Chapter 7

Ramanujan’s Cubic Continued Fraction

7.1 Introduction

Let, for $|q| < 1$,

$$G(q) := \frac{q^{1/3}}{1 + \frac{q + q^2}{1 + \frac{q^2 + q^4}{1 + \frac{q^3 + q^6}{1 + \cdots}}}},$$  \hspace{1cm} (7.1.1)

denote Ramanujan’s cubic continued fraction, first introduced by him in his second letter to Hardy [22]. Ramanujan also recorded this continued fraction on page 366 of his lost notebook [49], and claimed that there are many results of $G(q)$ which are analogous to Rogers-Ramanujan continued fraction $R(q)$. Motivated by Ramanujan’s claims, H.H. Chan [32] proved many new identities which probably were the identities vaguely referred by Ramanujan. He established some reciprocity theorems for $G(q)$, found relations between $G(q)$ and the three continued fractions $G(-q), G(q^2)$ and $G(q^3)$ and obtained some explicit evaluations of $G(q)$.

Note: Some parts of this chapter consist of our papers [8] and [9].

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We note that his proof of the relation between $G(q)$ and $G(q^3)$ is not satisfactory. In particular, the last deduction [32, (2.18), p. 347] is not an obvious one. In Section 7.2 of this chapter, we find an easy proof of this relation.

In Section 7.3, we establish some theta-function identities recorded by Ramanujan in the unorganized pages of both his first and second notebooks [48]. Berndt [17] also proved these identities via parameterization.

In Section 7.4, we give some more theorems for the explicit evaluation of the quotients of theta-functions by using the identities found in the previous section.

In Section 7.5, we combine the theorems found in Section 7.4 with some other theta-function identities to deduce a number of explicit evaluations for $G(q)$. In fact, we have found general formulas for the explicit evaluations of $G(-e^{-3\pi\sqrt{9}})$ and $G(e^{3\pi\sqrt{9}})$. General formulas for the explicit evaluations of $G(-e^{-\pi\sqrt{9}})$ and $G(e^{\pi\sqrt{9}})$, were established by Berndt, Chan and Zhang [24].

In Section 7.6, we give three new eta-function identities, and use them in our final section to find two new identities giving relations between $G(q)$ and the two continued fractions $G(q^5)$ and $G(q^7)$.

### 7.2 A Relation Between $G(q)$ and $G(q^3)$

H.H. Chan [32] found the following beautiful relation connecting $G(q)$ and $G(q^3)$. As we already mentioned in the Introduction, his proof is not satisfactory. Here we give a simple proof of his theorem.
7.2. A RELATION BETWEEN $G(q)$ AND $G(q^3)$

**Theorem 7.2.1** If $G(q)$ is as defined in (7.1.1), then

$$G^3(q) = G(q^3) \frac{1 - G(q^3) + G^2(q^3)}{1 + 2G(q^3) + 4G^2(q^3)}. \quad (7.2.1)$$

**Proof:** From Entry 1(i) [15, p. 345], we note that

$$1 + \frac{1}{G(q)} = \frac{\psi(q^{1/3})}{q^{1/3}\psi(q^3)}, \quad (7.2.2)$$

and

$$1 + \frac{1}{G^3(q)} = \frac{\psi^4(q)}{q^4\psi(q^9)}, \quad (7.2.3)$$

where $\psi(q)$ is as defined in (2.1.24).

Replacing $q$ by $q^3$ in (7.2.2), we find that

$$1 + \frac{1}{G(q^3)} = \frac{\psi(q)}{q\psi(q^9)}. \quad (7.2.4)$$

Now, from Entry 1(ii) [15, p. 345], we note that

$$\left(1 + 3q \frac{\psi(-q^3)}{\psi(-q)} \right)^3 = 1 + 9q \frac{\psi^4(-q^3)}{\psi^4(-q)}. \quad (7.2.5)$$

Replacing $q$ by $-q$ in (7.2.3) and (7.2.4), and then using the resultant identities in (7.2.5), we find that

$$\left(1 - \frac{3\psi}{1 + \psi}\right)^3 = 1 - \frac{9u}{1 + u}, \quad (7.2.6)$$

where $w = G(-q^3)$ and $u = G^3(-q)$.

Solving (7.2.6) for $u$, we find that

$$u = 1 - \frac{((1 - 2w)/(1 + w))^3}{8 + (\{(1 - 2w)/(1 + w))^3}. \quad (7.2.7)$$
Simplifying (7.2.7), we obtain
\[ u = w \frac{1 - w + w^2}{1 + 2w + 4w^2}. \] (7.2.8)

Replacing \( q \) by \(-q\) in (7.2.8), we complete the proof.

### 7.3 A Theta-Function Identity

The following theorem was recorded by Ramanujan on page 4 of his second notebook [48]. It is extremely useful in our calculations. Berndt [17, p. 202] proved this theorem via parameterization. Here we prove this from theta-function identities.

