Chapter 1

Introduction

1.1 Background

Several important classes of bounded Hilbert space operators were introduced around the year 1950. We refer to three such classes of operators, namely, weighted shift operators, subnormal operators and hyponormal operators. Weighted shifts are among the apparently simple but actually very rich examples of Hilbert space operators. They are related to subtle questions of function theory and constructive mathematics.

- If \( \{e_n\}_{n=0}^{\infty} \) denotes an orthonormal basis of the space of square summable complex sequences \( \ell^2(\mathbb{Z}_+) \), and \( \{\alpha_n\}_{n=0}^{\infty} \) is a bounded sequence of scalars, then the unilateral weighted shift \( W \) on \( \ell^2(\mathbb{Z}_+) \) is defined linearly such that \( We_n = \alpha_n e_{n+1} \) for all \( n \).

Though references to these definitions go back to the late 1950’s, the first systematic study of shift operators was undertaken by R. L. Kelly in his doctoral thesis in 1966 [68]. About ten years later A. L. Shields again compiled a thorough account of subsequent developments [81]. Since then this class of operators has received much attention. Initially it was used in the investigation of isometries, but slowly it emerged as a fertile domain for providing examples in the
study of general operators.

The other two classes of bounded Hilbert space operators that we have mentioned are subnormal and hyponormal operators. Motivated by the successful development of the theory of normal operators, in 1950 P.R. Halmos introduced the notion of subnormality and hyponormality for bounded Hilbert space operators.

- We recall that an operator $T$ is subnormal if it is the restriction of a normal operator to an invariant subspace.

- $T$ is hyponormal if $T^*T \geq TT^*$.

By simple matrix calculations it can be verified that subnormality implies hyponormality, but the converse is false. One reason is that subnormality is invariant under polynomial calculus or the calculus of analytic functions, while hyponormality is not. If we define $T$ to be polynomially hyponormal whenever $p(T)$ is hyponormal for every polynomial $p \in \mathbb{C}[\mathbb{Z}]$, then the natural question that follows is:

**Question A**: If $T$ is polynomially hyponormal, then must $T$ be subnormal?

In [75] it was shown that Question-A has an affirmative answer if and only if the corresponding problem for unilateral weighted shifts has an affirmative answer. In other words, it was proved that there exists a non subnormal polynomially hyponormal operator if and only if there exist a weighted shift operator with the same property. In [34] was given an example of an operator which is polynomially hyponormal but not subnormal. This means that there must also exist a
non subnormal polynomially hyponormal weighted shift operator. However, till date such a weighted shift operator has not yet been identified. The reason for this could be because the gap between subnormality and hyponormality is not clearly understood. In [24] it was pointed out that we can easily construct a non subnormal polynomially hyponormal weighted shift operator, if we can give an affirmative answer to the following question regarding perturbation of weighted shift operators:

**Question B:** Is polynomial hyponormality of the weighted shift stable under small perturbations of the weight sequence?

Let us assume that Question-B has an affirmative answer. Under this assumption, if we consider the recursively generated weighted shift $T_x$ with weight sequence: $1, \sqrt{x}, (\sqrt{3}, \sqrt{\frac{10}{3}}, \sqrt{\frac{17}{5}})^\wedge$, then it can be shown that $T_x$ is subnormal if and only if $x = 2$; whereas $T_x$ is polynomially hyponormal if and only if $2 - \delta_1 < x < 2 + \delta_2$ for some $\delta_1, \delta_2 > 0$. Thus for sufficiently small $\epsilon > 0$, the weight sequence $\alpha_\epsilon : 1, \sqrt{2 + \epsilon}, (\sqrt{3}, \sqrt{\frac{10}{3}}, \sqrt{\frac{17}{5}})^\wedge$ would induce a non subnormal polynomially hyponormal weighted shift operator, as desired.

