \ldots, x_5, s) \geq 0\) for \(0 < s \leq \sqrt{0.299}\) and Claim 1 is established.

Hence by (2.2.1),

\[
F_n(x_0, \ldots, x_n, s) \geq 0 \text{ for any } x_0, \ldots, x_n \in \mathbb{R}_+ \text{ and } 0 < s \leq \sqrt{0.299}.
\]

Again, \(z_n = \frac{v_n}{u_n} = \frac{4(n+1)}{n+2}, (n \geq 5)\) and also \(\{z_n\}_{n=6}^\infty\) is an increasing sequence converging to 4. Thus,

\[
1 + \frac{1}{z_6t} + \frac{1}{z_6z_7t^2} + \cdots + \frac{1}{z_6z_7\cdots z_n t^{n-5}} \\
\leq 1 + \frac{1}{z_6t} + \left(\frac{1}{z_6t}\right)^2 + \cdots + \left(\frac{1}{z_6t}\right)^{n-5} \\
\leq \sum_{n=0}^{\infty} \left(\frac{1}{z_6t}\right)^n = \frac{1}{1 - \frac{1}{z_6t}}
\]

Hence if \(t > 0.299\), then

\[
G_5 + \frac{1}{1 + \frac{1}{z_6t} + \frac{1}{z_6z_7t^2} + \cdots + \frac{1}{z_6z_7\cdots z_n t^{n-5}}} v_5 t x_5^2 \\
\geq G_5 + \left(1 - \frac{1}{z_6t}\right) v_5 t x_5^2 \\
= G_5 + \left(1 - \frac{7}{24t}\right) \frac{t}{6} x_5^2 \quad \text{(since } z_6 = \frac{24}{7} \text{ and } v_5 = \frac{1}{6}) \\
= G_5 + \left(\frac{24t - 7}{144}\right) x_5^2
\]

Now we consider the corresponding symmetric matrix \(B(t)\) to the quadratic form \(G_5 + \left(\frac{24t - 7}{144}\right) x_5^2\) as follows:
As was done in Claim 1, $d_i > 0$ for $t \geq 0.299$ and $i = 0, 1, 2, 3, 4$. Also, $d_5$ of $B(t)$ is

\[
B(t) = \begin{pmatrix}
u_0 + tv_0 & -\sqrt{w_0 t} & 0 & 0 & 0 & 0 \\
-\sqrt{w_0 t} & u_1 + tv_1 & -\sqrt{w_1 t} & 0 & 0 & 0 \\
0 & -\sqrt{w_1 t} & u_2 + tv_2 & -\sqrt{w_2 t} & 0 & 0 \\
0 & 0 & -\sqrt{w_2 t} & u_3 + tv_3 & -\sqrt{w_3 t} & 0 \\
0 & 0 & 0 & -\sqrt{w_3 t} & u_4 + tv_4 & -\sqrt{w_4 t} \\
0 & 0 & 0 & 0 & -\sqrt{w_4 t} & u_5 + \frac{24t-7}{144}
\end{pmatrix}
\]

for $t \geq 0.299$

This is because $d_5$ is an increasing graph as is seen from the following Mathematica graph of $d_5$:

![Graph of d5](image)

Figure 1

Therefore, $F_n \geq 0$ for $n \geq 5$ and $s \geq \sqrt{0.299}$

Thus for all $t \geq 0$, $F_n \geq 0$ ($n \geq 5$). So by Theorem 2.2.2, $W_\alpha$ is quadratically hyponormal.
Example 2.2.2. Let $\alpha$ be the positive weight sequence given by $\alpha : \sqrt{\frac{1}{2}}, \sqrt{\frac{1}{2}}, \sqrt{\frac{47}{80}}, \sqrt{\frac{3}{4}}, \ldots$ Then the weighted shift operator $W_\alpha$ with weight sequence $\alpha$ is quadratically hyponormal.

This can be shown by a method similar to that used in Example 2.2.1.

**Proposition 2.2.3.** Let $\gamma(z)$ denote the positive weight sequence $\{\alpha_n\}$ given by $\sqrt{\frac{1}{2}}, \sqrt{\frac{1}{2}}, \sqrt{\frac{2}{3}}, \sqrt{\frac{3}{4}}, \ldots$ Then the weighted shift operator $W_{\gamma(z)}$ is quadratically hyponormal for all $z \in \left[\frac{43}{80}, \frac{47}{80}\right]$.

**Proof.** As $\alpha_{n+1} = \frac{n}{n+1}$ for all $n \geq 2$, therefore we have, $w_n = u_{n+1}v_n$ for all $n \geq 5$.

In view of Theorem 2.2.2, it is sufficient to show that $F_n \geq 0$ for all $n \geq 5$.

For $x_0, x_1, \ldots, x_5, s$ reals, we denote a function $G_5 = G_5(x_0, x_1, \ldots, x_5, s)$ by

$$G_5 := F_5 - v_5tx_5^2 \quad \text{where} \quad s^2 = t$$

$$= \sum_{i=0}^{4} (u_i + tv_i)x_i^2 - 2\sum_{i=0}^{4} \sqrt{w_i t} x_i x_{i+1} + u_5x_5^2$$

Then,

$$F_6 = G_5(x_0, \ldots, x_5, s) + \frac{z_6t}{1 + z_6t}v_5tx_5^2 \quad \text{(since} \quad w_n = u_{n+1}v_n \quad \text{for all} \quad n \geq 5 \quad \text{and} \quad x_n = \frac{v_n}{u_n})$$

Hence,

$$F_6(x_0, \ldots, x_6, s) \geq 0 \quad \text{for any} \quad x_0, \ldots, x_6, s \in \mathbb{R}_+$$

$$\Leftrightarrow G_5(x_0, \ldots, x_5, s) + \frac{z_6t}{1 + z_6t}v_5tx_5^2 \geq 0 \quad \text{for any} \quad x_0, \ldots, x_5, s \in \mathbb{R}_+$$

