Chapter 1

Introduction

Let $K$ be a separable complex Hilbert space with an orthonormal basis \{\(e_0, e_1, \ldots\)\}. The simple unilateral shift is the operator $U$ defined as $Ue_n = e_{n+1}$. If \(\{\alpha_0, \alpha_1, \ldots\}\) is a bounded sequence of scalars, the operator $S$ defined by $Se_n = \alpha_ne_{n+1}$ is called the scalar weighted shift operator with weight sequence \(\{\alpha_n\}\). References to these definitions go back to the late 1950's. However, the first systematic study of shift operators was undertaken by R. L. Kelley in his doctoral thesis in 1966 [44]. A fairly thorough account of the subsequent development was compiled by A. L. Shields in 1974 [70].

A natural generalization of weighted shifts is given by considering the Hilbert space $K \oplus K \oplus \ldots$ denoted by $\ell_+^2(K)$ and the operator $S$ defined on $\ell_+^2(K)$ by $S(x_0, x_1, \ldots) = (0, A_0x_0, A_1x_1, \ldots)$, where \(\{A_n\}_{n=0}^{\infty}\) is a uniformly bounded sequence of operators on $K$. The shift $U_+$ defined in this way with each $A_n = I$, the identity operator on $K$, is of great interest and importance. Initially it was used in the investigations of isometries and later in the study of general operators, specially
in providing counter examples. These operator weighted shifts are also referred to as shifts of higher multiplicity. In our present work, we have further investigated these operators.

1.1 Basic spaces and operators

Let $K$ be a separable complex Hilbert space.

1. $\ell^2_+(K)$ is the Hilbert space of sequences $\langle x_n \rangle_{n=0}^{\infty}$ in $K$ such that $\sum_{n=0}^{\infty} \|x_n\|^2 < \infty$, with the inner product $\langle x, y \rangle = \sum_n \langle x_n, y_n \rangle$, where $x = \langle x_n \rangle, y = \langle y_n \rangle$.

2. $\ell^2(K)$ is the Hilbert space of all the two way sequences $x := \langle x_n \rangle_{n=-\infty}^{\infty}$ in $K$ such that $\sum_{n=-\infty}^{\infty} \|x_n\|^2 < \infty$ with the inner product $\langle x, y \rangle = \sum_n \langle x_n, y_n \rangle$.

3. Let $C$ be the unit circle in the complex plane and $\mu$ be Lebesgue measure on the Borel subsets of $C$, normalized so that $\mu(C) = 1$. Let $L^2(\mu)$ denote the class of all Lebesgue measurable functions that are square integrable with respect to $\mu$. The functions $e_n$ defined on $C$ as $e_n(z) = z^n$ for all $n \in \mathbb{Z}$ form a complete orthonormal set in $L^2(\mu) = L^2$. Let $H^2$ be the closed linear span of the $e_n$ with $n \geq 0$. Thus $H^2$ consists of all $f \in L^2$ all of whose negative Fourier coefficients vanish, i.e., $\int_0^{2\pi} f(\theta)e^{in\theta}d\theta = 0 \ \forall \ n > 0$.

Remark: Some authors define the Hardy spaces so as to make them
honest function spaces (consisting of functions analytic on the unit disc). If \( f \in H^2 \) with Fourier coefficients \( f = \sum_{n=0}^{\infty} \alpha_n e_n \), then \( \sum_{n=0}^{\infty} |\alpha_n|^2 < \infty \). Therefore, the radius of convergence of the power series \( \sum_{n=0}^{\infty} \alpha_n z^n \) is greater than or equal to 1. So the power series \( \sum_{n=0}^{\infty} \alpha_n z^n \) defines an analytic function \( \tilde{f} \) in the open unit disc \( D \). Let \( \widetilde{H}^2 \) denote those analytic functions in \( D \) whose series of Taylor coefficients is square summable. The linear mapping \( \varphi : H^2 \rightarrow \widetilde{H}^2 \) defined as \( \varphi(f) = \tilde{f} \) establishes a one-one correspondence between \( H^2 \) and \( \widetilde{H}^2 \). Thus, the functions in \( H^2 \), are often called the analytic elements in \( L^2 \). That is, \( H^2 \) consists of analytic functions in the unit disc with square summable Taylor series.

