Chapter 3

New Proofs of Ramanujan's Modular Equations of degree 9

3.1 Introduction

In this chapter, we find proofs of Ramanujan's modular equations of degree 9 by using theta function identities. Ramanujan recorded 14 modular equations of degrees 1, 3, 9 in Chapter 20 of his second notebook [28]. He also recorded two more equations on pages 286 and 296 of his first notebook [28], but the second equation is incorrect as shown by Berndt [13, p. 370, Entry 28]. All of Ramanujan's modular equations of degrees 1, 3, 9 have been proved by Berndt (See [11, pp. 352-358, Entry 3] and [13, p. 370, Entry 27]). As we have already mentioned in Chapter 1, modular equations can be expressed as identities involving the theta-functions \( \phi, \psi \) and \( f \). Therefore, often one first tries to derive a theta-function identity and then transcribes it into an equivalent modular equation. But, proofs of some modular equations of composite degree 9 given by Berndt are quite unlike this method. He sometimes reversed the process. In this chapter 3, we find new proofs of these modular equations by using theta-function identities. In the meantime, we also find new proofs of some of the theta-function identities. Earlier these identities were proved by Berndt by using modular equations and a method of parameterizations.

In Section 3.2, we state some preliminary results.

In Section 3.3, we state the theta-function identities and present new proofs of some of the identities.
In the final section, we prove the modular equations by using results from the previous two sections.

3.2 Preliminary Results

In this section, we state some results which will be used to derive our theta-function identities.

Lemma 3.2.1. [11, p. 51, Example (v)] We have

\[ f(q,q^5) = \psi(-q^3) \chi(q). \]  

(3.2.1)

Lemma 3.2.2. [11, p. 350, (2.3)] We have

\[ f(q,q^2) = \frac{\phi(-q^3)}{\chi(-q)}. \]  

(3.2.2)

Lemma 3.2.3. [11, p. 49, Entry 31 {Corollary (i) and (ii)}] We have

\[ \phi(q) = \phi(q^9) + 2qf(q^3,q^{15}), \]  

(3.2.3)

\[ \psi(q) = f(q^3,q^6) + q\psi(q^9). \]  

(3.2.4)

3.3 Theta-function identities

In this section, we state and prove some theta-function identities recorded by Ramanujan. We think that the proofs presented here are more transparent than those found by Berndt [11]-[13]. We also mention that the identities in [12] were likely unknown to the author when [11] was written.

Theorem 3.3.1. [11, p. 345, Entry 1(i)]

\[ 1 + \frac{\chi^3(-q^3)}{q^{1/3}\chi(-q)} = \frac{\psi(q^{1/3})}{q^{1/3}\psi(q^3)}. \]  

(3.3.1)

and

\[ 1 + \frac{\chi^9(-q^3)}{q\chi^3(-q)} = \frac{\psi^4(q)}{q\psi^4(q^3)}. \]  

(3.3.2)
Here we present a proof of (3.3.1) slightly different from Berndt [11, p. 345].

**Proof:** By (3.2.4) and (3.2.2), we find that

$$\psi(q) - q\psi(q^9) = \frac{\phi(-q^9)}{\chi(-q^3)}. \quad (3.3.3)$$

Dividing both sides by $q\psi(q^9)$ and then using (2.2.3), we obtain

$$\frac{\psi(q)}{q\psi(q^9)} - 1 = \frac{\chi^3(-q^3)}{q\chi(-q^3)}. \quad (3.3.4)$$

Replacing $q$ by $q^{1/3}$, we easily arrive at (3.3.1).

For a proof of (3.3.2) see [11, p. 346, Entry 1(ii)].

**Theorem 3.3.2.** [11, p. 345, Entry 1(ii)]

$$1 + \frac{\psi(-q^{1/3})}{q^{1/3}\psi(-q^3)} = \left(1 + \frac{\psi^3(-q)}{q\psi^3(-q^3)}\right)^{1/3}. \quad (3.3.5)$$

Berndt [11, p. 347] proved that this theorem follows from (3.3.1) and (3.3.2).

**Theorem 3.3.3.** [11, p. 345, Entry 1(ii)]

$$\frac{\phi(q^{1/3})}{\phi(q^3)} = 1 + \left(\frac{\phi^4(q)}{\phi^4(q^3)} - 1\right)^{1/3}. \quad (3.3.6)$$

Berndt [11, p. 218 and p. 347] offered two proofs for this theorem. Here we offer an alternative proof.

**Proof:** Berndt [12, Entry 8, pp. 144-146] proved the following beautiful theta-function identity due to Ramanujan.

$$\{f(a, b) - f(a^6b^3, a^3b^6)\}^3 = \frac{f(a^3, b^3)}{f(a^6b^3, a^3b^6)}f^3(a^2b, ab^2) - f^3(a^6b^3, a^3b^6). \quad (3.3.7)$$

Putting $a = b = q$ in (3.3.7), we find that

$$\{\phi(q) - \phi(q^3)\}^3 = \frac{\phi(q^3)}{\phi(q^9)}\phi^3(q^3) - \phi^3(q^9). \quad (3.3.8)$$
Simplifying (3.3.8), we obtain

\[
\frac{\phi^3(q)}{\phi^3(q^9)} - 3 \frac{\phi^2(q)}{\phi^2(q^9)} + 3 \frac{\phi(q)}{\phi(q^9)} = \frac{\phi^4(q^3)}{\phi^4(q^9)}.
\]  

(3.3.9)

Replacing \( q \) by \( q^{1/3} \) in (3.3.9), we can easily arrive at (3.3.6).