Similarly,

$$F_7 = G_5 + \left( v_5t - \frac{w_5t}{(u_6 + tv_6) - \left( \frac{w_6t}{u_7 + tv_7} \right)} \right) x_5^2 + \left( \frac{\sqrt{w_5t}}{\sqrt{(u_6 + tv_6) - \left( \frac{w_6t}{u_7 + tv_7} \right)}} \right) x_5$$

$$- \frac{(u_6 + tv_6) - \left( \frac{w_6t}{u_7 + tv_7} \right)}{x_6} \sqrt{(u_6 + tv_6) - \left( \frac{w_6t}{u_7 + tv_7} \right)} x_6 - \left( \frac{\sqrt{w_6t}}{\sqrt{u_7 + tv_7}} \right) x_6 \sqrt{u_7 + tv_7} x_7$$
and so

\[ F_7(x_0, \ldots, x_7, s) \geq 0 \text{ for any } x_0, \ldots, x_7, s \in \mathbb{R}_+ \]

\[ \Leftrightarrow G_5(x_0, \ldots, x_5, s) + \left( v_5 t - \frac{w_{5 t}}{(u_6 + tv_6) - \left( \frac{w_{4 t}}{u_7 + tv_7} \right)} \right) x_5^2 \geq 0 \]

for any \( x_0, \ldots, x_5, s \in \mathbb{R}_+ \)

\[ \Leftrightarrow G_5(x_0, \ldots, x_5, s) + \frac{z_7 z_6 t^2}{1 + z_7 t + z_7 z_6 t^2} v_5 t x_5^2 \geq 0 \text{ for any } x_0, \ldots, x_5, s \in \mathbb{R}_+ \]

So, by Mathematical induction, for \( n \geq 6 \) we have

\[
F_n(x_0, \ldots, x_n, s) \geq 0 \Leftrightarrow G_5 + \frac{(z_n x_{n-1} \ldots z_6 t^{n-5})}{1 + z_n t + z_n x_{n-1} t^2 + \cdots + z_n x_{n-1} \ldots z_6 t^{n-5}} v_5 t x_5^2 \geq 0 \\
\Leftrightarrow G_5 + \frac{1}{1 + \frac{1}{z_6 t} + \frac{1}{z_6 z_7 t^2} + \cdots + \frac{1}{z_6 \ldots z_n t^{n-5}}} v_5 t x_5^2 \geq 0
\]

(2.2.2)

Claim 1: \( G_5(x_0, \ldots, x_5, s) \geq 0 \text{ for } 0 \leq s \leq \sqrt{0.299} \)

The corresponding symmetric matrix to the quadratic form \( G_5 \) is

\[
A(t) = \begin{pmatrix}
  v_0 + tv_0 & -\sqrt{w_0 t} & 0 & 0 & 0 & 0 \\
-\sqrt{w_0 t} & u_1 + tv_1 & -\sqrt{w_1 t} & 0 & 0 & 0 \\
0 & -\sqrt{w_1 t} & u_2 + tv_2 & -\sqrt{w_2 t} & 0 & 0 \\
0 & 0 & -\sqrt{w_2 t} & u_3 + tv_3 & -\sqrt{w_3 t} & 0 \\
0 & 0 & 0 & -\sqrt{w_3 t} & u_4 + tv_4 & -\sqrt{w_4 t} \\
0 & 0 & 0 & 0 & -\sqrt{w_4 t} & u_5 \\
\end{pmatrix}
\]

We discuss the positivity of \( A(t) \) by Nested Determinant Test. By direct Computation, we have

\[ d_0 = \alpha_0^2 + t \alpha_0^2 \alpha_1^2 \]

\[ d_1 = t \alpha_0^2 \alpha_1^2 (\alpha_2^2 - \alpha_1^2) + t^2 \alpha_0^2 \alpha_1^4 \alpha_2^2 \]
$$d_2 = t^2 \left\{ \alpha_0 \alpha_1^3 (\alpha_0^2 - \alpha_0^2) + \alpha_0^2 \alpha_1^2 (\alpha_0^2 - \alpha_1^2) (\alpha_0^2 \alpha_0^2 - \alpha_1^2 \alpha_0^2) \right\}$$

$$+ t^3 \left\{ \left( \alpha_0^2 \alpha_0^2 - \alpha_1^2 \alpha_0^2 \right) \alpha_0^2 \right\}$$

$$d_3 = t^2 \left( \frac{3z^2}{16} - \frac{5z}{96} - \frac{z^3}{6} \right) + t^3 \left( \frac{5z^2}{24} - \frac{z}{16} - \frac{z^3}{6} \right) + t^4 \left( \frac{11z^2}{192} - \frac{z}{64} - \frac{z^3}{24} \right)$$

$$= t^2 P_2(z) + t^3 P_3(z) + t^4 P_4(z)$$

$$d_4 = t^2 \left( \frac{z^2}{64} - \frac{11z}{1152} - \frac{z^3}{72} \right) + t^3 \left( \frac{3z^2}{160} - \frac{z}{192} - \frac{z^3}{60} \right)$$

$$+ t^4 \left( \frac{251z^2}{2304} - \frac{13z}{480} - \frac{199z^3}{1440} + \frac{z^4}{18} \right) + t^5 \left( \frac{43z^2}{960} - \frac{3z}{320} - \frac{91z^3}{1440} + \frac{z^4}{36} \right)$$

$$= t^2 Q_2(z) + t^3 Q_3(z) + t^4 Q_4(z) + t^5 Q_5(z)$$

All $d_0, d_1, d_2$ are positive by their expressions for $\alpha_0 = \alpha_1$ and $d_3, d_4$ are positive for all $z \in [\frac{13}{80}, \frac{47}{80}]$ and $\forall t \geq 0$, since all $P_i(z)$ ($i = 2, 3, 4$) and $Q_i(z)$ ($i = 2, 3, 4, 5$) are positive for all $z \in [\frac{13}{80}, \frac{47}{80}]$.