4. Consider functions defined on the unit circle \( C \) with values in \( K \). Such a function \( f \) will be called measurable if the complex valued function \( \langle f(.), x \rangle \) on the circle is measured in the usual sense for each \( x \in K \). Consider all such measurable functions \( f \) for which \( \frac{1}{2\pi} \int_{0}^{2\pi} \|f(\theta)\|_K^2 d\theta < \infty \). For two such functions \( f \) and \( g \), inner product is defined as \( \langle f, g \rangle_{L^2(K)} = \frac{1}{2\pi} \int_{0}^{2\pi} \langle f(\theta), g(\theta) \rangle_K d\theta \). This is a Hilbert space denoted by \( L^2(K) \). Two functions \( f \) and \( g \) in \( L^2(K) \) are considered equal if they differ only on a set of measure zero.

5. Each \( f \in L^2(K) \) admits a unique expansion \( f(\theta) = \sum_{-\infty}^{\infty} x_k e^{ik\theta} \), with \( x_k \in K \) and \( \|f\|_{L^2(K)}^2 = \sum \|x_k\|_K^2 \). This is to be understood in the weak sense: for each \( x \in K \), the Fourier expansion of \( \langle f(.), x \rangle \) is
\[ \sum (x_k, x) e^{ik \theta}. \] Then \( H^2(K) := \{ f \in L^2(K) : f(\theta) = \sum_{k=0}^{\infty} x_k e^{ik \theta} \}. \)

The shift operator on these Hilbert spaces are defined as follows:

1. On \( \ell^2_+(K) \) and \( \ell^2_-(K) \):

The unilateral shift \( U_+ \) on \( \ell^2_+(K) \) is defined as

\[ U_+(x_0, x_1, \ldots) = (0, x_0, x_1, \ldots). \]

\( U_+ \) is an isometry and the multiplicity of \( U_+ \) is the cardinal number \( n = \dim K \). Its adjoint, called the backward shift is defined as

\[ U_+^*(x_0, x_1, \ldots) = (x_1, x_2, \ldots). \]

The bilateral shift \( W \) on \( \ell^2(K) \) is defined as

\[ W(\ldots, x_{-1}, [x_0], x_1, \ldots) = (\ldots, x_{-2}, [x_{-1}], x_0, \ldots). \]

\( W \) is unitary with multiplicity equal to \( \dim K \).

For a uniformly bounded sequence of operators \( \{A_n\} \) on \( K \), the operator \( S \) defined on \( \ell^2_+(K) \) as \( S(x_0, x_1, \ldots) = (0, A_0 x_0, A_1, x_1, \ldots) \) is called operator weighted shift.

2. On \( H^2(K) \):

Define \( \varphi : H^2(K) \to \ell^2_+(K) \) as \( \varphi f = (x_0, x_1, \ldots) \), where \( f(\theta) = \sum_{k=0}^{\infty} x_k e^{ik \theta} \) is the unique expansion of \( f \). Then \( \varphi \) is unitary. Let \( \{A_n\}_{n=0}^{\infty} \) be a uniformly bounded sequence of operators on \( K \). If \( S \) on \( H^2(K) \) is such that \( \varphi(S f) = (0, A_0 x_0, A_1 x_1, \ldots) \) whenever \( \varphi f = (x_0, x_1, \ldots) \), then \( S \) is called the unilateral operator weighted shift on \( H^2(K) \) with weight sequence \( \{A_n\}_{n=0}^{\infty} \).
Thus, if \( f \in H^2(K) \) and \( f(\theta) = \sum_{n=0}^{\infty} f_n e^{in\theta} \), then for all practical purposes we use the notation \( f = (f_0, f_1, \ldots) \) to denote an element of \( H^2(K) \). For a uniformly bounded sequence of operators \( \{A_n\}_{n=0}^{\infty} \) on \( K \), we define the operator-weighted shift \( S \) on \( H^2(K) \) as \( S(f_0, f_1, \ldots) = (0, A_0f_0, A_1f_1, \ldots) \). Note that the sequence \( \{A_n\}_{n=0}^{\infty} \) being uniformly bounded \( \sum \|A_n f_n\|^2 \leq (\sup_n \|A_n\|^2)(\sum \|f_n\|^2) \)
\( = (\sup_n \|A_n\|^2)\|f\|^2 < \infty \). Also in this case, \( \|S\| = \sup_n \|A_n\| \). We use the notation \( S \sim< A_n \) to denote that \( S \) is the unilateral shift on \( H^2(K) \) with weight sequence \( \{A_n\}_{n=0}^{\infty} \). If each \( A_n \) is an invertible operator on \( K \), we refer to \( S \) as an invertibly weighted operator shift.