**Theorem 3.3.4.** [11, p. 349, Entry 2(i)]

\[
\phi(q)\phi(q^9) - \phi^2(q^3) = 2q\phi(-q^2)\psi(q^3)\chi(q^3).
\]  

(3.3.10)

**Proof:** At first, we prove the following lemma.

**Lemma 3.3.5.**

\[
\phi^2(q) - \phi^2(q^3) = 4q\chi(q)\chi(-q^2)\psi(q^3)\psi(q^6).
\]  

(3.3.11)

Transcribing by using (2.2.11), (2.2.16), (2.2.18), (2.2.25), and (2.2.27) it can be seen that the above theta-function identity is equivalent to a modular equation of degree 3 [11, Entry 5(iii), first equation, p. 230]. Here we present a more direct proof of this theta-function identity.

**Proof:** Putting \( a = q^2 \), \( b = q^4 \), \( c = q \), and \( d = q^3 \) in (2.2.9), and then using (3.2.1), (3.2.2), (1.1.9), (1.1.10), and (2.2.4), we find that

\[
\frac{\phi(-q^6)\psi(-q^3)}{\chi(-q)} + f(-q^2)\chi(-q)\psi(q^3) = 2\psi(q^3)f(q^5, q^7).
\]  

(3.3.12)

Using (2.2.6) and (2.2.3), we can rewrite the above identity as

\[
\frac{\psi^2(-q^3)}{\psi(q^6)\chi(-q)} + f(-q) = 2f(q^5, q^7).
\]  

(3.3.13)

Replacing \( q \) by \(-q\) in (3.3.13) and then employing (2.2.7), we obtain

\[
\frac{\phi(q^3)}{\chi(q)} + f(q) = 2f(-q^5, -q^7).
\]  

(3.3.14)

Similarly, putting \( a = q^2 \), \( b = q^4 \), \( c = q \), and \( d = q^5 \) in (2.2.10), and then proceeding as above, we find that

\[
-\frac{\phi(q^3)}{\chi(q)} + f(q) = 2qf(-q, -q^{11}).
\]  

(3.3.15)
Multiplying (3.3.14) and (3.3.15), and then using (2.2.3), we obtain
\[ \phi^2(q) - \phi(q^3) = 4q\chi^2(q)f(-q^5, -q^7)f(-q, -q^{11}). \]  
(3.3.16)

Now, using the Jacobi triple product identity (1.1.6), we find that
\[ f(-q^5, -q^7)f(-q, -q^{11}) = (q; q^{12})_\infty(q^5; q^{12})_\infty(q^7; q^{12})_\infty(q^{11}; q^{12})_\infty(q^{12}; q^{12})_\infty \]
\[ = \frac{(q; q^2)_\infty(q^{12}; q^{12})^3_\infty}{(q^3; q^{12})_\infty(q^8; q^{12})_\infty(q^{11}; q^{12})_\infty}. \]
(3.3.17)

Using (1.1.11), (1.1.9), and then (1.1.10), we obtain
\[ f(-q^5, -q^7)f(-q, -q^{11}) = \frac{\chi(-q)f^3(-q^{12})}{\psi(-q^8)}. \]
(3.3.18)

Now, from (2.2.3), we note that
\[ f(-q^9) = \chi(-q)\psi(q) = \sqrt[3]{\phi(-q)\psi^2(q)}. \]
(3.3.19)

Replacing \( q \) by \( q^6 \) in (3.3.19), we deduce that
\[ f^3(-q^{12}) = \phi(-q^6)\psi^2(q^6). \]
(3.3.20)

Using (2.2.6), we find that
\[ f^3(-q^{12}) = \psi(q^3)\psi(-q^3)\psi(q^6). \]
(3.3.21)

Thus, (3.3.18) can be written as
\[ f(-q^5, -q^7)f(-q, -q^{11}) = \chi(-q)\psi(q^3)\psi(-q^3)\psi(q^6). \]
(3.3.22)

Employing (3.3.22) in (3.3.16), and then using (2.2.4), we arrive at (3.3.11), which completes the proof of the lemma.

**Proof:** From (3.2.3) and (3.2.1), we find that
\[ \phi(q) = \phi(q^9) + 2q\psi(-q^9)\chi(q^3). \]
(3.3.23)

Multiplying both sides of (3.3.23) by \( \phi(q^9) \), we obtain
\[ \phi(q)\phi(q^9) = \phi^2(q^9) + 2q\phi(q^9)\psi(-q^9)\chi(q^3). \]
(3.3.24)
Now, by (2.2.6) and (2.2.7), we deduce that

$$\phi(q)\psi(-q) = \psi(q)\phi(-q^2).$$  \hspace{1cm} (3.3.25)

Replacing $q$ by $q^9$ in (3.3.25) and then using (3.3.24), we find that

$$\phi(q)\phi(q^9) = \phi^2(q^9) + 2q\psi(q^9)\phi(-q^{18})\chi(q^3).$$  \hspace{1cm} (3.3.26)

Using (3.3.23) in (3.3.26), we obtain

$$\phi(q)\phi(q^9) = \phi^2(q^9) + 2q\psi(q^9)\chi(q^3)\{\phi(-q^2) + 2q^2\psi(q^{18})\chi(-q^6)\}.$$  \hspace{1cm} (3.3.27)

Employing (3.3.11) in (3.3.27) we easily arrive at Theorem 3.3.1 to complete the proof. Berndt et al. [16, lemma 9.1, p. 20] have also given a proof of Theorem (3.3.1). They used this modular equation to prove some results on Ramanujan's famous forty identities for the Rogers-Ramanujan functions.