We use Mathematica graph to show the positivity for $d_5$ of the matrix $A(t)$.

$$d_5(z, t) = -\frac{t^2 z}{4608} + \frac{t^3 z}{2304} - \frac{t^4 z}{1920} - \frac{t^5 z}{3840} + \frac{t^2 z^2}{1280} - \frac{t^3 z^2}{640} + \frac{11t^4 z^2}{15360} + \frac{t^5 z^2}{11520}$$

$$- \frac{t^2 z^3}{1440} + \frac{t^3 z^3}{720} - \frac{t^4 z^3}{640} - \frac{t^5 z^3}{384} + \frac{t^6 z^3}{360} + \frac{t^7 z^3}{720}$$

Figure 2
From the above Mathematica graph it is clear that if $0 < t \leq 0.299$ then $d_5 > 0$, which implies that $A(t) \geq 0$ for $0 < t \leq 0.299$ and $G_5(x_0, \ldots, x_5, s) \geq 0$ for $0 < s \leq \sqrt{0.299}$ and Claim 1 is established. Hence by (2.2.2) $F_n(x_0, \ldots, x_n, s) \geq 0$ ($n \geq 5$) for any $x_0, \ldots, x_n \in \mathbb{R}^+$ and $0 < s \leq \sqrt{0.299}$.

Now we will show for $t \geq 0.299$.

As, $z_n = \frac{\varphi_n}{\omega_n} = \frac{4(n+1)}{n+2}$, ($n \geq 5$), so $\{z_n\}_{n=6}^{\infty}$ is an increasing sequence and hence

\[ 1 + \frac{1}{z_6 t} + \frac{1}{z_6^2 t^2} + \cdots + \frac{1}{z_6^n t^n} \leq \frac{1}{1 - \frac{1}{z_6 t}} \]

Now,

\[ G_5 + \frac{1}{z_6 t} + \frac{1}{z_6^2 t^2} + \cdots + \frac{1}{z_6^n t^n} v_0 t x_5^2 \geq G_5 + \left( \frac{24 t - 7}{144} \right) x_5^2 \]

\[ \therefore z_6 = \frac{24}{7} \text{ and } v_5 = \frac{1}{6} \]

Now considering the corresponding symmetric matrix $B(t)$ to the quadratic form $G_5 + \left( \frac{24 t - 7}{144} \right) x_5^2$, we have

\[ B(t) = \begin{pmatrix}
    u_0 + t v_0 & -\sqrt{w_0 t} & 0 & 0 & 0 & 0 \\
    -\sqrt{w_0 t} & u_1 + t v_1 & -\sqrt{w_1 t} & 0 & 0 & 0 \\
    0 & -\sqrt{w_1 t} & u_2 + t v_2 & -\sqrt{w_2 t} & 0 & 0 \\
    0 & 0 & -\sqrt{w_2 t} & u_3 + t v_3 & -\sqrt{w_3 t} & 0 \\
    0 & 0 & 0 & -\sqrt{w_3 t} & u_4 + t v_4 & -\sqrt{w_4 t} \\
    0 & 0 & 0 & 0 & -\sqrt{w_4 t} & u_5 + \frac{24 t - 7}{144}
\end{pmatrix} \]

\[ d_5(z, t) = -\frac{t^2 z}{165888} - \frac{t^3 z}{27648} - \frac{t^4 z}{13824} - \frac{199 t^5 z}{46080} - \frac{t^6 z}{640} + \frac{t^2 z^2}{46080} + \frac{t^3 z^2}{7680} + \frac{827 t^4 z^2}{1658880} + \frac{2413 t^5 z^2}{138240} + \frac{43 t^6 z^2}{5760} - \frac{t^2 z^3}{51840} - \frac{t^3 z^3}{8640} - \frac{31 t^4 z^3}{41472} - \frac{4679 t^5 z^3}{207360} + \frac{91 t^6 z^3}{8640} - \frac{t^4 z^4}{12960} + \frac{241 t^5 z^4}{25920} + \frac{t^6 z^4}{216} \]
Figure 3

From the above Mathematica graph it is clear that the graph is an increasing graph and hence $d_5 > 0$ for $t \geq 0.299$ and $z \in [\frac{43}{80}, \frac{47}{80}]$.

Therefore $F_n(x_0, \ldots, x_n, s) \geq 0$ for all $t > 0$ and $z \in [\frac{43}{80}, \frac{47}{80}]$. So by Theorem 2.2.2, $W_{\gamma}(z)$ is quadratically hyponormal for $z \in [\frac{43}{80}, \frac{47}{80}]$.

\[\square\]

2.3 NASC for weak $k$-hyponormality

Let $\mathbb{C}[z, w]$ denote the polynomials in two variables and $\mathbb{C}[z]$ denote all polynomials with one variable $z$. We first give a construction given by J. Agler [2] which associates operators $T$ on a Hilbert space $H$ with linear functionals $\lambda : \mathbb{C}[z, w] \rightarrow \mathbb{C}$ which obey certain positivity conditions.