**Theorem 7.3.1** If \( \chi \) and \( \psi \) are as defined in (2.1.26) and (2.1.24), respectively, then

\[ \frac{\chi^3(q)}{\chi(q^3)} = 1 + 3q \frac{\psi(-q^9)}{\psi(-q)}, \] (7.3.1)

**Proof:** From Corollary (ii) of Chapter 16 in Berndt's book [15, p. 49], we find that

\[ \psi(q) - q\psi(q^9) = f(q^3, q^6). \] (7.3.2)

Using Jacobi's triple product identity [15, Entry 19, p. 35], Berndt [15, p. 350] proved that

\[ f(q, q^2) = \frac{\phi(-q^3)}{\chi(-q)}. \] (7.3.3)

Replacing \( q \) by \( q^3 \) in (7.3.3), and then using the resultant identity in (7.3.2), we find that

\[ \psi(q) - q\psi(q^9) = \frac{\phi(-q^9)}{\chi(-q^3)}. \] (7.3.4)

Now, from Corollary (i) [15, p. 49] and (2.1.43), we find that

\[ \phi(-q^9) = \phi(-q) + 2q\psi(q^9)\chi(-q^3). \] (7.3.5)
7.4. EXPLICIT EVALUATIONS OF THETA-FUNCTIONS

Invoking (7.3.5) in (7.3.4), we deduce that

\[ \psi(q) - 3q\psi(q^0) = \frac{\phi(-q)}{\chi(-q^3)}. \]  

(7.3.6)

Thus,

\[ 1 - 3q\frac{\psi(q^0)}{\psi(q)} = \frac{\phi(-q)}{\chi(-q^3)\psi(q)}. \]  

(7.3.7)

Now from Entry 24(iii) [15, p. 39], we note that

\[ \chi(q) = \sqrt[3]{\frac{\phi(q)}{\psi(-q)}}. \]  

(7.3.8)

Replacing \( q \) by \( -q \) in (7.3.7) and then using (7.3.8), we complete the proof of the theorem.

7.4. Explicit evaluations of theta-functions

Theorem 7.4.1

(i) \( e^{-\pi \sqrt{n}} \frac{\psi(-e^{-\pi \sqrt{n}})}{\psi(-e^{-\pi \sqrt{n}})} = \frac{1}{3} \left( \sqrt{2} \frac{G_n^3}{G_{9n}} - 1 \right) \)  

(7.4.1)

and

(ii) \( e^{-\pi \sqrt{n}} \frac{\psi(e^{-\pi \sqrt{n}})}{\psi(e^{-\pi \sqrt{n}})} = \frac{1}{3} \left( 1 - \sqrt{2} \frac{g_n^3}{g_{9n}} \right) \)  

(7.4.2)

Proof: From Theorem 7.3.1 and the definition of \( G_n \) from (3.1.2), we easily arrive at (7.4.1). To prove (ii), we replace \( q \) by \( -q \) in Theorem 7.3.1 and then use the definition of \( g_n \) from (3.1.2).

Since, \( G_{9n} \) and \( g_{9n} \) can be calculated from the respective values of \( G_n \) and \( g_n \) [24], from the above theorem, we see that the certain quotients of theta-functions on the right sides can be evaluated if the corresponding values of \( G_n \) and \( g_n \) are known. We give only a couple of examples below.
Corollary 7.4.2

\[ e^{-\pi} \frac{\psi(-e^{-9\pi})}{\psi(-e^{-\pi})} = \frac{\sqrt{2}(\sqrt{3} - 1) - 1}{3}. \] (7.4.3)

**Proof:** Putting \( n = 1 \) in Theorem 7.4.1(i), we find that

\[ e^{-\pi} \frac{\psi(-e^{-9\pi})}{\psi(-e^{-\pi})} = \frac{1}{3} \left( \sqrt{2} \frac{G_1^3}{G_9} - 1 \right). \] (7.4.4)

From Berndt’s book [16, p. 189],

\[ G_1 = 1 \quad \text{and} \quad G_9 = \left( \frac{1 + \sqrt{3}}{\sqrt{2}} \right)^{1/3}. \] (7.4.5)

Employing (7.4.5) in (7.4.4), and simplifying we complete the proof.

From Entry 11(ii) [15, p. 123], we find that

\[ \psi(-e^{-\pi}) = \phi(e^{-\pi}) 2^{-3/4} e^{\pi/8}. \] (7.4.6)

Since

\[ \phi(e^{-\pi}) = \frac{\pi^{1/4}}{\Gamma \left( \frac{3}{4} \right)} \]

is classical [62], (7.4.3) and (7.4.6) provide an explicit evaluation for \( \psi(-e^{-9\pi}) \).

Corollary 7.4.3

\[ e^{-\pi \sqrt{5/3}} \frac{\psi(-e^{-3\pi\sqrt{5}/5})}{\psi(-e^{-\pi\sqrt{5}/3})} = \frac{(3 + \sqrt{5})(\sqrt{5} - \sqrt{3}) - 2}{6}. \] (7.4.7)

**Proof:** Putting \( n = 5/9 \) in Theorem 7.4.1(i), we obtain

\[ e^{-\pi \sqrt{5/3}} \frac{\psi(-e^{-3\pi\sqrt{5}/5})}{\psi(-e^{-\pi\sqrt{5}/3})} = \frac{1}{3} \left( \sqrt{2} \frac{G_{5/9}^3}{G_5} - 1 \right). \] (7.4.8)
7.5. **EXPLICIT FORMULAS FOR $G(-e^{-3\pi\sqrt{n}})$ AND $G(e^{-3\pi\sqrt{n}})$**

Now, from Berndt's book [18, pp. 189 and 345], we note that

$$G_5 = \left(\frac{1 + \sqrt{5}}{2}\right)^{1/4} \quad \text{and} \quad G_{5/9} = \left(\frac{\sqrt{5} - \sqrt{3}}{2}\right)^{1/3} \quad (7.4.9)$$

Employing (7.4.9) in (7.4.8), and then simplifying we arrive at (7.4.7).