Theorem 3.3.6. [11, p. 349, Entry 2(ii)]

$$\psi(q) - 3qq\psi(q^9) = \frac{\phi(-q)}{\chi(-q^2)}.$$  \hspace{1cm} (3.3.28)

**Proof:** Replacing $q$ by $-q$ in (3.3.23) and then using the resulting identity in (3.3.3), we easily deduce (3.3.28).

Theorem 3.3.7. [13, p. 357, Entry 4]

$$\{3\phi(-q^9) - \phi(-q)\}^3 = 8\psi^3(q)\frac{\phi(-q^3)}{\psi(q^3)}.$$  \hspace{1cm} (3.3.29)

**Proof:** Replacing $q$ by $-q$ in (3.3.23), we find that

$$\phi(-q^9) - \phi(-q) = 2q\psi(q^9)\chi(-q^3).$$  \hspace{1cm} (3.3.30)

Using (2.2.3), this can be written as

$$\phi(-q^9) - \phi(-q) = 2q\psi(q^9)\sqrt[3]{\frac{\phi(-q^3)}{\psi(q^3)}}.$$  \hspace{1cm} (3.3.31)

Now, using (2.2.3) in (3.3.3), we deduce that

$$\psi(q) - q\psi(q^9) = \phi(-q^9)\sqrt[3]{\frac{\psi(q^3)}{\phi(-q^3)}}.$$  \hspace{1cm} (3.3.32)
Using (3.3.32) in (3.3.31), we find that

\[ 3\phi(-q^9) - \phi(-q) = 2\psi(q)\sqrt{\frac{\phi(-q^3)}{\psi(q^3)}}. \]  

(3.3.33)

So, we complete the proof by cubing (3.3.33).

Theorem 3.3.8. [11, p. 349, Entry 2(iii)]

\[ \phi(q)\phi(q^9) + \phi^2(q^3) = 2\psi(q)\phi(-q^{18})\chi(q^3). \]  

(3.3.34)

Proof: Replacing \( q \) by \( q^3 \) in (3.3.6) and then simplifying, we obtain

\[ 1 + 3\frac{\phi^2(q)\phi^2(q^3)}{\phi^4(q^3)} = \frac{\phi(q)\phi(q^9)}{\phi^4(q^3)} (\phi^2(q) + 3\phi^2(q^3)). \]  

(3.3.35)

Now, (3.3.23) can be rewritten as

\[ \phi(q) - \phi(q^9) = 2q\psi(-q^9)\chi(q^3). \]  

(3.3.36)

Again, replacing \( q \) by \(-q\) in (3.3.33) and then using (2.2.3), we deduce that

\[ 3\phi(q^9) - \phi(q) = 2\psi(-q)\chi(q^3). \]  

(3.3.37)

Multiplying (3.3.36) and (3.3.37), we obtain

\[ 3\phi^2(q^9) + \phi^2(q) = 4\phi(q)\phi(q^9) - 4q\psi(-q)\psi(-q^9)\chi^2(q^3). \]  

(3.3.38)

Using (3.3.38) in (3.3.35), we find that

\[ 1 - \frac{\phi^2(q)\phi^2(q^3)}{\phi^4(q^3)} = -4q\frac{\phi(q)\phi(q^9)\psi(-q)\psi(-q^9)\chi^2(q^3)}{\phi^4(q^3)}. \]  

(3.3.39)

With the aid of (3.3.10) the above identity can be written as

\[ \phi(q)\phi(q^9) + \phi^2(q^3) = 2\chi(q^3)\frac{\phi(q)\psi(-q)\phi(q^9)\psi(-q^9)}{\phi(-q^2)\psi(q)}. \]  

(3.3.40)

We complete the proof by employing (3.3.25) in (3.3.40).

Theorem 3.3.9. [11, p. 358, Entry 4(i)]

\[ \frac{\phi(-q^{18})}{\phi(-q^2)} + q\left(\frac{\psi(q^9)}{\psi(q)} - \frac{\psi(-q^9)}{\psi(-q)}\right) = 1. \]  

(3.3.41)
Berndt [11, p. 359] proved this by using the modular equation (3.4.12). Here we give an alternative proof.

Proof: Replacing \( q \) by \(-q\) in (3.3.3), we obtain

\[
\psi(-q) + q\psi(-q^9) = \frac{\phi(q^9)}{\chi(q^9)}. \tag{3.3.42}
\]

From (3.3.3) and (3.3.42), we obtain

\[
q \left\{ \frac{\psi(q^9)}{\psi(q)} - \frac{\psi(-q^9)}{\psi(-q)} \right\} = 2 - \frac{\phi(-q^9)}{\chi(-q^3)\psi(q)} - \frac{\phi(q^9)}{\chi(q^3)\psi(-q)}. \tag{3.3.43}
\]

Now, adding (3.3.10) and (3.3.34), we obtain

\[
\phi(q)\phi(q^9) = \psi(q)\phi(-q^{18})\chi(q^3) + q\phi(-q^2)\psi(q^9)\chi(q^3). \tag{3.3.44}
\]

Using (2.2.5), we obtain

\[
\sqrt{\frac{\phi(q)}{\phi(-q)} \frac{\phi(q^9)}{\phi(-q^9)}} = \psi(q)\phi(-q^{18})\chi(q^3) + q\psi(q^9)\chi(q^3). \tag{3.3.45}
\]

Employing (2.2.1) in (3.3.45), we find that

\[
\phi(q^9) = \frac{\chi(q^3)\phi(-q^{18})\psi(-q)}{\phi(-q^2)} + q\frac{\psi(q^9)\chi(q^3)\psi(-q)}{\psi(q)}. \tag{3.3.46}
\]

Replacing \( q \) by \(-q\) in (3.3.46), we obtain

\[
\phi(-q^9) = \frac{\chi(-q^3)\phi(-q^{18})\psi(q)}{\phi(-q^2)} - q\frac{\psi(-q^9)\chi(-q^3)\psi(-q)}{\psi(-q)}. \tag{3.3.47}
\]

Using (3.3.46) and (3.3.47) in (3.3.43), we deduce (3.3.41) to complete the proof.