For $h(z, w) = \sum_{i,j} h_{ij} z^i w^j \in \mathbb{C}[z, w]$ and an operator $T \in B(H)$, define $h(T, T^*) = \sum_{i,j} h_{ij} T^{*j} T^i$. In particular, $(z^i w^j)(T, T^*) = T^* T$ and $(z^i w^j)(T, T^*) = T^{*j} T^i$. If $x \in H$, then define a linear functional $\Lambda_T : \mathbb{C}[z, w] \rightarrow \mathbb{C}$ by the formula
\( \Lambda_T(h) = \langle h(T, T^*)x, x \rangle. \)

**Lemma 2.3.1.** For a polynomial \( p \in \mathbb{C}[z] \), \( \left( \overline{p(\bar{z})} \right)(T^*) = (p(T))^*. \)

**Proof.** Let \( p(z) = \sum_{i=0}^{n} a_i z^i \). Then \( \overline{p(\bar{z})} = \sum_{i=0}^{n} \bar{a}_i z^i \) and so \( \overline{p(\bar{z})}(T^*) = \sum_{i=0}^{n} \bar{a}_i T^{*i}. \)

\[ \therefore p(T)^* = \left( \sum_{i=0}^{n} a_i T^i \right)^* = \sum_{i=0}^{n} \bar{a}_i T^{*i} = \overline{p(\bar{z})}(T^*). \]

\( \square \)

Observed that for \( p \in \mathbb{C}[z] \) with \( p(z) = \sum_{i=0}^{n} a_i z^i \),

\[ p(w)p(z) = \left( \sum_{i=0}^{n} a_i w^i \right) \left( \sum_{i=0}^{n} a_i z^i \right) \in \mathbb{C}[z, w]. \]

**Lemma 2.3.2.** If \( x \) is a cyclic vector for \( T \), then \( \|T\| \leq 1 \) if and only if

\[ \Lambda_T \left( \overline{p(w)}(1 - zw)p(z) \right) \geq 0, \quad \forall p(z) \in \mathbb{C}[z]. \]

**Proof.** Since \( x \) is a cyclic vector for \( T \) so \( H = \text{cl}\{ p(T)x : p \in \mathbb{C}[z] \} \).

Now,

\[ \|T\| \leq 1 \iff \|Ty\|^2 \leq \|y\|^2 \quad (\forall y \in H) \]

\[ \iff \langle (I - TT^*)y, y \rangle \geq 0 \quad (\forall y \in H) \]

\[ \iff \langle (I - TT^*)p(T)x, p(T)x \rangle \geq 0 \]

\[ \iff \langle p(T)^*(I - TT^*)p(T)x, x \rangle \geq 0 \]

\[ \iff \langle \overline{p(\bar{w})}(1 - zw)p(z)(T, T^*)x, x \rangle \geq 0 \]

\[ \iff \Lambda_T \left( \overline{p(\bar{w})}(1 - zw)p(z) \right) \geq 0 \quad (\forall p(z) \in \mathbb{C}[z]) \]

\( \square \)

The following result was given by Agler [2]:
Lemma 2.3.3. [2] Let $T$ be a cyclic contract in $B(H)$. Then $T$ is weakly $k$-hyponormal if and only if

$$\Lambda_T \left[ \left( \overline{q(w)} + \overline{p(w)} \phi(z) \right) \left( q(z) + p(z) \phi(w) \right) \right] \geq 0$$

for all polynomials $p(z), q(z)$ and $\phi(z)$ with degree $\phi(z) \leq k$.

Now let us consider a weighted shift $W_\alpha$. Then $e_0$ is the standard cyclic vector for $W_\alpha$. Thus, taking $T = W_\alpha$, we get $\Lambda_T (z^*w) = \langle (z^*w)(T, T^*) e_0, e_0 \rangle = \langle T^* T e_0, e_0 \rangle = 0$ (for $i \neq j$).

Using this fact, a reformulation of Lemma 2.3.3 was given in [45] for the case where $T$ is a contractive weighted shift.

Lemma 2.3.4. [45] Suppose $W_\alpha$ is a contractive hyponormal weighted shift with weight sequence $\alpha := \{\alpha_i\}_{i=0}^{\infty}$. Then $W_\alpha$ is weakly $k$-hyponormal if and only if

$$\Delta_k^\alpha (\phi, p, q) := \gamma_k |\phi_j p_0|^2 + \left\langle \begin{pmatrix} \gamma_1 & \gamma_2 & \cdots & \gamma_k \\ \gamma_2 & \gamma_3 & \cdots & \gamma_{k+1} \\ \vdots & \vdots & \ddots & \vdots \\ \gamma_k & \gamma_{k+1} & \cdots & \gamma_{2k-1} \end{pmatrix} \begin{pmatrix} \phi_1 p_0 \\ \phi_2 p_1 \\ \vdots \\ \phi_k p_{k-1} \end{pmatrix}, \begin{pmatrix} \phi_{k-1} p_0 \\ \phi_k p_1 \\ \vdots \\ \phi_{2k-1} p_{k-1} \end{pmatrix} \right\rangle$$

$$+ \cdots + \left\langle \begin{pmatrix} \gamma_j \\ \gamma_{j+1} \\ \vdots \\ \gamma_{j+k} \end{pmatrix} \begin{pmatrix} q_j \\ q_{j+1} \\ \vdots \\ q_{j+k} \end{pmatrix}, \begin{pmatrix} n_j p_{j+k+1} \\ \phi_{j+1} p_{j+1} \\ \vdots \\ \phi_k p_{j+k} \end{pmatrix} \right\rangle$$

$$+ \sum_{j=0}^{\infty} \left\langle \begin{pmatrix} \gamma_j \\ \gamma_{j+1} \\ \vdots \\ \gamma_{j+2k} \end{pmatrix} \begin{pmatrix} q_j \\ q_{j+1} \\ \vdots \\ q_{j+k} \end{pmatrix}, \begin{pmatrix} n_j p_{j+k+1} \\ \phi_{j+1} p_{j+1} \\ \vdots \\ \phi_k p_{j+k} \end{pmatrix} \right\rangle \geq 0$$

for $\phi := \{\phi_i\}_{i=1}^{k}$, $p := \{p_i\}_{i=0}^{\infty}$ and $q := \{q_i\}_{i=0}^{\infty} \in \mathbb{C}$.