### 1.2 Motivation and aim of the research

The basic motivating factor behind our research is the vast potential of applications of operator weighted shifts in some of the very important aspects of operator theory. Here, we briefly describe some of these areas indicating why applications of the operator weighted shifts make them a high potential area of research.

1. In [70], Shields made a detailed study of the unilateral (scalar) weighted shift operator \( T \) on \( H^2(K) \), where \( \dim K = 1 \). In section 3 of the same paper, he defines weighted sequence space \( H^2_\beta \) as follows:
Let \( \{\beta(n)\} \) be a sequence of positive numbers with \( \beta(0) = 1 \). Consider the space of sequences \( f = \{\hat{f}(n)\} \) such that \( \|f\|^2 = \|f\|^2_{\beta} = \sum |\hat{f}(n)|^2(\beta(n))^2 < \infty \). The notation \( f(z) = \sum_{n=0}^{\infty} \hat{f}(n)z^n \) is used, whether or not the series converges for any complex values of \( z \). This space is denoted by \( H^2_{\beta} \). It was established that "multiplication by \( z \) on \( H^2_{\beta} \) is unitarily equivalent to an injective weighted shift". Using this relation the commutant and the spectrum of an injective weighted shift operator are examined.

This naturally motivates us to investigate the situation of an operator weighted shift \( S \) on \( H^2(K) \), where \( \text{dim} \, K > 1 \) and is possibly infinite.

2. The most fundamental unsolved problem in operator theory is "The invariant subspace problem" which goes as follows: Does every operator have a non trivial invariant subspace? In 1949, A. Beurling put forward his now famous "Invariant Subspace Theorem". The backbone of his result was the simple unilateral shift operator on a separable Hilbert space \( H \). In 1966, Lax and Halmos proved Beurling's invariant subspace theorem for the vector valued case (that is, on \( H^2(K) \) where \( \text{dim} \, K = \infty \)). Meanwhile, a parallel study for the reducing subspaces also got evolved. In fact, the invariant and reducing subspaces of the unilateral shift (both scalar and vector valued versions) have been studied extensively ([2], [35], [36], [49], [57], [70]). As a natural extension of
this, a study of the invariant and reducing subspaces of weighted shift operators gained momentum. In section 10 of [70], Shields studied the invariant subspaces of scalar weighted shift operators. In [48], Alan Lambert established conditions for reducibility of invertibly weighted operator shifts. Recently, Kehe Zhu and M. Stessin [72] also addressed a similar problem and classified the reducing subspaces of weighted unilateral shift operators of finite multiplicity.

These developments naturally raise the question “is it possible to relax the condition of invertible weights or is it absolutely necessary”? As a response to it, we consider the case where the weights are quasi-invertible. Another line of approach has been to try and link up shifts on $H^2(K)$ and the shifts on operator weighted sequence spaces, and use that knowledge to draw possible conclusions regarding minimal reducing subspaces of such operators.

3. Another area that arouses our interest and have great potential in the application of operator weighted shifts, concerns the hypercyclic operator weighted shifts. Rolewicz; in `1969, was the first to isolate the concept of hypercyclicity [65]. In 1982, Kitai [46] gave a criterion implying hypercyclicity for general operators that came to be known as the “hypercyclicity criterion”. This criterion has been widely used to show that many different types of operators are hypercyclic. In [38], Herrero posed the following problem:
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Does $T$ hypercyclic imply that $T \oplus T$ is also hypercyclic?

In [66] Salas proved that Herrero's question has an affirmative answer in case $T$ is a bilateral weighted shift on $L^2(K)$ or a unilateral backward weighted shift on $H^2(K)$, where $\dim K = 1$. He also gave a characterization for hypercyclic bilateral weighted shifts in terms of their weight sequences. In [19] Feldman provided a simpler characterization for hypercyclic bilateral weighted shifts that are invertible.

The question also arises as to what happens when $\dim K = \infty$. In this case, it needs to be investigated whether it is possible to give a characterization of $S$ in terms of its weight sequence $\{A_n\}$, possibly in such a way so that the characterizations of Salas and Feldman come out as particular cases when $\dim K = 1$.