Lemma 3.3.10. [11, p. 358, Entry 4(vi)]

\[
\frac{\phi(-q^2)}{\phi(-q^{18})} + \frac{1}{q} \left( \frac{\psi(q)}{\psi(q^9)} - \frac{\psi(-q)}{\psi(-q^9)} \right) = 3. \tag{3.3.48}
\]
This identity was proved by Berndt [11, p. 359] by using the modular equation (3.4.13). Here, we present an alternative proof.

**Proof:** Replacing $q$ by $-q$, in (3.3.28), we obtain

$$
\psi(-q) + 3q\psi(-q^9) = \frac{\phi(q)}{\chi(q^2)}.
$$

(3.3.49)

From (3.3.28) and (3.3.49), we find that

$$
\frac{\psi(q)}{q\psi(q^9)} - \frac{\psi(-q)}{q\psi(-q^9)} = 6 + \frac{\phi(-q)}{q\chi(-q^3)\psi(q^9)} - \frac{\phi(q)}{q\chi(q^3)\psi(q^9)}.
$$

(3.3.50)

Now, from (3.3.44), we deduce that

$$
\frac{\phi(q)\phi(q^9)}{\phi(-q^{18})} = \psi(q)\chi(q^3) + q\frac{\phi(-q^2)\psi(q^9)\chi(q^3)}{\phi(-q^{18})}.
$$

(3.3.51)

Using (2.2.5) and (2.2.1) in (3.3.51), we find that

$$
\frac{\phi(q)}{q\chi(q^3)\psi(q^9)} = \frac{\psi(q)}{q\psi(q^9)} + \frac{\phi(-q^2)}{\phi(-q^{18})}.
$$

(3.3.52)

Replacing $q$ by $-q$, we obtain

$$
\frac{\phi(-q)}{q\chi(-q^3)\psi(q^9)} = \frac{\psi(-q)}{q\psi(-q^9)} - \frac{\phi(-q^2)}{\phi(-q^{18})}.
$$

(3.3.53)

Using (3.3.52) and (3.3.53) in (3.3.50), we obtain (3.3.48). Thus, we complete the proof.

The theta-function identities in the following theorem were recorded by Ramanujan in the unorganized portions of his second notebook [28, p. 310]. Berndt [12, p. 185] proved this theorem by using parameterizations. Here we give alternative proofs by using other simple theta-function identities of Ramanujan.

**Theorem 3.3.11.** [12, p. 185, Entry 39] For $|q| < 1$

(i) 
$$
\frac{\phi^3(q^{1/3})}{\phi(q)} = \frac{\phi^3(q)}{\phi(q^3)} + 6q^{1/3}f^3(q^3) + 12q^{2/3}f^3(-q^6)\frac{f(-q^2)}{f(-q^2)},
$$

(3.3.54)

(ii) 
$$
\frac{\psi^3(q^{1/3})}{\psi(q)} = \frac{\psi^3(q)}{\psi(q^3)} + 3q^{1/3}f^3(-q^3) + 3q^{2/3}f^3(-q^6)\frac{f(-q)}{f(-q^2)}.
$$

(3.3.55)
Proof of (i): Replacing \( q \) by \( q^{1/3} \) in (3.3.9), we find that

\[
\frac{\phi^3(q^{1/3})}{\phi(q)} - 3 \frac{\phi^2(q^{1/3})}{\phi^2(q)} + 3 \frac{\phi(q^{1/3})}{\phi(q)} = \frac{\phi^4(q)}{\phi(q)}. \tag{3.3.56}
\]

Multiplying both sides of (3.3.56) by \( \phi^3(q^3)/\phi(q) \), we obtain

\[
\frac{\phi^3(q^{1/3})}{\phi(q)} = \frac{\phi^3(q)}{\phi(q^3)} + 3 \frac{\phi(q^{1/3})\phi(q)}{\phi(q)} \left( \phi(q^{1/3}) - \phi(q^3) \right). \tag{3.3.57}
\]

Now, replacing \( q \) by \( q^{1/3} \) in (3.3.23), we obtain

\[
\phi(q^{1/3}) = \phi(q^3) + 2q^{1/3} x(q) \psi(-q^3). \tag{3.3.58}
\]

Employing (3.3.58), we deduce from (3.3.57) that

\[
\frac{\phi^3(q^{1/3})}{\phi(q)} = \frac{\phi^3(q)}{\phi(q^3)} + 6q^{1/3} \frac{\phi^2(q^3) x(q) \psi(-q^3)}{\phi(q)} + 12q^{2/3} \frac{x^2(q) \psi^2(-q^3) \phi(q^3)}{\phi(q)}. \tag{3.3.59}
\]

From (2.2.3), we note that

\[
\frac{x(q)}{\phi(q)} = \frac{1}{f(q)} \quad \text{and} \quad \frac{\chi(q)}{f(q)} = \frac{1}{f(-q^2)}. \tag{3.3.60}
\]

Using (3.3.60), (2.2.2) and (2.2.4) in (3.3.59), we arrive at

\[
\frac{\phi^3(q^{1/3})}{\phi(q)} = \frac{\phi^3(q)}{\phi(q^3)} + 6q^{1/3} \frac{f^3(q^3)}{f(q)} + 12q^{2/3} \frac{f^3(-q^3)}{f(-q^2)},
\]

which completes the proof of (3.3.54).