Hence it needs to be investigated whether Question-B does have an affirmative answer or not. And for this we need to develop the perturbation theory of weighted shift operators. In fact, a proper investigation of the notion of perturbation of weighted shift operators would help us to bridge the gap between subnormality and hyponormality, and to understand the position of the subnormals within the class of hyponormals.
1.2 Objectives

The basic problem we refer to is understanding the gap between the classes of subnormal and hyponormal operators. In the recent past several new classes of operators like $k$-hyponormal and weakly $k$-hyponormal operators have been introduced and studied in an attempt to bridge the gap between subnormality and hyponormality. We refer the following papers for details [12] [13] [16] [17] [22] [26] [36] [44] [51] [75].

Most of this work is carried out on the class of weighted shift operators, this being a prototype to the original question. A huge volume of literature deals with characterizations of these intermediate classes of operators by establishing necessary and sufficient conditions. These conditions are always in terms of weights of the weighted shift. This motivated us to raise the following questions: “suppose we have a $k$-hyponormal weighted shift $W$ with sequence $\{a_n\}$. To what extent can the weight sequence be perturbed, so that the corresponding perturbed shift still retain the property of $k$-hyponormality?” The ability to answer this question would contribute much towards a proper understanding of the class of $k$-hyponormal weighted shift operators, and also to distinguish it from the other subclasses.

Again it is known from the existing literature that the class of $k$-hyponormals is within the class of weakly $k$-hyponormals. This motivate us to ask “whether a perturbed $k$-hyponormal remains weakly $k$-hyponormal.” The present work attempts to address such kinds of questions.

Hence, the objective of the present work is to contribute to the development of the theory of perturbation of weighted shift operators, with reference to the
notion of hyponormality, $k$-hyponormality, weak $k$-hyponormality and subnormality. Our work aims to carry forward the ongoing research in this area and also to plug some of the holes in the existing literature.

1.3 Review of literature

We begin by taking a look at the class of weighted shift operators with reference to the classes of subnormal and hyponormal operators. We denote by $W_\alpha$ the weighted shift on $\ell^2(\mathbb{Z}_+)$ with a bounded weight sequence $\alpha = \{\alpha_n\}$. If, in particular, each $\alpha_n$ is equal to 1, then $W_\alpha$ is referred to as the simple unilateral shift and denoted by $U_+$. Since the bilateral shift on $\ell^2(\mathbb{Z})$ is a natural normal extension of $U_+$, hence $U_+$ is subnormal, and therefore also hyponormal. However, the weighted shift $W_\alpha$ need not always be subnormal or even hyponormal.

In fact we have the following results:

- $W_\alpha$ is hyponormal if and only if $|\alpha_n| \leq |\alpha_{n+1}|$ for all $n$.

- (Berger's Theorem) $W_\alpha$ is subnormal if and only if there exists a Borel probability measure $\mu$ supported in $[0, \|W_\alpha\|^2]$, with $\|W_\alpha\|^2 \in \text{supp} \mu$, such that $\gamma_n = \int t^n d\mu(t)$ for all $n \geq 0$, where $\gamma_0 := 1$ and $\gamma_{n+1} := \alpha_n^2 \alpha_{n-1}^2 \ldots \alpha_0^2$ for $n \geq 0$.

In [51, Problem 203] Halmos asked for an example of a hyponormal operator that is not subnormal. Later on he himself comes up with one such example namely, the weighted shift operator $W_\alpha$ with weight sequence $\alpha = \{\alpha_n\}$, where $\alpha_0 = a, \alpha_1 = b, \alpha_n = 1$ for all $n > 1$ and $a < b < 1$. In fact Stampfli [80] was the first to address the question "which monotone shifts are subnormal?"
conditions for subnormality of \( W_\alpha \) in terms of the weights \( \alpha_n \). These conditions make it evident that even the first four weights \( (\alpha_0 < \alpha_1 < \alpha_2 < \alpha_3) \) may 'prevent' a shift from being subnormal.

Again in [51, Problem 209], Halmos asked for an example of a hyponormal operator whose square is not hyponormal. The example was duly provided but with much difficulty. We now recall subsequent development in the theory by which such examples can now be generated with much ease.

Let \( H \) be an infinite dimensional separable complex Hilbert space and let \( B(H) \), denote the algebra of bounded linear operators on \( H \).