Since by Theorem 5.3.3 of Chapter 5, we know the explicit formula for $\phi(q^9)/\phi(q)$, for $q = e^{-\pi\sqrt{n}}$, a positive rational, we now derive an identity by which the corresponding values of the quotients $\psi(-q^9)/\psi(-q)$ may be found.

**Theorem 7.4.4**

$$q \frac{\psi(-q^9)}{\psi(-q)} = \frac{1 - \phi(q^9)/\phi(q)}{(3\phi(q^9)/\phi(q)) - 1}. \quad (7.4.10)$$

**Proof:** Replacing $q$ by $-q$ in (7.3.4) and (7.3.6) and then dividing the first resulting identity by the second, we find that

$$\frac{\phi(q^9)}{\phi(q)} = \frac{\psi(-q) + q\psi(-q^9)}{\psi(-q) + 3q\psi(-q^9)}. \quad (7.4.11)$$

It is now easy to see that (7.4.10) and (7.4.11) are equivalent.

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**7.5 Explicit formulas for $G(-e^{-3\pi\sqrt{n}})$ and $G(e^{-3\pi\sqrt{n}})$**

Berndt, Chan and Zhang [24] have found general formulas for $G(-e^{-\pi\sqrt{n}})$ and $G(e^{-\pi\sqrt{n}})$ by employing the formulas connecting $G_n$ and $G_{b_n}$, and $g_n$ and $g_{b_n}$, respectively. Using the formulas for the explicit evaluations of the quotients of theta-functions found in the previous section, we can find the general formulas for $G(-e^{-3\pi\sqrt{n}})$ and $G(e^{-3\pi\sqrt{n}})$

From Entry 1(i) [15, p. 345], we find that

$$G(-q^3) = \frac{-q\psi(-q^9)/\psi(-q)}{1 + q\psi(-q^9)/\psi(-q)}. \quad (7.5.1)$$

There are assumed results under need of Bennett, Chan, Zhang.
Replacing $q$ by $-q$ in (7.5.1), we find that

$$G(q^3) = \frac{q\psi(q^9)/\psi(q)}{1 - q\psi(q^9)/\psi(q)}.$$ (7.5.2)

Taking $q = e^{-\pi\sqrt{n}}$ in (7.5.1) and (7.5.2), we find the following formulas for $G(-e^{-3\pi\sqrt{n}})$ and $G(e^{-3\pi\sqrt{n}})$.

**Theorem 7.5.1**

(i) \( G(-e^{-3\pi\sqrt{n}}) = \frac{-e^{-\pi\sqrt{n}}\psi(-e^{-9\pi\sqrt{n}})/\psi(-e^{-\pi\sqrt{n}})}{1 + e^{-\pi\sqrt{n}}\psi(-e^{-9\pi\sqrt{n}})/\psi(-e^{-\pi\sqrt{n}})} \) (7.5.3)

and

(ii) \( G(e^{-3\pi\sqrt{n}}) = \frac{e^{-\pi\sqrt{n}}\psi(e^{-9\pi\sqrt{n}})/\psi(e^{-\pi\sqrt{n}})}{1 - e^{-\pi\sqrt{n}}\psi(e^{-9\pi\sqrt{n}})/\psi(e^{-\pi\sqrt{n}})} \). (7.5.4)

Combining with Theorem 7.4.1, a number of explicit evaluations follow. We give a couple of examples below.

**Corollary 7.5.2**

$$G(-e^{-3\pi}) = \frac{1 - \sqrt{2}2(\sqrt{3} - 1)}{2 + \sqrt{2}(\sqrt{3} - 1)}.$$ (7.5.5)

**Proof:** Putting $n = 1$ in Theorem 7.5.1 (i), and then using Corollary 7.4.2, we arrive at (7.5.5).

**Corollary 7.5.3**

$$G(-e^{-\pi\sqrt{5}}) = \frac{(\sqrt{5} - \sqrt{3})(\sqrt{5} - 3)}{4}.$$ (7.5.6)

**Proof:** In this case we put $n = 5/9$ in Theorem 7.5.1 (i), and then use Corollary 7.4.3, to obtain

$$G(-e^{-\pi\sqrt{5}}) = \frac{2 - (\sqrt{5} - \sqrt{3})(3 + \sqrt{5})}{4 + (\sqrt{5} - \sqrt{3})(3 + \sqrt{5})}.$$ (7.5.7)
Simplifying (7.5.7), we complete the proof.

**Remark:** For different proofs of Corollary 7.5.3, see [24] and [32].

### 7.6 Three eta-function Identities

In this section, we prove three eta-function identities which we will use in our next section.