Proof of (ii): From (3.3.5), we obtain

\[
\frac{\psi^3(-q^{1/3})}{q \psi^3(-q^3)} + 3 \frac{\psi^2(-q^{1/3})}{q^{2/3} \psi^2(-q^3)} + 3 \frac{\psi(-q^{1/3})}{q^{1/3} \psi(-q^3)} = \frac{\psi^4(-q)}{q \psi^4(-q^3)}. \tag{3.3.61}
\]

Multiplying both sides of (3.3.61) by \( q \psi^3(-q^3)/\psi(-q) \), we deduce that

\[
\frac{\psi^3(-q^{1/3})}{\psi(-q)} = \frac{\psi^3(-q)}{\psi(-q^3)} - 3q^{1/3} \frac{\psi(-q^{1/3}) \psi(-q^3)}{\psi(-q)} \left( \psi(-q^{1/3}) + q^{1/3} \psi(-q^3) \right). \tag{3.3.62}
\]
Replacing $q$ by $-q$, in (3.3.62) we find that

$$\frac{\psi^3(q^{1/3})}{\psi(q)} = \frac{\psi^3(q)}{\psi(q^3)} + 3q^{1/3} \frac{\psi(q^{1/3})\psi(q^3)}{\psi(q)} (\psi(q^{1/3}) - q^{1/3}\psi(q^3)).$$  \hspace{1cm} (3.3.63)

Now, by (3.2.4) and (3.2.2), we obtain

$$\psi(q) = \frac{\phi(-q^3)}{\chi(-q^3)} + q\psi(q^3).$$  \hspace{1cm} (3.3.64)

Replacing $q$ by $q^{1/3}$, we rewrite (3.3.64) as

$$\psi(q^{1/3}) = \frac{\phi(-q^3)}{\chi(-q)} + q^{1/3}\psi(q^3).$$  \hspace{1cm} (3.3.65)

Employing (3.3.65), we obtain from (3.3.63) that

$$\frac{\psi^3(q^{1/3})}{\psi(q)} = \frac{\psi^3(q)}{\psi(q^3)} + 3q^{1/3} \frac{\psi(q^{1/3})\psi(-q^3)}{\psi(q)\chi(-q)} + 3q^{2/3} \frac{\psi^2(q^3)\phi(-q^3)}{\psi(q)\chi(-q)}. \hspace{1cm} (3.3.66)$$

From (2.2.3), we now note that

$$\psi(q)\chi(-q) = f(-q^2) \quad \text{and} \quad \chi(-q)f(-q^3) = f(-q).$$  \hspace{1cm} (3.3.67)

Using (3.3.67), (2.2.2) and (2.2.4), we conclude that

$$\frac{\psi^3(q^{1/3})}{\psi(q)} = \frac{\psi^3(q)}{\psi(q^3)} + 3q^{1/3} \frac{f^3(-q^3)}{f(-q)} + 3q^{2/3} \frac{f^3(-q^6)}{f(-q^2)},$$

which is (3.3.55)

### 3.4 Modular Equations

In this section, we find, except for two modular equations, new proofs of Ramanujan's modular equations of composite degree 9. Throughout this section, suppose $\beta$ and $\gamma$ are of the third and ninth degrees, respectively, with respect to $\alpha$ and $m = z_1/z_3$ and $m' = z_3/z_9$ are the corresponding multipliers.

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Theorem 3.4.1. [11, p. 352, Entry 3(i)] We have

\[ 1 + 4^{1/3} \left( \frac{\alpha^3(1 - \alpha)^3}{\beta(1 - \beta)} \right)^{1/24} = \frac{3}{\sqrt{mn'}}. \]  
\tag{3.4.1}

Proof: From (3.3.29), we obtain

\[ \{3\phi(-q^3) - \phi(-q)\} = 8\psi^3(q)\psi(-q^3). \]  
\tag{3.4.2}

Replacing \( q \) by \( -q \), we obtain,

\[ 3\phi(q^3) - \phi(q) = 2\psi(-q) \left\{ \frac{\phi(q^3)}{\psi(-q^3)} \right\}^{1/3}. \]  
\tag{3.4.3}

Transcribing (3.4.3) by using (3.2.3) and (3.2.1), we readily obtain (3.4.1).

Theorem 3.4.2. [11, p. 352, Entry 3(ii)]

\[ 1 + 4^{1/3} \left( \frac{\gamma^3(1 - \gamma)^3}{\beta(1 - \beta)} \right)^{1/24} = \sqrt{mn'}. \]  
\tag{3.4.4}

Proof: By using (3.2.3) and (3.2.1), we find that

\[ \phi(q) = \phi(q^9) + 2q\chi(q^3)\psi(q^9). \]  
\tag{3.4.5}

Transcribing this by employing (2.2.11), (2.2.17), and (2.2.25), we easily deduce (3.4.4).

Theorem 3.4.3. [11, p. 352, Entry 3(iii)] We have

\[ 1 - 2^{4/3} \left( \frac{\alpha^3(1 - \alpha)^3(1 - \gamma)^3}{\beta^2(1 - \beta)^2} \right)^{1/24} = \frac{m'}{m}. \]  
\tag{3.4.6}

Proof: Multiplying (3.3.10) and (3.3.34), and then transcribing the resulting identity by using (2.2.11), (2.2.13), (2.2.16), and (2.2.25), we easily arrive at (3.4.6).