- For \( S, T \in B(H) \), \([S, T] := ST - TS\).

- An \( n \)-tuple \( T = (T_1, \ldots, T_n) \) is hyponormal or the operators \( T_1, \ldots, T_n \) are jointly hyponormal if

\[
[T^*, T] := \begin{pmatrix}
[T_1^*, T_1] & [T_2^*, T_1] & \cdots & [T_n^*, T_1] \\
[T_1^*, T_2] & [T_2^*, T_2] & \cdots & [T_n^*, T_2] \\
\vdots & \vdots & \ddots & \vdots \\
[T_1^*, T_n] & [T_2^*, T_n] & \cdots & [T_n^*, T_n]
\end{pmatrix} \succeq 0.
\]

- For \( k \geq 1 \), \( T \in B(H) \) is \( k \)-hyponormal if \( (T, T^2, \ldots, T^k) \) is hyponormal i.e.,

\[
\begin{pmatrix}
\vdots & \vdots & \ddots & \vdots \\
[T^*, T^k] & [T^2, T^k] & \cdots & [T^k, T^k]
\end{pmatrix} \succeq 0.
\]

- (Bram-Halmos)

\( T \in B(H) \) is subnormal

\( \Leftrightarrow T \) is \( k \) - hyponormal for all \( k \geq 1 \)

\( \Leftrightarrow (T, T^2, \ldots, T^k) \) is hyponormal for all \( k \geq 1 \).
An $n$-tuple $T = (T_1, \ldots, T_n)$ is weakly hyponormal if $LS(T) := \{ \sum_{i=1}^{n} \lambda_i T_i : \lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{C}^n \}$ consists only of hyponormal operators.

For $k \geq 1$, $T \in B(H)$ is weakly $k$-hyponormal if $(T, T^2, \ldots, T^k)$ is weakly hyponormal.

$T \in B(H)$ is said to be polynomially hyponormal if $T$ is weakly $k$-hyponormal for all $k \geq 1$.

$W_\alpha$ is $k$-hyponormal $\iff (\gamma_{n+1+j})_{j=0}^k \geq 0$ for all $n \geq 0$, where $\gamma_0 := 1$ and $\gamma_{n+1} = \alpha_n^2 \gamma_n$ for $n \geq 0$, defines the moment sequence of $W_\alpha$.

With this last characterization at hand, it is possible to distinguish between $k$-hyponormality and $(k+1)$-hyponormality for every $k \geq 1$. But while $k$-hyponormality of weighted shift admits a simple characterization, the same is not true for weak $k$-hyponormality.

In an effort to unravel how $k$-hyponormality and weak $k$-hyponormality are interrelated, different researchers have adopted different line of thoughts:

(a) A number of papers have been written describing the links for specific families of weighted shifts e.g., those with recursively generated tails and those obtained by restricting the Bergman shift to suitable invariant subspaces. Some of the relevant references are the following [5] [12] [13] [17] [19] [20] [21] [24] [25] [32] [65] [66] [71].

(b) Another approach has been to take a closer look at weighted shifts whose first few weights are unrestricted but whose tails are subnormal and recursively generated, refer to [3] [15] [16] [45] [46] [60] [63] [80].
As such we have a whole range of results leading to a better understanding of the problem in hand.

- [80] If $W_\alpha$ is subnormal weighted shift with weight sequence $\alpha = \{\alpha_n\}_{n=0}^\infty$ and $\alpha_n = \alpha_{n+1}$ for some $n \geq 0$, then $\alpha_1 = \alpha_2 = \ldots$ i.e., $W_\alpha$ is flat.

- [6] Let $W_\alpha$ be a unilateral weighted shift with weight sequence $\{\alpha_n\}_{n=0}^\infty$ and assume that $W_\alpha$ is quadratically hyponormal (that is, weakly 2-hyponormal). If $\alpha_n = \alpha_{n+1}$ for some $n \geq 1$, then $\alpha_1 = \alpha_2 = \ldots$ i.e. $W_\alpha$ is subnormal.