Weighted shift operators have occurred sporadically in the literature for a number of years as examples and counter examples. It was possibly Beurling's Invariant Subspace Theorem in 1949 that first brought to light the potential of this entire class of operators. The first systematic study was R. L. Kelley's doctoral thesis [44] which was, unfortunately, never published. For weighted shifts of higher multiplicity (that is, shifts with operator weights) some of the early references are: [18], [20], [37], [48], [62].

Though this area of research is being vigorously pursued, much less is known about operator weighted shifts than is still unknown. In
our present work we attempt to carry forward the ongoing research; to plug some of the holes in the existing literature; to make particular case studies to gain better insight; with the aim of answering at least some of the queries regarding the various aspects of operator weighted shifts.

1.3 Terminology used throughout the thesis

1. *Orthonormal basis*: A subset $M$ of an inner product space $X$ is said to be an orthonormal basis for $X$ if $M$ is an orthonormal set and the span of $M$ is dense in $X$. We note the result: A separable Hilbert space $H$ always has a countable orthonormal basis.

2. *Commutant of an operator*: Commutant of an operator $A$ is the set of all operators $B$ such that $AB = BA$.

3. *Spectrum of an operator*: An operator on a Hilbert space $H$ is said to be invertible if it is bounded from below and has dense range. Spectrum of an operator $A$, denoted by $\sigma(A)$, consists of all scalars $\lambda$ for which the operator $A - \lambda I$ is not invertible. If $\Gamma(A)$ denotes the compression spectrum of $A$ which is the collection of all scalars $\lambda$ such that the operator $A - \lambda I$ does not have a dense range in $H$, and $\Pi(A)$ denotes the approximate point spectrum of $A$ which is the collection
of all scalars $\lambda$ such that the operator $A - \lambda I$ is not bounded from below, then we have $\sigma(A) = \Gamma(A) \cup \Pi(A)$. An important subset of $\Pi(A)$ is the point spectrum of $A$, denoted by $\Pi_0(A)$; $\Pi_0(A)$ consists of all those scalars $\lambda$ for which there exists a unit vector $f \in H$ such that $Af = \lambda f$. If $A^*$ denotes the Hilbert adjoint of the operator $A$, then $\Pi_0(A^*) = \Gamma(A)^*$ and $\Pi(A^*) \cup \Pi(A)^* = \sigma(A^*)$, cf. problem 58 [28].

4. **Similar operators**: The operators $A$ and $B$ are similar if there exists an invertible operator $X$ such that $A = XBX^{-1}$.

5. **Quasi similar operators**: The operators $A$ and $B$ are quasi similar if there exist operators $S$ and $T$ which are one to one and have dense range such that $AS = SB$ and $TA = BT$.

6. **Unitarily equivalent operators**: The operators $A$ and $B$ are unitarily equivalent if there exists a unitary operator $U$ such that $A = UBU^{-1}$.

7. **Quasi invertible operator**: An operator $A$ is said to be quasi invertible if it has zero kernel and dense range.

8. **Hyponormal operator**: An operator $A$ on a Hilbert space $H$ is said to be hyponormal if $A^*A \geq AA^*$. Equivalently $A$ is hyponormal if $\|Ax\| \geq \|A^*x\|$ $\forall x \in H$. 
9. \textit{l-hyponormal} : Let $T$ be a Hilbert space operator, and let $l$ be a positive integer. We call $T$ \textit{l-hyponormal} if $(T, T^2, \ldots, T^l)$ is (jointly) hyponormal. That is, if
\[
\begin{bmatrix}
\vdots & \vdots & \ddots & \vdots \\
\end{bmatrix} \geq 0.
\]

10. \textit{Diagonal operator} : An operator is said to be diagonal if it can be represented as a diagonal matrix with respect to some ordered basis.

11. \textit{Invariant subspace} : A subspace $M$ is invariant under an operator $A$ if $Ax \in M$ for every $x \in M$.

12. \textit{Reducing subspace} : A subspace $M$ reduces $A$ if $M$ and its orthogonal complement $M^\perp$ are both invariant subspaces of $A$. Equivalently, $M$ reduces $A$ if $M$ is an invariant subspace of both $A$ and $A^*$.

13. \textit{Lattice of an operator} : For an operator $A$ on $H$, lattice of $A$ is the collection of all subspaces of $H$ invariant under $A$, and it is denoted by $\text{Lat}A$.

14. \textit{Cyclic operator} : A bounded linear operator $T$ on a Hilbert space
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\( H \) is cyclic if there exists an \( x \in H \) such that its orbit under the operator \( \text{orb}(T, x) = \{x, Tx, T^2x, \ldots\} \) generates a dense linear manifold.

