Chapter-3

HALF-SPACE PROBLEMS IN POTENTIAL THEORY

3.1 Representation of Harmonic Function as a Simple and Double Layer Potentials

A harmonic function $\phi$ may be represented throughout the interior domain $B_i$ by Green's formula. On the other hand this may also be represented by a simple layer potential or by a double layer potential. To bring out the connection between these different representations, following Jaswon and Symm (1977), we introduce an arbitrary regular exterior harmonic function $f$ into the exterior domain $B_e$, such that it satisfies the expression (2.3.20) i.e.,

$$\int g(P,q) f(q) dq - \int g(P,q) f'(q) dq = 0, \quad P \in B_i. \quad (3.1.1)$$

Superposition of this on Green's formula (2.3.15) for interior domain, i.e., on

$$\int g(P,q) \phi(q) dq - \int g(P,q) \phi'(q) dq = 4\pi \phi(P), \quad P \in B_i, \quad (3.1.2)$$

yields the more general continuation formula

$$\int g(P,q) (\phi(q) - f(q)) dq - \int g(P,q) (\phi'(q) + f'(q)) dq = 4\pi \phi(P), \quad P \in B_i. \quad (3.1.3)$$

Now, we consider two distinct possibilities for $f$.

The first is $f = \phi$ over $\partial B$, providing the representation

$$4\pi \phi(P) = - \int g(P,q) [\phi(q) + f'(q)] dq, \quad P \in B_i \quad (3.1.4)$$
which is similar to the simple layer potential generated by the source density

\[ \sigma = -\frac{1}{4\pi}(\phi_i + f_e'). \]  

(3.1.5)

This possibility hinges upon the existence of a unique regular \( f \) in \( B_e \) satisfying \( f = \phi \) over \( \partial B \), as it is in fact ensured by the exterior Dirichlet existence theorem. The second possibility is \( f_e' = -\phi_i' \) over \( \partial B \), providing the representation

\[ 4\pi\phi(P) = \int_{\partial B} g(P,q)\left[\phi(q) - f(q)\right]dq, \quad P \in B_i \]  

(3.1.6)

which may be identified as the double layer potential generated by source density

\[ \mu = \frac{1}{4\pi}(\phi - f). \]  

(3.1.7)

This possibility hinges upon the existence of a unique regular \( f \) in \( B_e \) satisfying \( f_e' = -\phi_i' \) over \( \partial B \), as it is in fact ensured by the exterior Neumann existence theorem.

In the case of a harmonic function \( \phi \) vanishing at infinity at least in \( O(r^{-n}) \), \( n \geq 1 \), the above statements remain valid as these can be derived as a special case of close domain exterior as well as interior problem following article (2.3.5). So, analogous to statements (3.1.4) and (3.1.6), considering \( f = \phi \) and \( f_e' = -\phi_i' \) respectively, the expression for exterior half-space problem will be

\[ 4\pi\phi(P) = -\int_{\partial B} g(P,q)\left[\phi_e'(q) + f_e'(q)\right]dq, \quad P \in B_e \]  

(3.1.8)

which is similar to the simple layer potential generated by the source density

\[ \sigma = -\frac{1}{4\pi}(f_e' + \phi_e'). \]  

(3.1.9)
and

\[ 4\pi\phi(P) = \int_{\partial B} g(P,q)\left[f(q) - \phi(q)\right]dq, \quad P \in B_e \] (3.1.10)

which may be identified as the double layer potential generated by source density

\[ \mu = \frac{1}{4\pi}(f - \phi). \] (3.1.11)

The representation (3.1.8) remains valid at \( \partial B \), so yielding the boundary relation

\[ \int_{\partial B} g(p,q)\sigma(q)dq = \phi(p), \quad p \in \partial B \] (3.1.12)

where, \( \sigma \) is given by (3.1.9). This may be regarded as an integral equation in \( \sigma \) in terms of \( \phi \), to which a unique solution exists since \( \phi \) and \( f \) uniquely exist. Similarly the representation (3.1.10) jumps by \(-2\pi\mu(p)\) at \( \partial B \), so yielding the boundary relation

\[ \int_{\partial B} g(p,q)\mu(q)dq + 2\pi\mu(p) = \phi(p), \quad p \in \partial B. \] (3.1.13)

where \( \mu \) is given by (3.1.11). This may be regarded as integral equation for \( \mu \) in terms of \( \phi \), to which a unique solution exists since \( f \) uniquely exists.

### 3.2 Half-space Problems in Simple and Double Layer Potentials

For a regular harmonic function \( \phi \) defined in an exterior domain \( B_e \) bounded at the interior by a closed boundary \( \partial B (= S + S_w, \text{Fig. 2.3.4}) \), we may write \( \phi \) following (3.1.8) and (3.1.10) as

\[ \phi(P) = \int_{S} g(P,q)\sigma dq + \int_{S_w} g(P,q)\sigma dq, \quad P \in B_e \] (3.2.1)
where,
\[ \sigma = -\frac{1}{4\pi} (f' + \phi_e') = O(\phi_e) \]

and
\[ \phi(P) = \int_{S} g(P, q) \sigma dq + \int_{S^0} g(P, q) \epsilon \mu dq, \quad P \in B_e \]

(3.2.2)

where,
\[ \mu = \frac{1}{4\pi} (f - \phi) = O(\phi). \]

Following the same analysis as carried out in (2.3.26) for \( P \in B_e \) the above formula yields
\[ \phi(P) = \int_{S} g(P, q) \sigma dq, \quad P \in B_e \]

(3.2.3)

and
\[ \phi(P) = \int_{S} g(P, q) \epsilon \mu dq, \quad P \in B_e. \]

(3.2.4)

The relation (3.2.3) is valid for the boundary \( S \), so yielding the boundary relation as
\[ \phi(p) = \int_{S} g(p, q) \sigma dq, \quad p \in S. \]

(3.2.5)

For \( \phi \) given on \( S \) the expression (3.2.5) may be regarded as integral equation in \( \sigma \). This has unique \( \sigma \) as shown in (3.1.6). Similarly the representation (3.2.4) jumps by \(-2\pi \mu(p)\) at \( S \), so yielding the boundary relation
\[ \phi(p) = \int_{S} g(p, q) \epsilon \mu dq + 2\pi \mu(p), \quad p \in S. \]

(3.2.6)

For given \( \phi \), this may also be regarded as integral equation in \( \mu \) to which unique solution exists as shown in (3.1.7) since \( f \) exists uniquely.