Lemma 2.3.5.

$$\Delta_{k+1}^\alpha (\phi, p, q) = \Delta_k^\alpha (\phi, p, q) + \sum_{j=k+1}^{\infty} \gamma_j \left[ |\phi_{k+1} p_j - k - 1|^2 + 2Re \left\{ \overline{\phi_{k+1} p_j - k - 1} \left( \sum_{l=1}^{k} \phi_l p_j - l \right) \right\} \right]$$

for all $k \geq 1$ and $\phi := \{\phi_i\}_{i=1}^{k}$, $p := \{p_i\}_{i=0}^{\infty}$ and $q := \{q_i\}_{i=0}^{\infty} \in \mathbb{C}$.
Proof.

\[
\Delta_{k+1}^2(\phi, p, q) := \gamma_{k+1} |\phi_{k+1}p_0|^2 + \left( \begin{array}{ccc}
\gamma_k & \gamma_{k+1} \\
\gamma_{k+1} & \gamma_{k+2}
\end{array} \right) \left( \begin{array}{c}
\phi_{k+p_0} \\
\phi_{k+1}p_1
\end{array} \right) + \left( \begin{array}{c}
\phi_{k+p_0} \\
\phi_{k+1}p_1
\end{array} \right)
\]

\[
+ \ldots + \left( \begin{array}{ccc}
\gamma_1 & \gamma_2 & \gamma_{k+1} \\
\gamma_2 & \gamma_3 & \gamma_{k+2} \\
\vdots & \vdots & \vdots \\
\gamma_{k+1} & \gamma_{k+2} & \gamma_{2k+1}
\end{array} \right) \left( \begin{array}{c}
\phi_{k+p_0} \\
\phi_{k+1}p_1 \\
\vdots \\
\phi_{k+1}p_{k-1}
\end{array} \right) \left( \begin{array}{c}
\phi_{k+p_0} \\
\phi_{k+1}p_1 \\
\vdots \\
\phi_{k+1}p_{k-1}
\end{array} \right)
\]

\[
+ \sum_{j=0}^{\infty} \left( \begin{array}{ccc}
\gamma_j & \gamma_{j+1} & \gamma_{j+k+1} \\
\gamma_{j+1} & \gamma_{j+2} & \gamma_{j+k+2} \\
\vdots & \vdots & \vdots \\
\gamma_{j+k+1} & \gamma_{j+k+2} & \gamma_{j+2k+1}
\end{array} \right) \left( \begin{array}{c}
\phi_{k+p_0} \\
\phi_{k+1}p_1 \\
\vdots \\
\phi_{k+1}p_{k-1}
\end{array} \right) \left( \begin{array}{c}
\phi_{k+p_0} \\
\phi_{k+1}p_1 \\
\vdots \\
\phi_{k+1}p_{k-1}
\end{array} \right)
\]

\[
= \Delta_k^2(\phi, p, q) + \gamma_{k+1} \left[ |\phi_{k+1}p_0|^2 + 2Re \left( \phi_{k+1}p_0 \left( \sum_{l=1}^{k} \phi_l \bar{\phi}_{k+1-l} \right) + \phi_{k+1}p_{k+1} \bar{q}_0 \right) \right]
\]

\[
+ \gamma_{k+2} \left[ |\phi_{k+1}p_1|^2 + 2Re \left( \phi_{k+1}p_1 \left( \sum_{l=1}^{k} \phi_l \bar{\phi}_{k+2-l} \right) + \phi_{k+1}p_{k+2} \bar{q}_1 \right) \right]
\]

\[
+ \gamma_{k+3} \left[ |\phi_{k+1}p_2|^2 + 2Re \left( \phi_{k+1}p_2 \left( \sum_{l=1}^{k} \phi_l \bar{\phi}_{k+3-l} \right) + \phi_{k+1}p_{k+3} \bar{q}_2 \right) \right]
\]

\[
+ \ldots
\]

\[
= \Delta_k^2(\phi, p, q) + \sum_{j=k+1}^{\infty} \gamma_j \left[ |\phi_{k+1}p_{j-k-1}|^2 + 2Re \left( \phi_{k+1}p_{j-k-1} \left( \sum_{l=1}^{k} \phi_l \bar{\phi}_{j-l} \right) + \phi_{k+1}p_j \bar{q}_{j-k-1} \right) \right]
\]

Lemma 2.3.6. Let \( \alpha := \{\alpha_n\}_{n=0}^{\infty} \) be a positive weight sequence and \( x = \epsilon \alpha_n \) for \( 0 < \epsilon < 1, n \geq 0 \). Then for \( k \geq 1 \),

\[
\Delta_k^{[n]}(\phi, p, q) = \epsilon^2 \Delta_k^2(\phi, p, q) + (1 - \epsilon^2) \left( |q_0|^2 + \sum_{i=1}^{n} \gamma_i z_i \right)
\]

where

\[
z_i = \sum_{j=1}^{k} |\phi_{j}p_{i-j}|^2 + |q_i|^2 + 2Re \left[ p_i \left( \sum_{j=1}^{k} \phi_j \bar{q}_{i-j} \right) + \sum_{l=2}^{k} \phi_{i-l} \left( \sum_{j=1}^{l-1} \phi_j \bar{p}_{i-j} \right) \right]
\]