- [12] For $x > 0$ let $W_\alpha$ be the weighted shift whose weight sequence is given by $\alpha_0 := x$ and $\alpha_n = \sqrt{n+\frac{1}{2}}$ for $n \geq 1$. Then
  (i) $W_\alpha$ is subnormal $\iff 0 < x \leq \sqrt{\frac{1}{2}}$
  (ii) $W_\alpha$ is 2-hyponormal $\iff 0 < x \leq \frac{3}{4}$
  (iii) $W_\alpha$ is weakly 2-hyponormal $\iff 0 < x \leq \sqrt{\frac{5}{3}}$.

- [46] Let $\alpha(x) : \sqrt{x}, \sqrt{x}, \sqrt{\sqrt{\frac{1}{2}}}, \sqrt{\sqrt{\frac{1}{2}}}, \ldots$ be a weight sequence with Bergman tail. Then $\{x \in \mathbb{R}_+ | W_{\alpha(x)} \text{is q.h.}\}$ is a closed interval and is equal to $[\delta_1, \delta_2]$ where $\delta_1 \approx .1673$ and $\delta_2 \approx .7439$ approximate to four places after decimal.

- [23] Let $\{\delta_n\}_{n=0}^\infty$ be the weight sequence given by

$$
\delta_n = \begin{cases}
\frac{1}{3}, & \text{if } n = 0 \\
\frac{1}{2^n}, & \text{if } n = 2, 4, 6, \ldots \\
\frac{1}{2^n+2}, & \text{if } n = 1, 3, 5, \ldots
\end{cases}
$$

If $\alpha_n = \left(\sum_{k=0}^n \delta_k\right)^{1/2}$ for $n \geq 0$, then $W_\alpha$ is hyponormal but not 2-hyponormal.
• [66] Let \( \alpha(x) : \sqrt{\frac{2}{3}}, \sqrt{\frac{3}{4}}, \sqrt{\frac{4}{5}}, \ldots \) There exists \( \delta \in (\frac{9}{16}, \frac{2}{3}) \) such that
  
  (i) \( W_{\alpha(x)} \) is cubically but not 2-hyponormal if \( \frac{9}{16} < x \leq \delta \).
  
  (ii) \( W_{\alpha(x)} \) is quadratically hyponormal but not cubically hyponormal if \( \delta < x < \frac{2}{3} \).

Inspite of this huge repertoire of established results and generated examples, it should however be mentioned that the overall problem still remains largely unsolved.

The study of the multivariable analogue to these problems have also received much attention in the last few years [27] [28] [29] [30] [31] [37] [38] [39] [40].

• Consider double indexed positive bounded sequences \( \alpha_k, \beta_k \in \ell^{\infty}(\mathbb{Z}^2_+) \), \( k \equiv (k_1, k_2) \in \mathbb{Z}^2_+ := \mathbb{Z}_+ \times \mathbb{Z}_+ \) and let \( \ell^2(\mathbb{Z}^2_+) \) be the Hilbert Space of square summable complex sequences indexed by \( \mathbb{Z}^2_+ \). The 2-variable weighted shift \( T = (T_1, T_2) \) is defined by

\[
T_1 e_k = \alpha_k e_{k+\varepsilon_1}, T_2 e_k = \beta_k e_{k+\varepsilon_2}
\]

where \( \varepsilon_1 = (1, 0) \) and \( \varepsilon_2 = (0, 1) \). Here

\[
T_1 T_2 = T_2 T_1 \iff \beta_{k+\varepsilon_1} \alpha_k = \alpha_{k+\varepsilon_2} \beta_k \text{ for all } k \in \mathbb{Z}^2_+.
\]

Given \( k \equiv (k_1, k_2) \in \mathbb{Z}^2_+ \), the moments of \( T \) of order \( k \) are

\[
\gamma_k := \begin{cases}
1 & \text{if } k_1 = 0 = k_2 \\
\alpha^2_{(0,0)} \cdots \alpha^2_{(k_1-1,0)} & \text{if } k_1 \geq 1 \text{ and } k_2 = 0 \\
\beta^2_{(0,0)} \cdots \beta^2_{(0,k_2-1)} & \text{if } k_1 = 0 \text{ and } k_2 \geq 1 \\
\beta^2_{(0,0)} \cdots \beta^2_{(0,k_2-1)} \alpha^2_{(0,k_2)} \cdots \alpha^2_{(k_1-1,k_2)} & \text{if } k_1 \geq 1 \text{ and } k_2 \geq 1
\end{cases}
\]