15. Hypercyclic operator: A bounded linear operator \( T \) on a Hilbert space \( H \) is hypercyclic if there exists an \( x \in H \) such that its orbit under the operator \( \text{orb}(T, x) = \{x, Tx, T^2x, \ldots\} \) is dense in \( H \).

16. Equivalent norms: Two norms \( \|\cdot\| \) and \( \|\cdot\|_0 \) on a vector space \( X \) are said to be equivalent if there exist positive numbers \( a \) and \( b \) such that for all \( x \in X \) we have \( a\|x\|_0 \leq \|x\| \leq b\|x\|_0 \).

17. Power bounded operator: An operator \( A \) is said to be power bounded if there exists a positive real number \( c \) such that \( \|A^n\| \leq c \) for \( n = 0, 1, 2, \ldots \)

18. Algebra: An algebra \( \mathcal{U} \) over a field \( \mathbb{F} \) is a ring which is also a linear space over \( \mathbb{F} \) under the same addition and scalar multiplication and satisfies \((\lambda a)b = \lambda(ab) = a(\lambda b) \ \forall \lambda \in \mathbb{F}; \ a, b \in \mathcal{U} \).

19. Banach Algebra: An algebra \( \mathcal{U} \) is said to be a normed algebra if there is a norm defined on it which satisfies \( \|xy\| \leq \|x\| \|y\| \ \forall x, y \in \mathcal{U} \). A normed algebra which is complete in its norm is called a Banach algebra.
20. * algebra: If $\mathcal{U}$ is a Banach algebra, then an involution on $\mathcal{U}$ is a mapping $T \rightarrow T^*$ which satisfies:

(i) $T^{**} = T$ for $T$ in $\mathcal{U}$,

(ii) $(\alpha S + \beta T)^* = \alpha S^* + \beta T^*$ for $S, T$ in $\mathcal{U}$ and $\alpha, \beta$ in $\mathbb{C}$,

(iii) $(ST)^* = T^* S^*$ for $S, T$ in $\mathcal{U}$.

An algebra with a fixed involution is called a *-algebra.

21. $c^*$ algebra: A Banach algebra $\mathcal{U}$ in which the norm and the involution are related by the condition, $\|T^* T\| = \|T\|^2$ for $T$ in $\mathcal{U}$, is called a $c^*$-algebra.

22. Weak Operator Topology: Let $K$ be a Hilbert space and $\mathcal{B}(K)$ be the algebra of bounded linear operators on $K$. The weak operator topology is the weak topology defined by the collection of functions $T \rightarrow (Tf, g)$ from $\mathcal{B}(K)$ to $\mathbb{C}$ for $f, g$ in $K$.

23. $w^*$ algebra: If $K$ is a Hilbert space, then a subset $\mathcal{U}$ of $\mathcal{B}(K)$ is said to be a $w^*$-algebra on $K$ if $\mathcal{U}$ is a $c^*$-algebra which is closed in the weak operator topology.

24. Von Neumann Algebra: A Von Neumann algebra is a weakly closed subalgebra $\mathcal{U}$ of the algebra of operators on a Hilbert space $K$ which
contains the identity operator and is self adjoint.

1.4 Chapterwise outline of the thesis

Here we provide a brief chapter wise description of the work done in the thesis:

Chapter 1: This is an introductory chapter where we give a brief outline of our work, along with its background and motivation.

Chapter 2: We build up the operator weighted sequence space $H^2_B(K)$ which is used again in chapter 5. We first study similarity of operator weighted shifts and a necessary and sufficient condition, for similarity of two invertibly weighted operator shifts, is established. This leads to prove the following:

1. If an invertibly weighted operator shift is power bounded, then it is similar to a contraction.

2. The unilateral shift on the operator weighted sequence space $H^2_B(K)$ is unitarily equivalent to an invertibly weighted operator shift on $H^2(K)$.

In the second part of this chapter, we define the set of multiplications on $H^2_B(K)$. For this we first consider the space $H^\infty_B(K) := \{ f \in$
Chapter 3: The work in this chapter is intended to offer an understanding of the spectrum of operator weighted shifts. We assume the weights to be diagonal operators. We consider the restriction of the operator $T$ to closed linear subspaces of $H^2(K)$, and determine the spectrum and its parts. This process is repeated to give an approximation of the spectrum of the original operator $T$. A more specific analysis is conducted under the assumption that $T$ is hyponormal.