**Theorem 7.6.1** If

\[
P = \frac{\psi(q)}{q^{1/4}\psi(q^3)} \quad \text{and} \quad Q = \frac{\psi(q^5)}{q^{5/4}\psi(q^{15})},
\]

then

\[
(PQ)^2 + \frac{9}{(PQ)^2} = \left(\frac{Q}{P}\right)^3 + 5 \left(\frac{Q}{P}\right)^2 + 5 \left(\frac{P}{Q} - \frac{P}{Q} \right) - \left(\frac{P}{Q}\right)^3 \tag{7.6.1}
\]

**Proof.** We note from Entry 24(iii) [15, p. 39] that

\[
\psi(q) = \frac{f^2(-q^2)}{f(-q)}. \tag{7.6.2}
\]

Therefore \(P\) and \(Q\) can be reformulated as

\[
P = \frac{f(-q^3)f^2(-q^2)}{q^{1/4}f(-q)f^2(-q^6)} \quad \text{and} \quad Q = \frac{f(-q^{15})f^2(-q^{10})}{q^{5/4}f(-q^5)f^2(-q^{30})}.
\]

Now we set

\[
L_1 := \frac{f(-q)}{q^{1/12}f(-q^3)}, \quad L_2 := \frac{f(-q^5)}{q^{5/12}f(-q^{15})},
\]

\[
M_1 := \frac{f(-q^2)}{q^{1/6}f(-q^6)} \quad \text{and} \quad M_2 := \frac{f(-q^{10})}{q^{5/6}f(-q^{30})}, \tag{7.6.3}
\]

so that

\[
P = \frac{M_1^2}{L_1} \quad \text{and} \quad Q = \frac{M_2^2}{L_2}. \tag{7.6.4}
\]
Employing (7.6.3) in Entry 51 [17, p. 204], we obtain

\[(L_1 M_1)^2 + \frac{9}{(L_1 M_1)^2} = \left(\frac{L_1}{M_1}\right)^6 + \left(\frac{M_1}{L_1}\right)^6.\]  \hfill (7.6.5)

Replacing \(q\) by \(q^5\) in the same entry, and then using (7.6.3), we find that

\[(L_2 M_2)^2 + \frac{9}{(L_2 M_2)^2} = \left(\frac{L_2}{M_2}\right)^6 + \left(\frac{M_2}{L_2}\right)^6.\] \hfill (7.6.6)

Using (7.6.4) we may rewrite (7.6.5) in the form

\[\frac{M_1^6}{P^2} + \frac{9P^2}{M_1^6} = \left(\frac{M_1}{P}\right)^6 + \left(\frac{P}{M_1}\right)^6.\] \hfill (7.6.7)

Thus we arrive at

\[M_1^{12} = \frac{P^8(P^4 - 9)}{P^4 - 1}.\] \hfill (7.6.8)

Similarly from (7.6.4) and (7.6.6), we deduce that

\[M_2^{12} = \frac{Q^8(Q^4 - 9)}{Q^4 - 1}.\] \hfill (7.6.9)

Employing (7.6.3) in (59.10) [17, p. 215], we find that

\[\left(\frac{L_2}{L_1}\right)^3 + \left(\frac{M_2}{M_1}\right)^3 = \left(\frac{L_2 M_2}{L_1 M_1}\right)^2 - \left(\frac{L_2 M_2}{L_1 M_1}\right).\] \hfill (7.6.10)

Invoking (7.6.4) in (7.6.10), and then simplifying, we deduce that

\[\left(\frac{M_2}{M_1}\right)^3 = \frac{1 + P/Q}{\left(\frac{P}{Q}\right)^2 - \left(\frac{P}{Q}\right)^3}.\] \hfill (7.6.11)

From (7.6.8), (7.6.9), and (7.6.11), we find that

\[\frac{Q^8(Q^4 - 9)(P^4 - 1)}{P^8(P^4 - 9)(Q^4 - 1)} = \left[\frac{1 + P/Q}{\left(\frac{P}{Q}\right)^2 - \left(\frac{P}{Q}\right)^3}\right]^4.\] \hfill (7.6.12)
Setting $x := P/Q$ and $y := PQ$, and then simplifying, we deduce that

$$\frac{(y^2 - 9x^2)(x^2y^2 - 1)}{(y^2 - x^2)(x^2y^2 - 9)} = \left(\frac{1 + x}{1 - x}\right)^4. \tag{7.6.13}$$

Further simplifications give

$$(1 + x^2)(9x^3 - y^2 - 5xy^2 - 5x^2y^2 + 5x^4y^2 - 5x^5y^2 + x^6y^2 + x^3y^4) = 0. \tag{7.6.14}$$

Since the first factor never vanishes, we deduce that

$$9x^3 - y^2 - 5xy^2 - 5x^2y^2 + 5x^4y^2 - 5x^5y^2 + x^6y^2 + x^3y^4 = 0. \tag{7.6.15}$$

Thus,

$$y^2 + \frac{9}{y^2} = \frac{1}{x^3} + \frac{5}{x^2} + 5x^2 + 5\left(\frac{1}{x} - x\right) - x^3, \tag{7.6.16}$$

which is readily seen to be equivalent to (7.6.1).

