Chapter 4

Some New Proofs of Modular Relations for the Göllnitz-Gordon Functions

4.1 Introduction

We recall from Chapter 1, that for $|q| < 1$, the Rogers-Ramanujan functions, are defined by

$$G(q) := \sum_{n=0}^{\infty} \frac{q^{n^2}}{(q; q)_n} = \frac{1}{(q; q^2)_\infty (q^4; q^2)_\infty}, \quad (4.1.1)$$

$$H(q) := \sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(q; q)_n} = \frac{1}{(q^2; q^5)_\infty (q^3; q^5)_\infty}, \quad (4.1.2)$$

when the later equalities are the famous Rogers-Ramanujan identities. Ramanujan recorded forty modular relations for $G(q)$ and $H(q)$ in a manuscript published with the lost notebook [29]. These are now known as Ramanujan’s forty identities. Darling [23] established one of the identities in 1921 in the Proceedings of London Mathematical Society. Rogers [32] established ten of the forty identities including the one proved by Darling. Watson [36] proved eight of the forty identities, two of them from the list that Rogers proved. In 1977, Bressoud ([19], [20]) generalized Rogers’ results to prove fifteen from the list of forty. In 1989, Biagioli [17] used modular forms to prove seven of the remaining nine identities. Recently, Berndt et al. [16] have found proofs...
of 35 of the 40 identities in the spirit of Ramanujan\'s mathematics. For the remaining 5 identities, they also offered heuristic arguments showing that both sides of the identity have the same asymptotic expansions as $q \to 1^\pm$.

Now, we recall the definitions of Göllnitz-Gordon Functions,

$$S(q) := \sum_{n=0}^{\infty} \frac{(-q; q^2)_n q^{n^2}}{(q^2; q^2)_n} = \frac{1}{(q; q^2)_\infty}(q^4; q^8)_\infty(q^7; q^8)_\infty,$$  \hspace{1cm} (4.1.3)

$$T(q) := \sum_{n=0}^{\infty} \frac{(-q; q^2)_n q^{n^2+2n}}{(q^2; q^2)_n} = \frac{1}{(q^2; q^8)_\infty}(q^4; q^8)_\infty(q^8; q^8)_\infty.$$  \hspace{1cm} (4.1.4)

Many of the Göllnitz-Gordon identities are similar in character to the Rogers-Ramanujan\'s identities. For example, the quotient of $G(q)$ and $H(q)$ gives the Rogers-Ramanujan continued fraction, while the quotient of $S(q)$ and $T(q)$ gives the Ramanujan-Göllnitz-Gordon continued fraction [28, Vol. 2, p. 229].

Chan and Huang [21] succeeded in obtaining several relations involving the Ramanujan-Göllnitz-Gordon continued fraction. Motivated by the similarity between the Rogers-Ramanujan and Göllnitz-Gordon functions, S.-S. Huang [26] and Chen and Huang [22] derived 21 modular relations involving $S(q)$ and $T(q)$, one new relations for $G(q)$ and $H(q)$, and 9 relations involving both the pairs $G(q)$, $H(q)$ and $S(q)$ and $T(q)$. They used the methods of Rogers [32], Watson [36], and Bressoud [19]. In this Chapter, we find alternative proofs of the modular relations involving only $S(q)$ and $T(q)$ by employing Schröter\'s formulas and theta functions identities. We also find several new modular relations, and many more can be found by using the same method.

In Section 4.2, we give the list of the modular relations for the Göllnitz-Gordon functions which will be proved in this chapter.

In Section 4.3, we state some preliminary results.

In Sections 4.4-4.12, we present the proofs of the identities listed in section 4.2.

In our last section, we present the new modular relations for the Göllnitz-Gordon functions.
4.2 Modular Relations for the Göllnitz-Gordon Functions

For convenience, we denote \( f(-q^n) \) by \( f_n \).