Theorem 3.4.4. [11, p. 352, Entry 3(iv)] We have

\[ 1 - 4^{1/3} \left( \frac{\gamma^3(1 - \gamma)^3}{\beta(1 - \beta)} \right)^{1/24} = \sqrt{m'/m} = 4^{1/3} \left( \frac{\alpha^3(1 - \gamma)^3}{\beta(1 - \beta)} \right)^{1/24} - 1. \]  
\tag{3.4.7}
For a proof, see [11, p. 355].

Theorem 3.4.5. [11, p. 352, Entry 3(γ)] We have
\[
(\alpha\gamma)^{1/2} + \{(1 - \alpha)(1 - \gamma)\}^{1/2} \\
+ 2\{4\beta(1 - \beta)\}^{1/3} = 1 + 8\{\beta(1 - \beta)\}^{1/4}\{(\alpha\gamma(1 - \alpha)(1 - \gamma)\}^{1/8}. \tag{3.4.8}
\]

Proof: Using (3.4.6) in (3.4.36), we obtain
\[
(\alpha\gamma)^{1/2} + \{(1 - \alpha)(1 - \gamma)\}^{1/2} \\
+ 2\{4\beta(1 - \beta)\}^{1/3} \left\{1 - 2^{4/3}\left(\frac{\alpha^3\gamma^3(1 - \alpha)^3(1 - \gamma)^3}{\beta^2(1 - \beta)^2}\right)^{1/24}\right\} = 1. \tag{3.4.9}
\]

Simplifying this, we easily arrive at (3.4.8), to complete our proof.

Theorem 3.4.6. [11, p. 352, Entry 3(α)] We have
\[
\{\alpha(1 - \gamma)\}^{1/8} + \{\gamma(1 - \alpha)\}^{1/8} = 2\{\beta(1 - \beta)\}^{1/24}. \tag{3.4.10}
\]

Proof: Adding (3.3.10) and (3.3.34), we find that
\[
\phi(q)\phi(q^5) = \psi(q)\phi(-q^{18})\chi(q^3) + q\phi(-q^2)\psi(q^q)\chi(q^3). \tag{3.4.11}
\]

Transcribing this via (2.2.11), (2.2.13), (2.2.16), and (2.2.25), we deduce (3.4.10) to complete the proof.

Theorem 3.4.7. [11, p. 352, Entry 3(α)] We have
\[
\left(\frac{\gamma}{\alpha}\right)^{1/8} + \left(\frac{1 - \gamma}{1 - \alpha}\right)^{1/8} - \left(\frac{\gamma(1 - \gamma)}{\alpha(1 - \gamma)}\right)^{1/8} = \sqrt{m}. \tag{3.4.12}
\]

Proof: We transcribe (3.3.41) by (2.2.13), (2.2.16) and (2.2.17) to arrive at (3.4.12).

Theorem 3.4.8. [11, p. 352, Entry 3(α)]
\[
\left(\frac{\alpha}{\gamma}\right)^{1/8} + \left(\frac{1 - \alpha}{1 - \gamma}\right)^{1/8} - \left(\frac{\alpha(1 - \alpha)}{\gamma(1 - \gamma)}\right)^{1/8} = \frac{3}{\sqrt{mm'}} \tag{3.4.13}
\]

Proof: In this case, we transcribe (3.3.48) to deduce (3.4.13).
**Theorem 3.4.9.** [11, p. 352, Entry 3(a)] We have

\[
\left( \frac{\beta^2}{\alpha \gamma} \right)^{1/4} + \left( \frac{(1 - \beta)^2}{(1 - \alpha)(1 - \gamma)} \right)^{1/4} - \left( \frac{\beta^2(1 - \beta)^2}{\alpha \gamma(1 - \alpha)(1 - \gamma)} \right)^{1/4} = -3 \frac{m}{m'}.
\]

**(Proof):** Replacing \( q^{1/3} \) by \(-q\) in (3.3.5) and cubing both sides, we obtain

\[
\left( 1 - \frac{\psi(q)}{q\psi(q^9)} \right)^3 = 1 - \frac{\psi^4(q^3)}{q^2\psi^4(q^9)}.
\]

Simplifying this, we find that

\[
\frac{\psi^4(q^3)}{q\psi^2(q)\psi^2(q^9)} - 3q\frac{\psi(q^9)}{\psi(q)} + 3 - \frac{\psi(q)}{q\psi(q^9)} = 0.
\]

Replacing \( q \) by \(-q\) in (3.4.16), we obtain

\[
\frac{\psi^4(-q^3)}{q\psi^2(-q)\psi^2(-q^9)} + 3q\frac{\psi(-q^9)}{\psi(-q)} + 3 + \frac{\psi(-q)}{q\psi(-q^9)} = 0
\]

From (3.4.16) and (3.4.17), we deduce that

\[
\frac{\psi^4(q^3)}{q\psi^2(q)\psi^2(q^9)} - \frac{\psi^4(-q^3)}{q\psi^2(-q)\psi^2(-q^9)} = 3q \left( \frac{\psi(q^9)}{\psi(q)} - \frac{\psi(-q^9)}{\psi(-q)} \right) + \frac{1}{q} \left( \frac{\psi(q)}{\psi(q^9)} - \frac{\psi(-q)}{\psi(-q^9)} \right) - 6.
\]