**Remark** Since by Entry 24(iii) [15, p. 39],

$$\phi(-q) = \frac{f^2(-q)}{f(-q^2)},$$

proceeding as above, we see that, if

$$P = \frac{\phi(-q)}{\phi(-q^3)} \quad \text{and} \quad Q = \frac{\phi(-q^5)}{\phi(-q^{15})},$$

then (7.6.1) holds. Replacing $q$ by $-q$, we see that the same identity holds if

$$P = \frac{\phi(q)}{\phi(q^3)} \quad \text{and} \quad Q = \frac{\phi(q^5)}{\phi(q^{15})}.$$
Theorem 7.6.2 If

\[ P = \frac{f(-q)f(-q^7)}{q^{2/3}f(-q^3)f(-q^{21})} \quad \text{and} \quad Q = \frac{f(-q^2)f(-q^{14})}{q^{4/3}f(-q^5)f(-q^{42})}, \]

then

\[ \left( \frac{P}{Q} \right)^3 + \left( \frac{Q}{P} \right)^3 = 10 + PQ + \frac{9}{PQ} - 2 \left( 25 + 4PQ + \frac{36}{PQ} \right)^{1/2}. \quad (7.6.17) \]

Proof: Let

\[ R = \frac{f(q)f(q^7)}{q^{2/3}f(q^3)f(q^{21})}. \]

By Entries 12(i) and (iii) in Chapter 17 of [15, p. 124] we find that

\[ R = \sqrt{mm'} \left( \frac{\alpha\gamma(1-\alpha)(1-\gamma)}{\beta\delta(1-\beta)(1-\delta)} \right)^{1/24} \quad (7.6.18) \]

and

\[ Q = \sqrt{mm'} \left( \frac{\alpha\gamma(1-\alpha)(1-\gamma)}{\beta\delta(1-\beta)(1-\delta)} \right)^{1/12}, \quad (7.6.19) \]

where \( \beta, \gamma, \) and \( \delta \) have degrees 3, 7, 21, respectively, over \( \alpha \) and \( m \) and \( m' \) are the multipliers connecting \( \alpha, \beta \) and \( \gamma, \delta, \) respectively.

From (7.6.18) and (7.6.19), we readily see that

\[ \frac{Q}{R} = \left( \frac{\alpha\gamma(1-\alpha)(1-\gamma)}{\beta\delta(1-\beta)(1-\delta)} \right)^{1/24} \quad (7.6.20) \]

and

\[ \frac{R^2}{Q} = \sqrt{mm'}. \quad (7.6.21) \]

Now by Entries 13(v) and 13(vi) in Chapter 20 of [15, p. 384], we note the "mixed" modular
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equations

\[
\left(\frac{\beta \delta}{\alpha \gamma}\right)^{1/4} + \left(\frac{(1 - \beta)(1 - \delta)}{(1 - \alpha)(1 - \gamma)}\right)^{1/4} + \left(\frac{\beta \delta(1 - \beta)(1 - \delta)}{\alpha \gamma(1 - \alpha)(1 - \gamma)}\right)^{1/4} - 2 \left(\frac{\beta \delta(1 - \beta)(1 - \delta)}{\alpha \gamma(1 - \alpha)(1 - \gamma)}\right)^{1/8} \times \left\{ 1 + \left(\frac{\beta \delta}{\alpha \gamma}\right)^{1/8} + \left(\frac{(1 - \beta)(1 - \delta)}{(1 - \alpha)(1 - \gamma)}\right)^{1/8} \right\} = m m', \quad (7.6.22)
\]

and

\[
\left(\frac{\alpha \gamma}{\beta \delta}\right)^{1/4} + \left(\frac{(1 - \alpha)(1 - \gamma)}{(1 - \beta)(1 - \delta)}\right)^{1/4} + \left(\frac{\alpha \gamma(1 - \alpha)(1 - \gamma)}{\beta \delta(1 - \beta)(1 - \delta)}\right)^{1/4} - 2 \left(\frac{\alpha \gamma(1 - \alpha)(1 - \gamma)}{\beta \delta(1 - \beta)(1 - \delta)}\right)^{1/8} \times \left\{ 1 + \left(\frac{\alpha \gamma}{\beta \delta}\right)^{1/8} + \left(\frac{(1 - \alpha)(1 - \gamma)}{(1 - \beta)(1 - \delta)}\right)^{1/8} \right\} = \frac{9}{m m'}, \quad (7.6.23)
\]

respectively.

For simplicity, we set

\[
x := \left(\frac{\beta \delta}{\alpha \gamma}\right)^{1/8} + \left(\frac{(1 - \beta)(1 - \delta)}{(1 - \alpha)(1 - \gamma)}\right)^{1/8} \quad \text{and} \quad y := \left(\frac{\beta \delta(1 - \beta)(1 - \delta)}{\alpha \gamma(1 - \alpha)(1 - \gamma)}\right)^{1/8},
\]

so that

\[
\frac{R^3}{Q^3} = y. \quad (7.6.24)
\]

Then from (7.6.22), we find that

\[
x = y \pm \left(4y + \frac{m m'}{y}\right)^{1/2}. \quad (7.6.25)
\]

Also, from (7.6.23), we find the reciprocal equation of (7.6.25) as

\[
\frac{x}{y} = \frac{1}{y} \pm \left(\frac{4}{y} + \frac{9}{m m'}\right)^{1/2}. \quad (7.6.26)
\]

Combining (7.6.25) and (7.6.26), we obtain

\[
y \pm \left(4y + \frac{m m'}{y}\right)^{1/2} = 1 \pm y \left(\frac{4}{y} + \frac{9}{m m'}\right)^{1/2}. \quad (7.6.27)
\]
Employing (7.6.20), (7.6.21), and (7.6.24) in (7.6.27), we find that

\[
\frac{R^3}{Q^3} \pm \left( \frac{4R^3}{Q^3} + \frac{R^4}{Q^2} \right)^{1/2} = 1 \pm \frac{R^3}{Q^3} \left( \frac{4Q^3}{R^3} + \frac{9Q^2}{R^4} \right)^{1/2}. \tag{7.6.28}
\]