A multivariable weighted shift can be defined in an entirely similar way.
Chapter 1

• [67](Berger's Theorem: characterization of subnormality for 2-variable weighted shifts) $T$ admits a commuting normal extension if and only if there is a probability measure $\mu$ defined on the 2-dimensional rectangle $R = [0, a_1] \times [0, a_2]$, $(a_i := \|T_i\|^2)$ such that $\gamma_k = \int \int_R t^k d\mu(t) = \int \int_R t_1^{k_1} t_2^{k_2} d\mu(t_1, t_2) \ (\forall k \in \mathbb{Z}_+^2)$.

• [25] A 2-variable weighted shift $T = (T_1, T_2)$ is $k$-hyponormal $\iff (\gamma_u \gamma_{u+(m,n)+(p,q)} - \gamma_{u+(m,n)} \gamma_{u+(p,q)})_{1 \leq m+n \leq k, 1 \leq p+q \leq k} \geq 0$ for all $u \in \mathbb{Z}_+^2$.

• A 2-variable weighted shift $T$ is horizontally flat if $\alpha_{(k_1, k_2)} = \alpha_{(1,1)}$ for all $k_1, k_2 \geq 1$; vertically flat if $\beta_{(k_1, k_2)} = \beta_{(1,1)}$ for all $k_1, k_2 \geq 1$; flat if it is horizontally flat and vertically flat; symmetrically flat if $T$ is flat and $\alpha_{(1,1)} = \beta_{(1,1)}$.

1.4 Notations

We mention here a few standard notations to be followed throughout the sequel.

$\mathbb{N}$ : Set of natural numbers.

$\mathbb{Z}$ : Set of integers.

$\mathbb{Z}_+$ : Set of non-negative integers.

$\mathbb{R}$ : Set of real numbers.

$\mathbb{R}_+$ : Set of non-negative real numbers.

$\mathbb{C}$ : Set of complex numbers.

$\mathbb{Z}_+^2$ : Set of Ordered pairs of non-negative integers.

$\ell^2(\mathbb{Z}_+)$ : Hilbert space of square summable complex sequences indexed by the set $\mathbb{Z}_+$.

$\ell^2(\mathbb{Z}_+^2)$ : Hilbert space of square summable complex sequences indexed by the
Chapter 1

\[ \ell^\infty(\mathbb{Z}_+) : \text{Space of all bounded sequences of scalars indexed by the set } \mathbb{Z}_+. \]

\[ \ell^\infty(\mathbb{Z}_+^2) : \text{Space of all bounded sequences of scalars indexed by the set } \mathbb{Z}_+^2. \]

In addition to these we also often use the following abbreviations:

q.h.: Quadratic hyponormal or weak 2-hyponormal.

p.q.h.: Positive quadratic hyponormal.

NASC: Necessary and sufficient condition.

cl: Closure.

1.5 Chapterwise brief summary

Chapter 1: This chapter is introductory in nature. We include here the motivation and objectives of the present work, along with a brief review of literature leading to the same. A chapterwise brief summary of the work done in each chapter of the thesis is also included here.

Chapter 2: On convexity of weakly \( k \)-hyponormal region

Let \( \alpha = \{\alpha_n\}_{n=0}^\infty \) be a weight sequence. Let \( k \geq 1 \) and \( j \geq 0 \). Define

\[ \alpha[j : x] : \alpha_0, \alpha_1, \ldots, \alpha_{j-1}, x, \alpha_{j+1}, \ldots \]

We say, \( \alpha[j : x] \) is the perturbation of weight sequence \( \alpha \) where the \( j^{th} \) weight of \( \alpha \) namely, \( \alpha_j \) is perturbed to \( x \).