\[
\begin{align*}
S(q)S(q) + qT(q)T(q) &= \left\{ \frac{f_2 f_5}{f_1 f_4} \right\}^3, \\
S(q)S(q) - qT(q)T(q) &= \frac{f_3 f_4}{f_1 f_2} \left\{ \frac{f_4}{f_5} \right\}^2, \\
S(q^7)T(q) - q^3S(q)T(q^7) &= 1, \\
S(q^3)S(q) + q^2T(q^3)T(q) &= \frac{f_3 f_4}{f_1 f_2}, \\
S(q^3)T(q) - qS(q^3)T(q^3) &= \frac{f_1 f_{12}}{f_3 f_4}, \\
S(q^5)S(q) + q^3T(q^5)T(q) &= \frac{f_2 f_{10}}{f_1 f_{20}}, \\
S(q^5)T(q) - q^2S(q^5)T(q^5) &= \frac{f_2 f_{10}}{f_4 f_5}, \\
S(q^4)T(q^2) - qS(q^2)T(q^4) &= \frac{f_1 f_{32}}{f_2 f_{16}}, \\
S(q^8)T(q) - q^4S(q)T(q^8) &= \frac{f_5 f_6}{f_3 f_{12}}, \\
S(q^9)S(q) + q^5T(q^9)T(q) &= \frac{f_2 f_{12} f_{18}}{f_1 f_4 f_9 f_{36}}, \\
S(q^{15})S(q) + q^8T(q^{15})T(q) &= \frac{f_2 f_{12} f_{18} f_{30}}{f_1 f_5 f_{10} f_{15} f_{60}}, \\
S(q^5)T(q^3) - qS(q^3)T(q^5) &= \frac{f_1 f_3 f_{10}}{f_2 f_{12} f_{20} f_{30}}, \\
S(q^{13})T(q) - q^7S(q)T(q^{15}) &= 1, \\
S(q^{39})S(q) + q^{20}T(q^{39})T(q) &= \frac{f_2 f_{12} f_{13} f_{52} f_{78}}{f_1 f_4 f_{26} f_{39} f_{196}}, \\
S(q^{13})T(q^3) - q^5S(q^3)T(q^{13}) &= \frac{f_2 f_3 f_{12} f_{26} f_{39} f_{196}}{f_1 f_4 f_{10} f_{22} f_{55} f_{220}}, \\
S(q^{55})S(q) + q^{28}T(q^{55})T(q) &= \frac{f_2 f_{11} f_{20} f_{44} f_{110}}{f_1 f_4 f_{10} f_{22} f_{55} f_{220}}, \\
S(q^{11})T(q^5) - q^3S(q^5)T(q^{11}) &= \frac{f_2 f_3 f_{12} f_{26} f_{39} f_{196}}{f_1 f_4 f_{14} f_{18} f_{63} f_{252}}.
\end{align*}
\]
Lemma 4.3.1. [11, p. 40, Entry 25] We have
\[ \phi(q) + \phi(-q) = 2\phi(q^4), \] (4.3.1)
\[ \phi(q) - \phi(-q) = 4q\psi(q^8), \] (4.3.2)
\[ \phi^2(q) - \phi^2(-q) = 8q^2\psi^2(q^4), \] (4.3.3)
\[ \phi^2(q) + \phi^2(-q) = 2\phi^2(q^2). \] (4.3.4)

Lemma 4.3.2. [11, p. 48, Entry 31 with k = 2] We have
\[ f(a, b) = f(a^3b, ab^3) + af(b/a, a^5b^3). \] (4.3.5)

Lemma 4.3.3. [11, p. 46] We have
\[ f(a, b) + f(-a, -b) = 2f(a^3b, ab^3). \] (4.3.6)

Lemma 4.3.4. [11, p. 46] We have
\[ f(a, b) - f(-a, -b) = 2af(b/a, a^5b^4). \] (4.3.7)

Lemma 4.3.5. [11, p. 46] We have
\[ f(a, b)f(-a, -b) = f(-a^2, -b^2)\phi(ab). \] (4.3.8)
Lemma 4.3.6. [11, p. 51, Example (iv), with \(q\) replaced by \(-q\)] We have

\[
\phi(q) + \phi(q^2) = 2\frac{f^2(-q^3, -q^5)}{\psi(-q)},
\]

and

\[
\phi(q) - \phi(q^2) = 2q\frac{f^2(-q, -q^7)}{\psi(-q)}.
\]

Lemma 4.3.7. [26, Lemma 3.1] We have

\[
S(q)T(q) = \frac{f_2f_5^2}{f_1f_4^2}, \quad \phi(q) = \frac{f_5^2}{f_2^3f_4^2}, \quad \text{and} \quad \psi(q) = \frac{f_2^2}{f_1^2}.
\]

Lemma 4.3.8. We have

\[
\phi(-q) = \frac{f_1}{f_2}, \quad \psi(-q) = \frac{f_1f_4}{f_2}, \quad f(q) = \frac{f_3}{f_1f_4}, \quad \text{and} \quad \chi(q) = \frac{f_2^2}{f_1f_4}.
\]

Proof: These identities easily follow from Entries 24-25 [11, pp. 39-40].

Lemma 4.3.9. We have

\[
f(-q^3, -q^5) = \frac{f_1f_4}{f_2} S(q),
\]

and

\[
f(-q, -q^7) = \frac{f_1f_4}{f_2} T(q).
\]

Proof: By [22, Lemma 2.6] and (1.1.9), we note that

\[
S(q) = \frac{\psi(-q^2)f(-q^3, -q^5)}{f_1f_5^2},
\]

and

\[
T(q) = \frac{\psi(-q^2)f(-q, -q^7)}{f_1f_5}.
\]

Using (4.3.12) we complete the proof.