We show the following:

If $T$ is an hyponormal operator weighted shift on $H^2(K)$ with weight sequence $\{A_n\}_{n=1}^\infty$ of diagonal operators on $K$, then $\sigma(T) = \{\lambda : |\lambda| \leq \|T\|\}$,

\[ \Pi(T) = \bigcup_{i=1}^\infty \{\lambda : |\lambda| = \gamma_i\} , \{\lambda : |\lambda| < \|T\|\} \subseteq \Gamma(T) \subseteq \{\lambda : |\lambda| \leq \|T\|\} , \]

for both the following cases:

(i) $|\alpha_j^{(i)}| \leq |\alpha_j^{(i+1)}|$, and (ii) $|\alpha_j^{(i)}| \geq |\alpha_j^{(i+1)}|$ \forall i and \forall j.

Here $\{\alpha_n^{(m)}\}_{m=1}^\infty$ is the diagonal of the operator $A_n$ and $\gamma_i := \lim_{n \to \infty} |\Pi_{i=1}^n \alpha_j^{(i)}|$
"The Invariant Subspace Problem". In this chapter we concentrate on operator weighted shifts with quasi invertible weights, and find their unitary equivalence and reducibility conditions. Our main results are:

1. If $S \prec A_n$ is an operator weighted shift on $H^2(K)$ with quasi invertible weights $A_n$, then $S$ is unitarily equivalent to an operator weighted shift $T \prec B_n$ on $H^2(K)$, where each $B_n$ is positive quasi invertible.

2. Let $S \prec A_n$ be an operator weighted shift on $H^2(K)$ with quasi invertible weights $A_n$. Let $S_0 = I$ and $S_n = A_n A_{n-1} \ldots A_1 \forall n \geq 1$. For a closed subspace $M_0$ of $K$, let $M := \sum_{n=0}^{\infty} \oplus S_n M_0$. Then the following are equivalent:
   i) $M$ reduces $S$.
   ii) $S_n M_0$ is invariant for $A^*_n A_{n+1} \forall n$.
   iii) $(S_n M_0)^\perp = S_n (M_0^\perp) \forall n$.
   iv) $S^*_n S_n M_0 = M_0 \forall n$.

3. Let $S \prec A_n$ be an operator weighted shift on $H^2(K)$ with positive quasi invertible weights $A_n$. The lattice of reducing subspaces of $S$ is isomorphic to the lattice of invariant subspaces of the $w^*$ algebra generated by $\{I, A_1, A_2, \ldots\}$.
Chapter 5: In this chapter we give a complete description of the (minimal) reducing subspaces of the unilateral shift operator on $H^2_B(K)$. For an orthonormal basis $\{e_n\}_{n=0}^\infty$ of $K$, we denote by $x^iy^j$ the sequence in $K$ that has $e_i$ as the $(j+1)th$ entry and zero as all other entries. Let $X_n = \text{Span}\{x^ny^k : k \in \mathbb{N}\}$. A weight sequence $\{B_n\}$ is of type I if for each pair of distinct non negative integers $m$ and $n$ ($m \neq n$) there exists some positive integer $k$ such that $\frac{\beta_n^{(k)}}{\beta_m^{(k)}} \neq \frac{\beta_n^{(k)}}{\beta_m^{(k)}}$. A weight sequence $\{B_n\}$ is of type II if it is not of type I. Thus $\{B_n\}$ is of type II if there exists distinct non negative integers $m$ and $n$ ($m \neq n$) such that $\frac{\beta_n^{(k)}}{\beta_m^{(k)}} = \frac{\beta_n^{(k)}}{\beta_m^{(k)}}$ for every positive integer $k$.

We define a (equivalence) relation $\sim$ on the set $\{0, 1, 2, \ldots \}$ as follows: Two non negative integers $m$ and $n$ are said to be $B$-related (denoted by $m \sim n$) if for every positive integer $k$, we have $\frac{\beta_n^{(k)}}{\beta_m^{(k)}} = \frac{\beta_n^{(k)}}{\beta_m^{(k)}}$.