Proof. Let \( \gamma'_j \) denote the moment sequence of \( \alpha[n : x] \). Then

\[
\gamma'_j = \left\{ \begin{array}{ll}
\gamma_j, & \text{for } j \leq n \\
\epsilon^2 \gamma_j, & \text{for } j > n
\end{array} \right.
\]
Case I: If \( n = 0 \), that is \( \alpha_0 \) be perturbed to \( x = \varepsilon \alpha_0 \), then
\[
\gamma_j' = \begin{cases} 
\gamma_0 = 1, & \text{for } j = 0 \\
\varepsilon^2 \gamma_j, & \text{for } j > 0,
\end{cases}
\]
\[
\Delta_1^{(0, x)}(\phi, p, q) = \gamma_1'|\phi_1 p_0|^2 + \sum_{j=0}^{\infty} \left( \begin{pmatrix} \gamma'_j & \gamma'_{j+1} \\ \gamma'_{j+1} & \gamma'_{j+2} \end{pmatrix} \begin{pmatrix} q_j \\ \phi_1 p_{j+1} + q_{j+1} \end{pmatrix} \right) 
= \varepsilon^2 \Delta_1^{(0)}(\phi, p, q) + (1 - \varepsilon^2)|q_0|^2
\]

By Lemma 2.3.5, we get
\[
\Delta_2^{(0, x)}(\phi, p, q) = \Delta_1^{(0, x)}(\phi, p, q) + \sum_{j=2}^{\infty} \gamma_j' \left[ |\phi_2 p_{j-2}|^2 + 2 \text{Re} \{ \bar{\phi}_2 p_{j-2} \phi_1 p_{j-1} + \phi_2 p_j \bar{q}_{j-2} \} \right]
= \varepsilon^2 \Delta_1^{(0)}(\phi, p, q) + (1 - \varepsilon^2)|q_0|^2
+ \sum_{j=2}^{\infty} \varepsilon^2 \gamma_j \left[ |\phi_2 p_{j-2}|^2 + 2 \text{Re} \{ \bar{\phi}_2 p_{j-2} \phi_1 p_{j-1} + \phi_2 p_j \bar{q}_{j-2} \} \right]
= \varepsilon^2 \Delta_2^{(0)}(\phi, p, q) + (1 - \varepsilon^2)|q_0|^2
\]

Similarly,
\[
\Delta_k^{(0, x)}(\phi, p, q) = \varepsilon^2 \Delta_k^{(0)}(\phi, p, q) + (1 - \varepsilon^2)|q_0|^2
\]
for all \( k \geq 1 \)

Case II: If \( n = 1 \), that is \( \alpha_1 \) be perturbed to \( x = \varepsilon \alpha_1 \), then
\[
\gamma_j' = \begin{cases} 
\gamma_1, & \text{for } j \leq 1 \\
\varepsilon^2 \gamma_j, & \text{for } j > 1,
\end{cases}
\]
\[
\Delta_1^{(1, x)}(\phi, p, q) = \varepsilon^2 \Delta_1^{(1)}(\phi, p, q) + (1 - \varepsilon^2)(|q_0|^2 + \gamma_1 z_1),
\]
where \( z_1 = |\phi_1 p_0|^2 + |q_1|^2 + 2 \text{Re}(\phi_1 p_1 \bar{q}_0) \)

Also, as \( \gamma_j' = \varepsilon^2 \gamma_j \) for \( j \geq 2 \), so by Lemma 2.3.5, we get
\[
\Delta_2^{(1, x)}(\phi, p, q) = \varepsilon^2 \Delta_2^{(1)}(\phi, p, q) + (1 - \varepsilon^2)(|q_0|^2 + \gamma_1 z_1),
\]
for all \( k \geq 1 \)
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Case III: If \( n = 2 \), that is \( \alpha_2 \) be perturbed to \( x = \varepsilon \alpha_2 \), then

\[
\gamma_j' = \begin{cases} 
\gamma_j, & \text{for } j \leq 2 \\
\varepsilon^2 \gamma_j, & \text{for } j > 2,
\end{cases}
\]

\[
\Delta_1^{[2, z]}(\phi, p, q) = \varepsilon^2 \Delta_1^{[2]}(\phi, p, q) + (1 - \varepsilon^2) (|q_0|^2 + \gamma_1 z_1' + \gamma_2 z_2'),
\]

where \( z_1' = |\phi_1 p_0|^2 + |q_1|^2 + 2 \text{Re}(\phi_1 p_1 \bar{q}_0) \) and \( z_2' = |\phi_1 p_1|^2 + |q_2|^2 + 2 \text{Re}(\phi_1 p_2 \bar{q}_1) \).

Again, by Lemma 2.3.5

\[
\Delta_2^{[2, z]}(\phi, p, q) = \Delta_1^{[2, z]}(\phi, p, q) + \sum_{j=2}^{\infty} \gamma_j' [\phi_2 \tilde{p}_j - \phi_2 \tilde{p}_j - 1 + \phi_2 \tilde{p}_j - 2] + 2 \text{Re} \left\{ \tilde{p}_2 \phi_0 \phi_1 \tilde{q}_1 + \phi_2 \phi_0 \tilde{q}_0 \right\}
\]

\[
= \varepsilon^2 \Delta_2^{[2]}(\phi, p, q) + (1 - \varepsilon^2) (|q_0|^2 + \gamma_1 z_1' + \gamma_2 z_2') + (1 - \varepsilon^2) \gamma_2 [|\phi_2 p_0|^2
\]

\[
+ 2 \text{Re} \left\{ \tilde{p}_2 \phi_0 \phi_1 \tilde{q}_1 + \phi_2 \phi_0 \tilde{q}_0 \right\}
\]

\[
= \varepsilon^2 \Delta_2^{[2]}(\phi, p, q) + (1 - \varepsilon^2) (|q_0|^2 + \gamma_1 z_1'' + \gamma_2 z_2''),
\]

where

\[
z_1'' = |\phi_1 p_0|^2 + |q_1|^2 + 2 \text{Re}(\phi_1 p_1 \bar{q}_0)
\]

and

\[
z_2'' = |\phi_1 p_1|^2 + |\phi_2 p_0|^2 + |q_2|^2 + 2 \text{Re} \left\{ \tilde{p}_2 \phi_0 \phi_1 \tilde{q}_1 + \phi_2 \phi_0 \tilde{q}_0 + \phi_1 \phi_2 \tilde{q}_0 \right\}
\]