Let \( \Omega_{\alpha}(k,j) := \{x : W_{\alpha[j] x} \text{ is } k\text{-hyponormal}\} \)

and \( \omega_{\alpha}(k,j) := \{x : W_{\alpha[j] x} \text{ is weakly } k\text{-hyponormal}\} \).

If \( W_\alpha \) is a weighted shift then \( W_{\alpha[j] x} \) is referred to as a rank-one perturbation of \( W_\alpha \) where the \( j^{th} \) weight \( \alpha_j \) is perturbed to \( x \). If for \( i < j \), \( \alpha_i \) and \( \alpha_j \) are perturbed to \( x \) and \( y \) respectively, then \( W_{\alpha[(i,x),(j,y)]} \) is referred to as rank-two perturbation of \( W_\alpha \) Similarly, we can define any finite perturbation of \( W_\alpha \).

In [24, Theorem 6.5] it was shown that rank-one perturbations of \( k \)-hyponormal
weighted shifts which preserve $k$-hyponormality form a convex set. That is, if
$W_\alpha$ is $k$-hyponormal then $\Omega_\alpha(k,j)$ is a convex set. The natural question that
follows is "If $W_\alpha$ is weakly $k$-hyponormal, then is $\omega_\alpha(k,j)$ a convex set?"
In this chapter we answer this question in the affirmative.
For this we have used the characterization of weak $k$-hyponormality given in [45].

Chapter 3: On convexity of positive quadratic hyponormal region

This chapter is in continuation of Chapter 2. Here also we continue to investigate
the idea of convexity.
If $\alpha = \{\alpha_n\}_{n=0}^\infty$ be a positive weight sequence, $t \geq 0$, $k \geq 1$ and
$W_\alpha$ is weakly $k$-hyponormal, then we have shown that $\omega_\alpha(k,j)$ is a nonempty convex set, where
$\omega_\alpha(k,j) := \{x : W_\alpha[x] \text{ is weakly } k\text{-hyponormal}\}$.
Question: For $y \in \omega_\alpha(k,i+1)$, is there any relation between $\omega_\alpha(k,i)$ and
$\omega_\alpha[i+1,y](k,i)$? Here $\omega_\alpha[i+1,y](k,i) := \{x : W_\alpha[(i:x),(i+1:y)] \text{ is weakly } k\text{-hyponormal}\}$.

In this chapter we address this problem with reference to a positively quadrat­
ically hyponormal operator $W_\alpha$ with weight sequence $\alpha = \{\alpha_n\}_{n=0}^\infty$ where $\alpha_n = \sqrt{\frac{n+1}{n+2}}$ for all $n$.
We have proved the following:

1. For $x \in [k_1, k_2]$, the weighted shift $W_{[(0:x),(1:x)]}$ is p.q.h., where $k_1 = 0.630435$, $k_2 = 0.737144$.

2. For $y \in [k_1, k_2]$, $\{x : W_\alpha[(0:x),(1:y)] \text{ is p.q.h.}\} = (0, y]$.

3. If either $y < k_1$ or $y > k_2$, then there exists $0 < x \leq y$ such that
$W_\alpha[(0:x),(1:y)]$ is not p.q.h. If we represent the perturbations of $\alpha_0$ and
$\alpha_1$ as $x$ and $y$ respectively, and represent them in the 2-dimensional plane
then our result can be graphically represented as follows
Chapter 4: Finite rank perturbation of $2$-hyponormal weighted shifts

In [24, Theorem 2.1] it has been shown that a non-zero finite rank perturbation of a subnormal shift is never subnormal unless the perturbation occurs at the initial weight. However, this is not necessarily true for a $2$-hyponormal shift as shown in [24, Example 3.1(ii)]. In view of this, the question being addressed in this chapter is as follows:

"Given a $2$-hyponormal weighted shift $W_\alpha$ and $j \geq 0$, does there always exist $\varepsilon > 0$ such that for $x \in (\alpha_j - \varepsilon, \alpha_j + \varepsilon)$, $W_{\alpha[x]}$ is again $2$-hyponormal?"

