Chapter 1

Introduction

1.1 Introduction

The Indian mathematical genius Srinivasa Ramanujan Ayenger (1887-1920) recorded many spectacular mathematical results in his notebooks [48] and his lost notebook [49]. It is well known that Ramanujan rarely provided any proof for his stated results. Berndt ([11], [14], [15], [17], and [18]), Agarwal[1], and Andrews and Berndt [3] systematically discussed the claims made by Ramanujan and provided proofs for the results stated by Ramanujan. Some of their proofs are based on modern ideas and some of them are verified being knowing the result in advance. In this thesis, we prove some of those results regarding modular equations, class invariants, theta-functions, and continued fractions. In the course of our study, we have also discovered many new results.
1.2 Scope of the Thesis

The thesis has seven chapters including the introductory Chapter 1.

In Chapter 2, we deal with Ramanujan’s Schl"{a}fli-type “mixed” modular equations. On pages 86 and 88 of his first notebook [48], Ramanujan recorded 12 Schl"{a}fli-type “mixed” modular equations. 11 of these were not recorded in his second notebook [48]. One of these 11 equations follows from a modular equation recorded by Ramanujan in Chapter 20 of his second notebook. This was first observed by K. G. Ramanathan [41, pp. 419-420]. Berndt [18] proved the other 10 equations by modularity, a method with which Ramanujan was not familiar. We give alternate proofs for 8 of these equations. Two are proved by deriving some theta-function identities using Schr"{o}ter’s formulae, and the rest are proved by employing Ramanujan’s Schl"{a}fli-type modular equations of prime degrees and some other modular equations. In the process, we also find two new Schl"{a}fli-type “mixed” modular equations [(2.3.19) and (2.3.60)]. For example, in Lemma 2.3.1 of Section 2.3, we find that, if

\[ Q = \left( \frac{\alpha \delta (1 - \alpha)(1 - \delta)}{\beta \gamma (1 - \beta)(1 - \gamma)} \right)^{\frac{1}{h}} \]

and

\[ R = \left( \frac{\gamma \delta (1 - \gamma)(1 - \delta)}{\alpha \beta (1 - \alpha)(1 - \beta)} \right)^{\frac{1}{h}} , \]

then,

\[ R^2 + \frac{1}{R^2} = Q^4 + \frac{1}{Q^4} - 3 , \]

where \( \beta, \gamma, \) and \( \delta \) are of degrees 3, 7, and 21, respectively, over \( \alpha \).
1.2. SCOPE OF THE THESIS

In Chapter 3, we deal with Weber-Ramanujan class invariants.

Let

\[(a; q)_{\infty} = \prod_{n=0}^{\infty} (1 - aq^n), \quad |q| < 1,\]

and, after Ramanujan, we set

\[\chi(q) = (-q; q^2)_{\infty}.\]

If \(q = \exp(-\pi \sqrt{n})\), where \(n\) is any positive rational number, then Weber-Ramanujan's class invariant \(G_n\) is defined by

\[G_n := 2^{-1/4}q^{-1/24}\chi(q).\]

In section 3.3, we derive

\[G_{217} = \left(\sqrt{\frac{11 + 4\sqrt{7}}{2}} + \sqrt{\frac{9 + 4\sqrt{7}}{2}}\right)^{1/2}\left(\sqrt{\frac{12 + 5\sqrt{7}}{4}} + \sqrt{\frac{16 + 5\sqrt{7}}{4}}\right)^{1/2}\]

by using Ramanujan's modular equations of degrees 7 and 31. Berndt, Chan, and Zhang [26] also could not utilize the modular equations of degrees 7 and 31 recorded by Ramanujan to effect a proof for \(G_{217}\). In Section 3.4, we employ some of the Schl"afli-type "mixed" modular equations discussed in Chapter 2, along with some other Schl"afli-type modular equations of prime degrees to evaluate Ramanujan's class invariants \(G_{15}, G_{21}, G_{33}, G_{39}, G_{55},\) and \(G_{65}\). It is worthwhile to note that our evaluation of \(G_{65}\) is much easier than that of Berndt, Chan, and Zhang [26]. The most important feature of our method is that we can also simultaneously get the values of \(G_{5/3}, G_{7/3}, G_{11/3}, G_{13/3}, G_{11/5},\) and \(G_{13/5}\). Previously, these values were found by verifications. We also note that, these class invariants can be utilized to find some of the explicit values of certain \(q\)-continued fractions [25], certain values of Ramanujan's product of theta-functions [27], and some values of the quotient of eta-functions [30].
In Chapter 4, we deal with Ramanujan's eta-function identities.

If \( q = \exp(2\pi i z) \), then Ramanujan's eta-function \( f(-q) \) is defined by

\[
f(-q) := q^{-1/24} \eta(z),
\]

where \( \eta(z) \) is classical Dedekind eta-function defined by

\[
\eta(z) := e^{\pi z/12} \prod_{n=1}^{\infty} (1 - e^{2\pi inz}), \quad \text{Im} z > 0.
\]

In the unorganized portions of his second notebook, Ramanujan [48] recorded without proofs 25 beautiful identities involving quotients of only eta-functions and no other theta-functions. Berndt and Zhang [23] proved some of these identities. Proofs of all the 25 identities recorded by Ramanujan are given in Chapter 25 of Berndt's book [17]. Of these identities 19 were proved by employing modular equations and parameterizations and 6 were proved by invoking the theory of modular forms. But in many of their proofs via parameterizations, they used heavy amount of tedious algebra and the identities must be known beforehand. So those proofs may be merely called verifications. In Chapter 4, we deduce five of these identities [see Theorems 4.2.1-4.2.5] by using Ramanujan's other eta-functions identities and one of our newly derived identities [see Lemma 4.5.1]. The main advantage of our method is that one can find other identities of the same kind. For example, in Section 7.6 of our last chapter, we find three new identities of the same kind in connection with Ramanujan's cubic continued fraction.
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In chapter 5, we deal with explicit evaluations of Ramanujan's theta-function $\phi(q)$, defined by

$$
\phi(q) := 1 + 2 \sum_{k=1}^{\infty} q^{k^2},
$$

where $|q| < 1$.