Employing (3.3.41) and (3.3.48) in (3.4.18), we find that

\[
\frac{\psi^4(q^3)}{q\psi^2(q)\psi^2(q^9)} - \frac{\psi^4(-q^3)}{q\psi^2(-q)\psi^2(-q^9)} = -3 \frac{\phi(-q^{18})}{\phi(-q^2)} - \frac{\phi(-q^2)}{\phi(-q^{18})}.
\]

Now, replacing \( q^{1/3} \) by \( q \) in (3.3.6) and then simplifying, we deduce that

\[
3 \frac{\phi(q^9)}{\phi(q)} + \frac{\phi(q)}{\phi(q^9)} = 3 + \frac{\phi^4(q^3)}{\phi^2(q)\phi^2(q^9)}.
\]

Replacing \( q \) by \(-q^2\), in (3.4.20), we obtain

\[
3 \frac{\phi(-q^{18})}{\phi(-q^2)} + \frac{\phi(-q^2)}{\phi(-q^{18})} = 3 + \frac{\phi^4(-q^6)}{\phi^2(-q^2)\phi^2(-q^{18})}.
\]
Now, using (3.4.21) in (3.4.19), we find that
\[
\frac{\psi^4(q^3)}{q\psi^2(q)\psi^2(q^9)} - \frac{\psi^4(-q^3)}{q\psi^2(-q)\psi^2(-q^9)} + 3 = -\frac{\phi^4(-q^3)}{\phi^2(-q^3)\phi^2(-q^{18})}. \tag{3.4.22}
\]
Transcribing this by using (2.2.13), (2.2.17) and (2.2.17), we easily deduce (3.4.14).

**Theorem 3.4.10.** ([11, p. 352, Entry 3 (ii)] We have
\[
\left(\frac{\alpha\gamma}{\beta^2}\right)^{1/4} + \left(\frac{(1 - \alpha)(1 - \gamma)}{(1 - \beta)^2}\right)^{1/4} - \left(\frac{\alpha\gamma(1 - \alpha)(1 - \gamma)}{\beta^2(1 - \beta)^2}\right)^{1/4} = \frac{n'}{m}. \tag{3.4.23}
\]
**Proof:** Replacing \( q \) by \( q^3 \) in (3.3.5) and then simplifying, we find that
\[
1 - 3q\frac{\psi^2(-q)\psi^2(-q^9)}{\psi^4(-q^3)} = \frac{\psi(-q)\psi(-q^9)}{\psi^4(-q^3)}(\psi^2(-q) + 3q^2\psi^2(-q^9)). \tag{3.4.24}
\]
Again, multiplying (3.3.42) and (3.3.49), we find that
\[
\psi^2(-q) + 3q^2\psi^2(-q^9) = \frac{\phi(q)\phi(q^9)}{\chi^2(q^3)} - 4q\psi(-q)\psi(-q^9). \tag{3.4.25}
\]
From (3.4.24) and (3.4.25), we obtain
\[
1 + q\frac{\psi^2(-q)\psi^2(-q^9)}{\psi^4(-q^3)} = \frac{\psi(-q)\psi(-q^9)\phi(q)\phi(q^9)}{\psi^4(-q^3)\chi^2(q^3)}. \tag{3.4.26}
\]
Multiplying (3.3.39) and (3.4.26), we find that
\[
1 + q\frac{\psi^2(-q)\psi^2(-q^9)}{\psi^4(-q^3)} = \frac{\phi^2(q)\phi^2(q^9)}{\phi^4(q^3)} \left(1 - 3q\frac{\psi^2(-q)\psi^2(-q^9)}{\psi^4(-q^3)}\right). \tag{3.4.27}
\]
We transcribe this by employing (2.2.11) and (2.2.17) to arrive at
\[
\frac{n'}{m} = \frac{(\alpha\gamma(1 - \alpha)(1 - \gamma))^{1/4}}{(\beta(1 - \beta))^{1/2}} = 1 - 3\frac{m}{m'} \frac{(\alpha\gamma(1 - \alpha)(1 - \gamma))^{1/4}}{(\beta(1 - \beta))^{1/2}}. \tag{3.4.28}
\]
Using the expression of \(-3m/m'\) from (3.4.14) in (3.4.28), we readily deduce (3.4.23) to complete the proof.
Theorem 3.4.11. \[11, \text{p. 352, Entry 3(xvi)}\] We have

\[
\frac{2^{1/3} \{\beta(1-\beta)\}^{1/4}}{\{\alpha(1-\gamma)\}^{1/8} - \{\gamma(1-\alpha)\}^{1/8}} = \sqrt{\frac{m}{m'}}. \tag{3.4.29}
\]

**Proof:** Subtracting (3.3.10) from (3.3.34), dividing the result by 2, we obtain

\[
\phi^3(q^3) = \psi(q)\phi(-q^{18})\chi(q^3) - q\phi(-q^2)\psi(q^3)\chi(q^3). \tag{3.4.30}
\]

Transcribing (3.4.30) via (2.2.11), (2.2.13), (2.2.16), and (2.2.25), we easily arrive at (3.4.29).