We rewrite (7.6.28) as

\[
R^3 - Q^3 = \pm R^3 \left( \frac{4Q^3}{R^3} + \frac{9Q^2}{R^4} \right)^{1/2} \mp Q^3 \left( \frac{4R^3}{Q^3} + \frac{R^4}{Q^2} \right)^{1/2}. \tag{7.6.29}
\]

Squaring both sides of (7.6.29), and then simplifying, we arrive at

\[
R^6 + Q^6 = 10R^3Q^3 + 9R^2Q^2 + R^4Q^4 - 2R^3Q^3 \left( 25 + \frac{4RQ}{RQ} + \frac{36}{RQ} \right)^{1/2}. \tag{7.6.30}
\]

Dividing both sides of (7.6.30) by \(R^3Q^3\), we find that

\[
\left( \frac{R}{Q} \right)^3 + \left( \frac{Q}{R} \right)^3 = 10 + \frac{RQ}{RQ} + \frac{9}{RQ} - 2 \left( 25 + \frac{4RQ}{RQ} + \frac{36}{RQ} \right)^{1/2}. \tag{7.6.31}
\]

If we replace \(q\) by \(-q\) then \(RQ\) transforms to \(PQ\) and \((R/Q)^3\) transforms to \((P/Q)^3\). Thus (7.6.31) is transformed to (7.6.17), which completes the proof of the theorem.

**Theorem 7.6.3** If

\[
P = \frac{\psi(q)}{q^{1/4}\psi(q^3)} \quad \text{and} \quad Q = \frac{\psi(q^7)}{q^{7/4}\psi(q^{21})},
\]

then

\[
k_1(PQ)^3 + k_2(PQ) = k_3(PQ)^2 + k_4 \left( \frac{P}{Q} \right)^2 - k_5, \tag{7.6.32}
\]

where

\[
k_1 = \left( \frac{P}{Q} \right)^8 - 1, \quad k_2 = 14P^4 \left( \frac{P}{Q} \right)^4 - 1, \quad k_3 = P^4(7 - P^4),
\]

\[
k_4 = 7P^4(P^4 - 3), \quad \text{and} \quad k_5 = 27 \left( \frac{P}{Q} \right)^4 - 7P^4 \left( 3 + 3 \left( \frac{P}{Q} \right)^4 - P^4 \right). \tag{7.6.33}
\]
7.6. THREE ETA-FUNCTION IDENTITIES

**Proof.** Proceeding as in Theorem 7.6.1, if we set

\[ L_1 := \frac{f(-q)}{q^{1/12} f(-q^{3})}, \quad L_2 := \frac{f(-q^7)}{q^{7/12} f(-q^{21})}, \]

\[ M_1 := \frac{f(-q^2)}{q^{1/8} f(-q^{6})}, \quad \text{and} \quad M_2 := \frac{f(-q^{14})}{q^{7/8} f(-q^{42})}, \]  

(7.6.34)

so that

\[ P = \frac{M_1^2}{L_1} \quad \text{and} \quad Q = \frac{M_2^2}{L_2}, \]  

(7.6.35)

we find that

\[ M_1^{12} = \frac{P^8 (P^4 - 9)}{P^4 - 1}, \]  

(7.6.36)

and

\[ M_2^{12} = \frac{Q^8 (Q^4 - 9)}{Q^4 - 1}. \]  

(7.6.37)

Employing (7.6.34) and (7.6.35) in Theorem 7.6.2, we deduce that

\[ \left( \frac{M_1 M_2}{PQ} \right)^3 + \left( \frac{PQ}{M_1 M_2} \right)^3 = 10 + \frac{(M_1 M_2)^3}{PQ} + \frac{9PQ}{(M_1 M_2)^3} - 2 \left( 25 + \frac{4(M_1 M_2)^3}{PQ} + \frac{36PQ}{(M_1 M_2)^3} \right)^{1/2}. \]  

(7.6.38)

Simplifying (7.6.38), we find that

\[ ax + \frac{b}{x} + 10 = 2 \left( 25 + \frac{4x}{PQ} + \frac{36PQ}{x} \right)^{1/2}, \]  

(7.6.39)

where

\[ x = (M_1 M_2)^3, \quad a = \frac{1}{PQ} - \frac{1}{(PQ)^3}, \quad \text{and} \quad b = 9PQ - (PQ)^3. \]  

(7.6.40)

Squaring both sides of (7.6.39), and then simplifying, we deduce that

\[ a^2 k + b^2 + 2abx^2 = x \left( c + dx^2 \right), \]  

(7.6.41)
where

\[ k = x^2, \quad c = 144PQ - 20b \quad \text{and} \quad d = \frac{16}{PQ} - 20a. \]  

(7.6.42)

Squaring both sides of (7.6.41), and then rearranging the terms, we arrive at

\[ a^4k^2 + b^4 + 6a^2b^2k - 2cdk = x^2\left(c^2 + d^2k - 4a^3bk - 4ab^3\right). \]  

(7.6.43)

Squaring both sides of (7.6.43), and then transferring to one side, we find that

\[ \left(a^4k^2 + b^4 + 6a^2b^2k - 2cdk\right)^2 - k\left(c^2 + d^2k - 4a^3bk - 4ab^3\right)^2 = 0. \]  

(7.6.44)