In this chapter we establish a set of sufficient conditions under which there exists $\varepsilon > 0$ such that for $x \in (\alpha_j - \varepsilon, \alpha_j + \varepsilon)$, $W_{\alpha[x]}$ will again be $2$-hyponormal. Applying these conditions we can completely determine the situations where $2$-hyponormality preserving perturbations do not exist.

Moreover, in [24, Theorem 2.3], it was shown that a $2$-hyponormal weighted shift remains quadratically hyponormal under small non-zero finite rank perturbations. The proof was based on the definition of positive quadratic hyponormality. In this chapter we give an independent proof for the same result, using a different characterization of quadratic hyponormality.

Chapter 5: Perturbation of $2$-variable hyponormal weighted shift

In Chapter 4 we have addressed the question of finite rank perturbation of $2$-hyponormal weighted shift considering the unilateral weighted shift $W_\alpha$ on $\ell^2(\mathbb{Z}_+)$. In this chapter we initiate a parallel discussion for the $2$-variable weighted shift...
shift on \( \ell^2(\mathbb{Z}_+^2) \). For a unilateral weighted shift \( W_\alpha \) it is well known that \( W_\alpha \) is hyponormal if and only if \( \alpha_n \leq \alpha_{n+1} \) for all \( n \). Hence for a strictly increasing weight sequence, any slight perturbation of the \( n^{th} \) weight still retains the hyponormality property for the perturbed shift. "Is the same true for a two variable weighted shift?" The answer is negative as is shown in the work done in this chapter. We also frame a set of positivity conditions which can completely determine hyponormality of the perturbed shift.

**Chapter 6: On weak hyponormality of 2-variable weighted shifts**

In Chapter 5 it was shown that if for a 2-variable hyponormal shift \( T = (T_1, T_2) \), a weight \( \alpha_{(k_1,k_2)} \) is perturbed, then the resulting perturbed shift may not remain hyponormal. For example, say we have the 2-variable hyponormal shift \( T = (T_1, T_2) \) with respective weight sequences \( \{\alpha_{(k_1,k_2)}\} \) and \( \{\beta_{(k_1,k_2)}\} \), as shown in the following diagram

Suppose the weight \( \alpha_{(2,2)} \) is perturbed slightly to \( x \). Then to preserve commutativity, we need to perturb at least a minimum number of adjacent weights. So
accordingly, $\beta_{(2,2)}$ changes to $y$, $\alpha_{(1,2)}$ changes to $z$, and $\beta_{(2,1)}$ changes to $t$. The weight diagram of the perturbed shift $\tilde{T} = (\tilde{T}_1, \tilde{T}_2)$ will be as follows:

In Chapter 5 it was shown that $\tilde{T}$ may not remain hyponormal. In fact the conditions under which $\tilde{T}$ will still be hyponormal is completely given in that chapter.

In this chapter, we show that though $\tilde{T}$ may not be hyponormal, it will however still remain weakly hyponormal for sufficiently small perturbations $x$ of $\alpha_{(k_1,k_2)}$.

**Chapter 7: Back-step extension of weighted shifts**

In this chapter we address the question of perturbation of subnormal weighted shifts. It was shown in [24, Theorem 2.1] that a non-zero finite rank perturbation of a subnormal shift is never subnormal, unless the perturbation occurs at the initial weight $\alpha_0$. So the idea is to begin with a subnormal shift and create a back-step extension preserving subnormality. The necessary and sufficient conditions (NASC) for subnormal backward extension of a 1-variable weighted shift was first given by Curto [12, Proposition 8]. Later an improved version of this result
was given by Curto and Yoon [37, Proposition 1.5]. In the same paper, they have also given the NASC for subnormal backward extension of a 2-variable weighted shift [37, Proposition 2.9]. However, these results only deal with 1-step extension. In this chapter we extend these results to 2-step extension, and following a similar technique we propose NASC for \( n \)-step backward extension of 1-variable and 2-variable weighted shifts. In the last section we show how these results can also be derived applying Schur product technique.