As \( \gamma_j' = \varepsilon^2 \gamma_j \) for \( j > 2 \), so for \( k > 2 \), we get

\[
\Delta_k^{[2, z]}(\phi, p, q) = \varepsilon^2 \Delta_k^{[2]}(\phi, p, q) + (1 - \varepsilon^2) (|q_0|^2 + \gamma_1 z_1'' + \gamma_2 z_2''),
\]

Thus, for \( k \geq 1 \),

\[
\Delta_k^{[2, z]}(\phi, p, q) = \varepsilon^2 \Delta_k^{[2]}(\phi, p, q) + (1 - \varepsilon^2) (|q_0|^2 + \gamma_1 z_1 + \gamma_2 z_2),
\]

where

\[
z_1 = |\phi_1 p_0|^2 + |q_1|^2 + 2 \text{Re}(\phi_1 p_1 \bar{q}_0)
\]
and

\[ z_2 = \sum_{j=1}^{k} |\phi_j p_{2-j}|^2 + |q_2|^2 + 2Re \left[ p_2 \left( \sum_{j=1}^{k} \phi_j \bar{q}_{2-j} \right) + \bar{p}_2 p_1 \bar{p}_1 \right], \]

assuming \( p_m = 0 = q_m \) for \( m < 0 \).

**Case IV:** If \( n = 3 \), that is \( \alpha_3 \) be perturbed to \( x = \varepsilon \alpha_3 \), then

\[ \gamma'_j = \begin{cases} 
\gamma_j, & \text{for } j \leq 3 \\
\varepsilon^2 \gamma_j, & \text{for } j > 3,
\end{cases} \]

As in Case III, here we get for \( k \geq 1 \),

\[ \Delta_k^\alpha \varepsilon^3 \Delta_k^\alpha \left( \phi, p, q \right) = \varepsilon^2 \Delta_k^\alpha \left( \phi, p, q \right) + \left( 1 - \varepsilon^2 \right) \left( |q_0|^2 + \sum_{i=1}^{3} \gamma_i z_i \right), \]

where

\[ z_1 = |\phi_1 p_0|^2 + |q_1|^2 + 2Re(\phi_1 p_1 \bar{q}_0) \]

\[ z_2 = \sum_{j=1}^{k} |\phi_j p_{2-j}|^2 + |q_2|^2 + 2Re \left[ p_2 \left( \sum_{j=1}^{k} \phi_j \bar{q}_{2-j} \right) + \bar{p}_2 p_1 \bar{p}_1 \right] \]

and

\[ z_3 = \sum_{j=1}^{k} |\phi_j p_{3-j}|^2 + |q_3|^2 + 2Re \left[ p_3 \left( \sum_{j=1}^{k} \phi_j \bar{q}_{3-j} \right) + \sum_{l=2}^{k} \bar{p}_l p_{3-l} \left( \sum_{j=1}^{l-1} \phi_j \bar{p}_{3-j} \right) \right] \]

assuming \( p_m = 0 = q_m \) for \( m < 0 \). That is, for \( i = 1, 2, 3 \)

\[ z_i = \sum_{j=1}^{k} |\phi_j p_{i-j}|^2 + |q_i|^2 + 2Re \left[ p_i \left( \sum_{j=1}^{k} \phi_j \bar{q}_{i-j} \right) + \sum_{l=2}^{k} \bar{p}_l p_{i-l} \left( \sum_{j=1}^{l-1} \phi_j \bar{p}_{i-j} \right) \right] \]

Continuing in this way, if \( \alpha_n \) is perturbed to \( \varepsilon \alpha_n \) \( (n = 0, 1, 2, \ldots) \), then for all \( k \geq 1 \),

\[ \Delta_k^\alpha \varepsilon^n \Delta_k^\alpha \left( \phi, p, q \right) = \varepsilon^2 \Delta_k^\alpha \left( \phi, p, q \right) + \left( 1 - \varepsilon^2 \right) \left( |q_0|^2 + \sum_{i=1}^{n} \gamma_i z_i \right), \]

where

\[ z_i = \sum_{j=1}^{k} |\phi_j p_{i-j}|^2 + |q_i|^2 + 2Re \left[ p_i \left( \sum_{j=1}^{k} \phi_j \bar{q}_{i-j} \right) + \sum_{l=2}^{k} \bar{p}_l p_{i-l} \left( \sum_{j=1}^{l-1} \phi_j \bar{p}_{i-j} \right) \right]. \]
2.4 Perturbation and convexity

Lemma 2.4.1. Let $W_\alpha$ be a weakly $k$-hyponormal weighted shift and $\varepsilon \alpha_n \in \omega_\alpha(k, n)$ for some $\varepsilon \in (0, 1)$. Then $[\sqrt{\varepsilon} \alpha_n, \alpha_n] \subseteq \omega_\alpha(k, n)$.

Proof. Let $x = \varepsilon \alpha_n$ and for $0 < t < 1$, let $z_t = \sqrt{\delta} \alpha_n$ where $\delta = t \varepsilon + (1 - t)$.

Claim 1: $z_t \in \omega_\alpha(k, n)$ for all $0 < t < 1$. As $\varepsilon < \delta < 1$, so $\delta < \sqrt{\delta} < 1$ and therefore $z_t \in (x, \alpha_n)$.