From (7.6.36), (7.6.37), (7.6.40), and (7.6.42), we note that

\[ k = \frac{(PQ)^8(P^4 - 9)(Q^4 - 9)}{(P^4 - 1)(Q^4 - 1)}. \]  

(7.6.45)

Substituting the expressions for \(a, b, c, d,\) and \(k\) from (7.6.40), (7.6.42), and (7.6.45) in (7.6.44), and then factoring by Mathematica, we deduce that

\[ y^{10}(y^4 - 9)^4A(y, z)B(y, z) = 0, \]  

(7.6.46)

where \(y = PQ,\) \(z = P/Q,\)

\[ A(y, z) = -27z^4 + 21y^2z^2 - 21y^2z^4 + 21y^2z^6 - y^3 - 14y^3z^2 + 14y^3z^6 + y^3z^8 + 7y^4z^2 - 7y^4z^4 + 7y^4z^6 - y^6z^4, \]

and

\[ B(y, z) = 27z^4 - 21y^2z^2 + 21y^2z^4 - 21y^2z^6 - y^3 - 14y^3z^2 + 14y^3z^6 + y^3z^8 - 7y^4z^2 + 7y^4z^4 - 7y^4z^6 + y^6z^4. \]

It can be seen that the first three factors in (7.6.46) are not identically zero. Thus, we deduce that

\[ B(y, z) = 0. \]  

(7.6.47)
7.7. RELATIONS OF $G(q)$ WITH $G(q^5)$ AND $G(q^7)$

It is now easy to see that (7.6.32) and (7.6.46) are equivalent.

**Remark:** Since by Entry 24(iii) [15, p. 39]

$$\phi(-q) = \frac{f^2(-q)}{f(-q^2)},$$

proceeding as above, it can be seen that, if

$$P = \frac{\phi(-q)}{\phi(-q^3)} \quad \text{and} \quad Q = \frac{\phi(-q^7)}{\phi(-q^{21})},$$

then (7.6.32) holds. Replacing $q$ by $-q$, we also see that the same identity holds if

$$P = \frac{\phi(q)}{\phi(q^3)} \quad \text{and} \quad Q = \frac{\phi(q^7)}{\phi(q^{21})}.$$

7.7 Relations of $G(q)$ with $G(q^5)$ and $G(q^7)$

In this section we find relations between $G(q)$ and the two continued fractions $G(q^5)$ and $G(q^7)$.

**Theorem 7.7.1** Let for $|q| < 1$, $v = G(q)$ and $w = G(q^5)$. Then

$$v^6 - vw + 5vw(v^3 + w^3)(1 - 2vw) + w^6 = v^2w^2(16v^3w^3 - 20v^2w^2 + 20vw - 5). \quad (7.7.1)$$

**Proof.** From (7.2.3), we note that

$$P^4 = 1 + \frac{1}{v^3} \quad \text{and} \quad Q^4 = 1 + \frac{1}{w^3}, \quad (7.7.2)$$

where

$$P = \frac{\psi(q)}{q^{1/4}\psi(q^3)} \quad \text{and} \quad Q = \frac{\psi(q^5)}{q^{5/4}\psi(q^{15})}.$$
From the identity in Theorem 7.6.1, we see that

\[(PQ)^2 + \frac{9}{(PQ)^2} - 5 \left( \frac{P}{Q} \right)^2 - 5 \left( \frac{Q}{P} \right)^2 = \frac{P}{Q} \left( \left( \frac{Q}{P} \right)^4 + 5 \left( \frac{Q}{P} \right)^2 - \left( \frac{P}{Q} \right)^2 - 5 \right). \tag{7.7.3} \]

Squaring both sides of (7.7.3), and then simplifying, we deduce that

\[\begin{align*}
(PQ)^4 + \frac{81}{(PQ)^4} + 15 \left( \frac{Q}{P} \right)^4 + 15 \left( \frac{P}{Q} \right)^4 + 120 - 10Q^4 - 10P^4 - \frac{90}{P^4} - \frac{90}{Q^4}
\end{align*}\]

\[= \left( \frac{P}{Q} \right)^2 \left( \left( \frac{Q}{P} \right)^8 + \left( \frac{Q}{P} \right)^4 + 15 \left( \frac{Q}{P} \right)^4 + 15 \right). \tag{7.7.4} \]

Squaring both sides of (7.7.4), and then using (7.7.2), we can deduce that

\[G(v, w)H(v, w) = 0, \tag{7.7.5} \]

where

\[G(v, w) = v^6 - vw + 5v^4w + 5v^2w^2 - 10v^5w^2 - 20v^3w^3 + 5vw^4 + 20v^4w^4 - 10v^2w^6 - 16v^5w^5 + w^6, \]

and

\[H(v, w) = v^{12} + v^7w - 5v^{10}w + v^2w^2 - 10v^5w^2 + 20v^8w^2 + 10v^{11}w^2 + 5v^3w^3 - 35v^6w^3 + 10v^9w^3 + 5v^4w^4 - 5v^7w^4 + 80v^{10}w^4 - 10v^2w^5 + 110v^5w^5 + 10v^8w^5 + 16v^{11}w^5 - 35v^3w^6 + 386v^6w^6 + 280v^9w^6 + vw^7 - 5v^4w^7 + 440v^7w^7 + 320v^{10}w^7 + 20v^2w^8 + 10v^5w^8 + 80v^8w^8 + 10v^3w^9 + 280v^6w^9 + 320v^9w^9 - 5vw^10 + 80v^4w^10 + 320v^7w^10 + 256v^{10}w^10 + 1010v^2w^{11} + 16v^5w^{11} + w^{12}. \]