At different places of his notebooks [48], Ramanujan recorded several explicit values $\phi(q)$. Borwein and Borwein [31] first observed that Ramanujan's class invariants could be used to calculate certain explicit values of $\phi(e^{-n\pi})$. Berndt and Chan [21] verified all of Ramanujan's non-elementary values of $\phi(e^{-n\pi})$. They also derived some new values by combining Ramanujan's class invariants with his modular equations. We give simpler proofs for some of these evaluations and calculate some new values of $\phi(e^{-n\pi})$. We also find some new theorems for finding explicit values of quotients of theta-functions by deriving some theta-function identities.

In Chapter 6, we deal with the famous Rogers-Ramanujan continued fraction $R(q)$, defined by

$$
R(q) := \frac{q^{1/5}}{1 + \frac{q}{1 + \frac{q^2}{1 + \frac{q^3}{1 + \ldots}}}}, \quad |q| < 1.
$$

In his first and second letters to Hardy [22], Ramanujan communicated several explicit values of $R(q)$ and $S(q)$, where $S(q) = -R(-q)$. Watson [52]-[53] proved some of the results claimed by Ramanujan in those letters. In both his first [48] and lost notebooks [49], Ramanujan recorded several other evaluations. In particular, on page 210 of his lost notebook [49], Ramanujan provided a list of evaluations and intended evaluations. Ramanathan [42]-[46] made the first attempt to find a uniform method to evaluate $R(q)$ by using Kronecker's limit formula, with
which Ramanujan was not familiar. Berndt and Chan [20] and Berndt, Chan, and Zhang [25] completed the incomplete list of Ramanujan by using some modular equations recorded by Ramanujan [48] in his notebooks. Most importantly, Berndt, Chan, and Zhang [25] derived general formulas for evaluating $R(e^{-2\pi \sqrt{n}})$ and $S(e^{-\pi \sqrt{n}})$ in terms of Weber-Ramanujan class invariants. The lost notebook [49] also contains many formulas for $R(q)$ and theta-function identities giving more formulas for the explicit evaluation of $R(q)$. Kang [37]-[38] proved many of the claims made by Ramanujan. It appears that though Ramanujan’s formulas are interesting, they generally are not very much amenable in the calculation of elegant values of $R(q)$. Here we find some of the evaluations of $R(q)$ and $S(q)$, by using the values of the quotients of theta-functions found in Chapter 5 and some other theta-function identities. Our evaluations are much easier than those of the previous authors.

In Chapter 7, we deal with Ramanujan’s cubic continued fraction $G(q)$, defined by

$$G(q) := \frac{q^{1/3}}{1 + \frac{q + q^4}{1 + \frac{q^2 + q^4}{1 + \frac{q^3 + q^4}{1}}}} , \quad |q| < 1. \quad (1.2.5)$$

Ramanujan first introduced this continued fraction in his second letter to Hardy [22]. He also recorded this continued fraction on page 366 of his lost notebook [49], and claimed that there are many results of $G(q)$ which are analogous to $R(q)$. Motivated by Ramanujan’s claims, Chan [32] proved many new identities which probably were the identities vaguely referred by Ramanujan. He established some reciprocity theorems for $G(q)$, found relations between $G(q)$ and the three continued fractions $G(-q)$, $G(q^2)$ and $G(q^4)$, and obtained some explicit evaluations of $G(q)$. For example, he proved the following relation between $G(q)$ and $G(q^4)$

$$G^4(q) = G(q^4) \frac{1 - G(q^4) + G^2(q^4)}{1 + 2G(q^4) + 4G^2(q^4)} . \quad (1.2.6)$$
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But his proof of (1.2.6) is not satisfactory. In particular, the last deduction [32, (2.18), p. 347] is not an obvious one. In Section 7.2, we find an easy proof of (1.2.6).

By deriving some theta-function identities in Section 7.3 and Section 7.4, we give general formulas for the explicit evaluations of \( G(-e^{-3\pi \sqrt{n}}) \) and \( G(e^{-3\pi \sqrt{n}}) \) in Section 7.5. General formulas for the explicit evaluations of \( G(-e^{-\pi \sqrt{n}}) \) and \( G(e^{\pi \sqrt{n}}) \), were established by Berndt, Chan and Zhang [24].

In Section 7.6, we find three new beautiful eta-function identities [Theorems 7.6.1-7.6.3], and use them to derive two beautiful identities [Theorems 7.7.1-7.7.2] giving relations between \( G(q) \) and the two continued fractions \( G(q^5) \) and \( G(q^7) \). For example, in Theorem 7.7.1, we prove that, if \( v = G(q) \) and \( w = G(q^5) \), then

\[
v^6 - vw + 5vw(v^3 + w^3)(1 - 2vw) + w^6 = v^2w^2(16v^3w^3 - 20v^2w^2 + 20vw - 5).
\]