Lemma 4.3.10. [11, Corollary, p. 49] We have

\[
\psi(q) = f(q^6, q^{10}) + qf(q^2, q^{14}).
\]
Replacing $q$ by $-q$ in (4.3.17), we have the following useful result

**Lemma 4.3.11.** We have

$$\psi(-q) = f(q^6, q^{10}) - qf(q^2, q^{14})$$  \hspace{1cm} (4.3.18)

In the following six Schlotter's formulas, we assume that $\mu$ and $\nu$ are integers such that $\mu > \nu \geq 0$

**Lemma 4.3.12.** [11, p. 67, (36.1)] We have

$$\frac{1}{2}\left\{f(Aq^{\mu+\nu}, \frac{q^{\mu+\nu}}{A})f(Bq^{\mu-\nu}, \frac{q^{\mu-\nu}}{B}) + f(-Aq^{\mu+\nu}, -\frac{q^{\mu+\nu}}{A})f(-Bq^{\mu-\nu}, -\frac{q^{\mu-\nu}}{B})\right\}$$

$$= \sum_{m=0}^{\mu-1} \left(\frac{A}{B}\right)^m q^{2\mu m^2} f\left(\frac{A^{\mu-\nu}B^{\nu+\nu}q^{2\mu(4m+1)(\mu^2-\nu^2)}}{A^{\mu-\nu}B^{\nu+\nu}q^{2\mu(4m+1)(\mu^2-\nu^2)}}\right)$$

$$\times f\left(\frac{ABq^{2\mu+4\nu m}}{A^\nu B^{\mu+\nu}}\right).$$  \hspace{1cm} (4.3.19)

**Lemma 4.3.13.** [11, p. 68, (36.2)] We have

$$\frac{1}{2}\left\{f(Aq^{\mu+\nu}, \frac{q^{\mu+\nu}}{A})f(Bq^{\mu-\nu}, \frac{q^{\mu-\nu}}{B}) - f(-Aq^{\mu+\nu}, -\frac{q^{\mu+\nu}}{A})f(-Bq^{\mu-\nu}, -\frac{q^{\mu-\nu}}{B})\right\}$$

$$= A \sum_{m=0}^{\mu-1} (AB)^m q^{(2m+1)(\mu+\nu)+2\mu m^2}$$

$$\times f\left(A^{\mu-\nu}B^{\nu+\nu}q^{(2\mu+4m+2)(\mu^2-\nu^2)}q^{(2\mu-4m-2)(\mu^2-\nu^2)}}{A^{\mu-\nu}B^{\nu+\nu}}\right)$$

$$\times f\left(\frac{ABq^{4\mu+2\nu+4\nu m}}{A^\nu B^{\mu+\nu}}\right).$$  \hspace{1cm} (4.3.20)

**Lemma 4.3.14.** [11, p. 69, (36.7)] If $\mu$ is odd, then

$$\psi(q^{\mu+\nu})\psi(q^{\mu-\nu}) = \phi(q^{\mu-\nu^2})\psi(q^{2\mu})$$

$$+ \sum_{m=1}^{(\mu-1)/2} q^{\mu m^2-\nu m} f(q^{(\mu+2m)(\mu^2-\nu^2)}, q^{(\mu-2m)(\mu^2-\nu^2)})$$

$$\times f(q^{2\nu m}, q^{2\mu-2\nu m})$$  \hspace{1cm} (4.3.21)
Lemma 4.3.15. [11, p. 69, (36.8)] If \( \mu \) is even, then
\[
\psi(q^{\mu+\nu})\psi(q^{\mu-\nu}) = \phi(q^{\mu(\mu^2-\nu^2)})\psi(q^{2\mu}) + \sum_{m=1}^{\mu/2-1} q^{\mu m^2-\nu m} \\
\times f(q^{(\mu+2m)(\mu^2-\nu^2)}, q^{(\mu-2m)(\mu^2-\nu^2)}) f(q^{2\nu m}, q^{2\mu-2\nu m}) \\
+ q^{\mu^2/4-\mu \nu/2} \psi(q^{2\mu(\mu^2-\nu^2)}) f(q^{\mu \nu}, q^{2\mu-\nu \nu}).
\] (4.3.22)

Lemma 4.3.16. [11, p. 69, (36.9)] If \( \mu \) is odd, then
\[
\psi(q^{\mu+\nu})\psi(q^{\mu-\nu}) = q^{(\mu-3)/2} q^3 q^\nu \psi(q^{2\mu(\mu^2-\nu^2)}) f(q^{\mu+\nu}, q^{\mu-\nu}) \\
+ \sum_{m=0}^{(\mu-3)/2} q^{\mu m(m+1)} f(q^{(\mu+2m+1)(\mu^2-\nu^2)}, q^{(\mu-2m-1)(\mu^2-\nu^2)}) \\
\times f(q^{\mu+\nu+2\nu m}, q^{\mu-\nu-2\nu m}).
\] (4.3.23)