Now by Lemma 2.3.6, for $\phi := \{\phi_i\}_{i=1}^k, \ p := \{p_i\}_{i=0}^\infty$ and $q := \{q_i\}_{i=0}^\infty$ in $\mathbb{C}$,

\[
\Delta_k^{[n:x]}(\phi, p, q) = \varepsilon^2 \Delta_k^\alpha(\phi, p, q) + (1 - \varepsilon^2) \left( |q_0|^2 + \sum_{i=1}^n \gamma_i z_i \right) \tag{2.4.1}
\]

and

\[
\Delta_k^{[n:x]}(\phi, p, q) = \delta \Delta_k^\alpha(\phi, p, q) + (1 - \delta) \left( |q_0|^2 + \sum_{i=1}^n \gamma_i z_i \right) \tag{2.4.2}
\]

Thus from (2.4.1) and (2.4.1), we get

\[
\Delta_k^{[n:x]}(\phi, p, q) = \delta \Delta_k^\alpha(\phi, p, q) + \left( \frac{1 - \delta}{1 - \varepsilon^2} \right) \left[ \Delta_k^{[n:x]}(\phi, p, q) - \varepsilon^2 \Delta_k^\alpha(\phi, p, q) \right]
\]

\[
= (\delta - \varepsilon^2) \Delta_k^\alpha(\phi, p, q) + \left( \frac{1 - \delta}{1 - \varepsilon^2} \right) \Delta_k^{[n:x]}(\phi, p, q) \geq 0
\]
as $\alpha_1, x \in \omega_\alpha(k, n)$.

Therefore by Lemma 2.3.4, $z_t \in \omega_\alpha(k, n)$ and Claim 1 is established.

Now let $\xi \in [\sqrt{\varepsilon} \alpha_n, \alpha_n]$. Then

\[
\xi = \lambda \alpha_n \text{ for } \sqrt{\varepsilon} \leq \lambda \leq 1
\]

\[
\Rightarrow y = \lambda^2 \alpha_n \in [\varepsilon \alpha_n, \alpha_n]
\]

\[
\Rightarrow \lambda^2 = t \varepsilon + (1 - t) \text{ for some } 0 < t < 1
\]

\[
\Rightarrow \xi = \lambda \alpha_n \in \omega_\alpha(k, n), \text{ by Claim 1.}
\]
Corollary 2.4.2. If $x, y \in \omega_\alpha(k, n)$, $x < y$ and $x = \varepsilon y$ ($0 < \varepsilon < 1$) Then $[\sqrt{\varepsilon}y, y] \subset \omega_\alpha(k, n)$

Proof. If $\gamma_i'$ denotes the moment sequence of $\alpha[n : y]$, then by Lemma 2.3.6,

$$\Delta_k^{\alpha[n y]}(\phi, p, q) = \varepsilon^2 \Delta_k^{\alpha[n y]}(\phi, p, q) + (1 - \varepsilon^2) \left(|q_0|^2 + \sum_{i=1}^{n} \gamma'_i z_i\right)$$

and so the result follows as in Lemma 2.4.1.

Theorem 2.4.3. Let $W_\alpha$ be a contractive weakly k-hyponormal weighted shift and $\omega_\alpha(k, j) := \{x : W_\alpha[j z] \text{ is weakly k-hyponormal}\}$ Then $\omega_\alpha(k, j)$ is a convex set.

Proof. Let $x, y \in \omega_\alpha(k, j)$. Without loss of generality we choose $x < y$ Then $x = \varepsilon y$ for some $0 < \varepsilon < 1$ By Corollary 2.4 2,

$$[\varepsilon^{\frac{1}{2}}y, y] \subset \omega_\alpha(k, j) \quad (2.4.3)$$

Step 1:

Let $x_1 = \varepsilon^{\frac{1}{2}}y$ Then $x = \varepsilon y = \varepsilon^{\frac{1}{2}}x_1$ As $\varepsilon^{\frac{1}{2}}x_1, x_1 \in \omega_\alpha(k, j)$, so by Corollary 2.4.2,

$$[\varepsilon^{\frac{1}{2}}x_1, x_1] \subset \omega_\alpha(k, j).$$

That is,

$$[\varepsilon^{\frac{1}{2}}y, \varepsilon^{\frac{1}{2}}y] \subset \omega_\alpha(k, j) \quad (2.4.4)$$

Therefore from (2.4.3) and (2.4.4), we get $[\varepsilon^{\frac{1}{2}}y, y] \subset \omega_\alpha(k, j)$.

Step 2:

Let $x_2 = \varepsilon^{\frac{1}{2}}y$ Then $x = \varepsilon y = \varepsilon^{\frac{1}{2}}x_2$ As $\varepsilon^{\frac{1}{2}}x_2, x_2 \in \omega_\alpha(k, j)$, so again by Corollary 2.4.2, $[\varepsilon^{\frac{1}{2}}x_2, x_2] \subset \omega_\alpha(k, j)$ That is,

$$[\varepsilon y, \varepsilon^{\frac{1}{2}}y] \subset \omega_\alpha(k, j) \quad (2.4.5)$$
Therefore from (2.4.3), (2.4.4) and (2.4.5), we get \([\varepsilon^{\frac{3}{2}} y, y] \subset \omega_{\alpha}(k, j)\).

Continuing this process, after \(n^{th}\) step we have \(\left[\varepsilon^{\left(\frac{2n+1}{2n+1}\right)} y, y\right] \subset \omega_{\alpha}(k, j)\). But \(\frac{2n+1}{2n+1} \uparrow 1\) as \(n \to \infty\) and so \(\varepsilon^{\left(\frac{2n+1}{2n+1}\right)} \downarrow \varepsilon\) as \(n \to \infty\). Thus we get \(\left(\varepsilon y, y\right) \subset \omega_{\alpha}(k, j)\). Therefore if \(x, y \in \omega_{\alpha}(k, j)\), then \([x, y] \subset \omega_{\alpha}(k, j)\) and so \(\omega_{\alpha}(k, j)\) is convex. \(\square\)