Theorem 3.4.12. \[11, \text{p. 352, Entry 3(xvi)}\] We have

\[
\begin{align*}
(\alpha^{1/4} - \gamma^{1/4})^4 + & \{(1-\gamma)^{1/4} - (1-\alpha)^{1/4}\}^4 \\
= & \{(\alpha(1-\gamma))^{1/4} - (\gamma(1-\alpha))^{1/4}\}^4.
\end{align*} \tag{3.4.31}
\]

**Proof:** From \[10, \text{p. 338}\], we note the following general result of Ramanujan:

If the modular equation of degree \(n-1\) is

\[
\{\alpha\beta\}^{1/n} + \{(1-\alpha)(1-\beta)\}^{1/n} = 1, \tag{3.4.32}
\]

then

\[
\{(\alpha(1-\beta))^{1/n} - (\beta(1-\alpha))^{1/n}\}^n
= \{\alpha^{1/n} - \beta^{1/n}\}^n + \{(1-\beta)^{1/n} - (1-\alpha)^{1/n}\}^n, \tag{3.4.33}
\]

is a modular equation of degree \((n-1)^2\).

Now, we know from Entry 5 \[11, \text{p. 230}\] that

\[
(\alpha\beta)^{1/4} + \{(1-\alpha)(1-\beta)\}^{1/4} = 1. \tag{3.4.34}
\]

where \(\beta\) has degree 3 over \(\alpha\). Thus,

\[
\begin{align*}
(\alpha^{1/4} - \gamma^{1/4})^4 + & \{(1-\gamma)^{1/4} - (1-\alpha)^{1/4}\}^4 \\
= & \{(\alpha(1-\gamma))^{1/4} - (\gamma(1-\alpha))^{1/4}\}^4,
\end{align*} \tag{3.4.35}
\]

where \(\gamma\) has degree 9 over \(\alpha\). This completes the proof.

Theorem 3.4.13. \[11, \text{p. 352, Entry 3(xvi)}\] We have

\[
1 = (\alpha\gamma)^{1/2} + \{(1-\alpha)(1-\gamma)\}^{1/2} + 2\{4\beta(1-\beta)\}^{1/3}\frac{m'}{m}. \tag{3.4.36}
\]

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Proof: The proof of this modular equation is somewhat different in nature from the other proofs.

Setting \( \mu = 5, \nu = 4, A = 1, B = -1 \) in Schöter's formula (4.3.20), we deduce that

\[
\frac{1}{2} \{ \phi(Q)\phi(-q) - \phi(-Q)\phi(q) \} = \frac{-2qf(-Q^8, -Q^{12})f(-q^8, -q^{12})}{-2q^9f(-Q^4, -Q^{16})f(-q^4, -q^{16})}, \tag{3.4.37}
\]

where \( Q = q^9 \). Now, \( G(q) \) and \( H(q) \) are known as the Rogers-Ramanujan functions defined in (1.1.15) and (1.1.16). Ramanujan [29, pp. 236-237] found forty modular relations for \( G(q) \) and \( H(q) \), which are called Ramanujan's forty identities. The sixth of these forty identities is

\[
G(Q)G(q) + q^2H(Q)H(q) = \frac{f^2(-q^3)}{f(-q)f(-q^9)}, \tag{3.4.38}
\]

where \( f(-q) \) is as defined in (1.1.10). The first proof of (3.4.38) was given by Rogers [32]. Berndt et al. [16] and Baruah et al. [8] also found several new proofs.

Now, using the Jacobi triple product identity (1.1.6) in (1.1.15) and (1.1.16), we easily find that

\[
G(q) = \frac{f(-q^2, -q^3)}{f(-q)} \quad \text{and} \quad H(q) = \frac{f(-q, -q^4)}{f(-q)} \tag{3.4.39}
\]

Employing (3.4.39) in (3.4.37), we find that

\[
\frac{1}{2} \{ \phi(Q)\phi(-q) - \phi(-Q)\phi(q) \} = \frac{-2qf(-Q^4)G(q^4) + q^8H(Q^4)H(q^4)}{f(-q^4)f(-q^{36})}. \tag{3.4.40}
\]

Replacing \( q \) by \( q^4 \) in (3.4.38), and then using the resultant identity in (3.4.40), we deduce that

\[
\{ \phi(Q)\phi(-q) - \phi(-Q)\phi(q) \} = -2qf(-q^{12}). \tag{3.4.41}
\]

Replacing \( q \) by \( q^{1/2} \), we obtain

\[
\{ \phi(Q^{1/2})\phi(-q^{1/2}) - \phi(-Q^{1/2})\phi(q^{1/2}) \} = -2q^{1/2}f^2(-q^6). \tag{3.4.42}
\]

We complete the proof by transcribing (3.4.42) by employing (2.2.14), (2.2.15) and (2.2.21).
Theorem 3.4.14. [13, p. 370, Entry 27]

\[
\left(\frac{\alpha}{\gamma}\right)^{1/8} \left(\frac{1-\alpha}{1-\gamma}\right)^{1/4} \sqrt{\frac{m}{mm'}} + \frac{3}{mm'} = \left(\frac{\alpha}{\gamma}\right)^{1/8} + \left(\frac{1-\alpha}{1-\gamma}\right)^{1/4}. \tag{3.4.43}
\]

Proof: From (3.3.3) and (3.3.28), we deduce that

\[
\psi(q)\phi(-q^9) - 3q\psi(q^9)\phi(-q^9) = \psi(q)\phi(-q) - q\psi(q^9)\phi(-q). \tag{3.4.44}
\]

Employing (2.2.6) in (3.4.44), we obtain

\[
\psi(q)\phi(-q^9) - 3q\psi(q^{9/2})\psi(-q^{9/2}) = \psi(q^{1/2})\psi(-q^{1/2}) - q\psi(q^9)\phi(-q). \tag{3.4.45}
\]

Transcribing this by employing (2.2.12), (2.2.16), (2.2.19), and (2.2.20), we easily arrive at (3.4.43).