A weight sequence $\{B_n\}$ of type II is said to be of type III if $\sim$ partitions $\{0, 1, 2, \ldots \}$ into a finite number of equivalence classes. Our main results can now be stated as follows:

1. If $\{B_n\}$ is of type I, then $X_n$’s are the only reducing subspaces of $S$ in $H^2_B(K)$.

2. If $\{B_n\}$ is of type II, then $S$ has minimal reducing subspaces other than the $X_n$’s. In fact, $S$ has infinitely many reducing subspaces each generated by some $F = F_0(x)$ which is transparent.

3. If $\{B_n\}$ is of type III, then every reducing subspace of $S$ in $H^2_B(K)$
must contain a minimal reducing subspace.

Chapter 6: In this chapter, we deal with hypercyclic operator weighted shifts and give a characterization in terms of their weight sequences. We also make a particular study in case these shifts are invertible. As immediate corollaries we get the characterizations of Salas [66] and Feldman [19] as particular cases when dim \( K = 1 \). The main results can be stated as:

1. Let \( T \) be a bilateral (forward) operator weighted shift on \( L^2(K) \) with weight sequence \( \{A_n\}_{n=-\infty}^{\infty} \), where \( \{A_n\} \) is a uniformly bounded sequence of positive invertible diagonal operators on \( K \). Then \( T \) is hypercyclic iff given \( \epsilon > 0 \) and \( q \in \mathbb{N} \), there exists \( n \) arbitrarily large such that for all \( |j| \leq q \), \( \| \prod_{s=0}^{n-1} A_{s+j} \| < \epsilon \) and \( \| \prod_{s=1}^{n} A_{-s}^{-1} \| < \epsilon \).

2. Let \( T \) be an invertible bilateral operator weighted shift on \( L^2(K) \) with weight sequence \( \{A_n\}_{n=-\infty}^{\infty} \) of positive invertible diagonal operators on \( K \). Then \( T \) is hypercyclic if and only if there exists a sequence of integers \( n_k \to \infty \) such that \( \lim_{k \to \infty} \| \prod_{j=1}^{n_k} A_j \| = 0 \) and \( \lim_{k \to \infty} \| \prod_{j=1}^{n_k} A_j^{-1} \| = 0 \).

3. Let \( T \) be the bilateral operator weighted shift on \( L^2(K) \) with weight sequence \( \{A_n\}_{n=-\infty}^{\infty} \), where \( \{A_n\}_{n=-\infty}^{\infty} \) is a uniformly bounded sequence of positive invertible diagonal operators on \( K \), and there exists \( m > 0 \) such that \( \|A_n^{-1}\| \leq m \forall n < 0 \). Then \( T \) is hypercyclic if and only if there
exists a sequence of integers $n_k \to \infty$ such that $\lim_{k \to \infty} \| \prod_{j=1}^{n_k} A_j \| = 0$ and $\lim_{k \to \infty} \| \prod_{j=1}^{n_k} A_j^{-1} \| = 0$.

Chapter 7: In this chapter we address the "Subnormal Completion problem" taking into account the operator weighted shifts. Here, we discuss recursively generated subnormal shifts. Recursively generated weighted shifts have played a profound role in the study of the subnormal completion problem. It seeks necessary and sufficient conditions for the existence of a subnormal weighted shift $W$ whose first $(m+1)$ weights are $\alpha_0, \alpha_1, \ldots, \alpha_m$, where $\{\alpha_0, \alpha_1, \ldots, \alpha_m\}$ is a given family of positive numbers. In the papers [9], [11], [12], [13], subnormal weighted shifts whose moment sequences conform to multi step linear recursion relations has been studied and their primary findings are the following:

1. A necessary and sufficient condition for the existence of a subnormal weighted shift whose first $(m+1)$ weights are $\alpha_0, \alpha_1, \ldots, \alpha_m$.
2. It has been shown that every subnormal shift is the norm limit of recursively generated ones.

However, in all these cases the operator being considered is the scalar weighted shift of multiplicity 1 on the Hardy space $H^2$. Here we initiate a parallel study for operator weighted shifts of higher (finite) multiplicity defined on $H^2(K)$.

The questions that we address here are the following:
1. If $A_0, \ldots, A_m$ are $(m+1)$ positive diagonal operators on $K$, what are the necessary and sufficient conditions for the existence of a subnormal operator weighted shift $W$ whose first $(m+1)$ weights are $A_0, \ldots, A_m$?

2. In case $A_0, \ldots, A_m$ have a subnormal completion, we need to give an explicit method of construction for it.

3. Is every subnormal operator weighted shift recursively generated? If not, then can it be approximated by recursively generated ones?