From the definitions of \(v\) and \(w\), we note that \(v = O(q^{1/3})\) and \(w = O(q^{5/3})\) as \(q\) tends to 0. So the first factor in (7.7.5) vanishes for \(q\) sufficiently small. Hence by the identity theorem,

\[G(v, w) \text{ vanishes for } |q| < 1. \]

Thus,

\[v^6 - vw + 5v^4w + 5v^2w^2 - 10v^5w^2 - 20v^3w^3 + 5vw^4 + 20v^4w^4 - 10v^2w^6 - 16v^5w^5 + w^6 = 0, \tag{7.7.6} \]

which is equivalent to (7.7.1). Thus we complete the proof.
7.7. RELATIONS OF $G(q)$ WITH $G(q^5)$ AND $G(q^7)$

**Theorem 7.7.2** Let for $|q| < 1$, $v = G(q)$ and $w = G(q^7)$. Then

$$v^8 - vw - 56v^3w^3(v^2 + w^2) + 7vw(v^3 + w^3)(1 - 8v^3w^3) + 28v^2w^2(v^4 + w^4) = v^4w^4(21 - 64v^3w^3).$$

(7.7.7)

**Proof.** From (7.2.3), we find that

$$P^4 = 1 + \frac{1}{v^3} \quad \text{and} \quad Q^4 = 1 + \frac{1}{w^3},$$

(7.7.8)

where

$$P = \frac{\psi(q)}{q^{1/4}\psi(q^3)} \quad \text{and} \quad Q = \frac{\psi(q^7)}{q^{7/4}\psi(q^{21})}.$$

Now, squaring both sides of the identity in Theorem 7.6.3, we find that

$$(k_1^2 + k_1^2(PQ)^4 + 2k_3k_5)(PQ)^2 + 2k_4k_5 \left(\frac{P}{Q}\right)^2 = k_6,$$

(7.7.9)

where $k_1 - k_5$ are as given in Theorem 7.6.3, and

$$k_6 = k_3^2(PQ)^4 + k_4^2 \left(\frac{P}{Q}\right)^4 + k_5^2 + 2k_3k_4P^4 - 2k_1k_2(PQ)^4.$$

Squaring both sides of (7.7.9), and then using (7.7.8), we deduce that

$$(1 + v^3)^3 A(v, w) B(v, w) = 0,$$

(7.7.10)

where,

$$A(v, w) = v^8 - vw + 7v^4w + 28v^6w^2 - 56v^5w^3 + 7vw^4 + 21v^4w^4 - 56v^7w^4 - 56v^3w^5 + 28v^2w^6 - 56v^4w^7 - 64v^7w^7 + w^7,$$

and
\[ B(v, w) = v^{16} + v^9 w - 7v^{12} w + v^2 w^2 - 14v^5 w^2 + 49v^8 w^2 - 28v^{14} w^2 + 28v^7 w^3 - 196v^{10} w^3 - 112v^{13} w^3 - 56v^6 w^4 + 385v^9 w^4 + 763v^{12} w^4 + 56v^{15} w^4 - 14v^2 w^5 + 56v^5 w^5 + 406v^8 w^5 + 840v^{11} w^5 - 56v^4 w^6 + 196v^7 w^6 + 2604v^{10} w^6 + 1568v^{13} w^6 + 28v^3 w^7 + 196v^6 w^7 - 1960v^9 w^7 - 3080v^{12} w^7 + 64v^{15} w^7 + 49v^2 w^8 + 406v^5 w^8 - 4920v^8 w^8 - 3248v^{11} w^8 + 3136v^{14} w^8 + vw^9 + 385v^4 w^9 - 1960v^7 w^9 - 1568v^{10} w^9 + 1792v^{13} w^9 - 196v^3 w^{10} + 2604v^6 w^{10} - 1568v^9 w^{10} - 3584v^{12} w^{10} + 840v^5 w^{11} - 3248v^8 w^{10} + 3584v^{10} w^{11} + 7168v^{13} w^{11} - 7v^2 w^{12} + 763v^4 w^{12} - 3080v^7 w^{12} - 3584v^{10} w^{12} - 112v^3 w^{13} + 1568v^6 w^{13} + 1792v^9 w^{13} - 28v^2 w^{14} + 3136v^8 w^{14} + 7168v^{11} w^{14} + 4096v^{14} w^{14} + 56v^4 w^{15} + 64v^7 w^{15} + w^{16}. \]

From the definitions of \( v \) and \( w \), we see that \( v = O(q^{1/3}) \) and \( w = O(q^{7/3}) \) as \( q \) tends to 0. Hence the second factor of (7.7.10) vanishes for \( q \) sufficiently small. By the identity theorem that factor vanishes for \( |q| < 1 \). Thus we arrive at

\[ v^8 - vw + 7v^4 w + 28v^6 w^2 - 56v^5 w^3 + 7vw^4 + 21v^4 w^4 - 56v^7 w^4 - 56v^3 w^5 + 28v^2 w^6 - 56v^4 w^7 - 64v^7 w^7 + w^7 = 0, \]

which is equivalent to (7.7.7).