Lemma 4.3.17. [11, p. 69, (36.10)] If \( \mu \) is even, then
\[
\psi(q^{\mu+\nu})\psi(q^{\mu-\nu}) = \sum_{m=0}^{\mu/2-1} q^{\mu m(m+1)} f(q^{(\mu+2m+1)(\mu^2-\nu^2)}, q^{(\mu-2m-1)(\mu^2-\nu^2)}) f(q^{\mu+\nu+2\nu m}, q^{\mu-\nu-2\nu m}).
\] (4.3.24)

4.4 Proofs of (4.2.1) and (4.2.2):

Proof of (4.2.1): Adding (4.3.9) and (4.3.10), we find that
\[
f^2(-q^3, -q^5) + q f^2(-q, -q^7) = \psi(-q) \phi(q)
\] (4.4.1)

Employing (4.3.13), (4.3.14), (4.3.11), and (4.3.12) in (4.4.1), we easily arrive at (4.2.1).

Proof of (4.2.2): Subtracting (4.3.10) from (4.3.9), we obtain
\[
f^2(-q^3, -q^5) - q f^2(-q, -q^7) = \psi(-q) \phi(q^2).
\] (4.4.2)

Employing (4.3.13), (4.3.14), (4.3.11), and (4.3.12) in (4.4.2), we easily deduce (4.2.2).
4.5 Proofs of (4.2.3) - (4.2.5):

Proof of (4.2.3): Putting $\mu = 4$, $\nu = 3$ in (4.3.22), it can be shown [11, p. 315, (19.1)] that

$$\psi(q)\psi(q^7) = \phi(q^{28})\psi(q^8) + q\psi(q^{14})\psi(q^2) + q^6\psi(q^{56})\phi(q^4). \quad (4.5.1)$$

Replacing $q$ by $-q$ in (4.5.1), and then subtracting the resulting identity from (4.5.1), we find that

$$\psi(q)\psi(q^7) - \psi(-q)\psi(-q^7) = 2q\psi(q^{14})\psi(q^2). \quad (4.5.2)$$

Using (4.3.17) and (4.3.18) in (4.5.2), we obtain

$$f(q^2, q^{14})f(q^{42}, q^{70}) + q^6\psi(q^6, q^{10})f(q^{14}, q^{48}) = \psi(q^{14})\psi(q^2). \quad (4.5.3)$$

Replacing $q^2$ by $-q$ in (4.5.3), we deduce that

$$f(-q, -q^7)f(-q^{21}, -q^{35}) - q^6f(-q^3, -q^5)f(-q^7, -q^{49}) = \psi(-q^7)\psi(-q). \quad (4.5.4)$$

Employing (4.3.13), (4.3.14) and (4.3.12), we obtain (4.2.3). So, we complete the proof.

Proofs of (4.2.4) and (4.2.5): Putting $\mu = 2$ and $\nu = 1$ in (4.3.22), we find that

$$\psi(q)\psi(q^3) = \phi(q^9)\psi(q^4) + q^6\psi(q^{12})\phi(q^2). \quad (4.5.5)$$

Replacing $q$ by $-q$ in (4.5.5), we obtain

$$\psi(-q)\psi(-q^3) = \phi(q^6)\psi(q^4) - q^6\psi(q^{12})\phi(q^2). \quad (4.5.6)$$

Adding (4.5.5) and (4.5.6), we arrive at

$$\psi(q)\psi(q^3) + \psi(-q)\psi(-q^3) = 2\phi(q^6)\psi(q^4).$$

Using (4.3.17) and (4.3.18), this can be written as

$$f(q^6, q^{10})f(q^{18}, q^{30}) + q^4f(q^2, q^{14})f(q^6, q^{42}) = \phi(q^6)\psi(q^4). \quad (4.5.7)$$

Replacing $q^2$ by $-q$ in (4.5.7), we find that

$$f(-q^3, -q^5)f(-q^9, -q^{15}) + q^2f(-q, -q^7)f(-q^3, -q^{21}) = \phi(-q^3)\psi(q^2). \quad (4.5.8)$$
Using (4.3.13), (4.3.14), (4.3.11) and (4.3.12) in (4.5.8), we easily obtain
\[ S(q^3)S(q) + q^2T(q^3)T(q) = \frac{f_3f_4}{f_1f_{12}}. \]

Thus, we complete the proof of (4.2.4).

Now, subtracting (4.5.6) from (4.5.5), we find that
\[ \psi(q)\psi(q^3) - \psi(-q)\psi(-q^3) = 2q\psi(q^{12})\phi(q^2) \quad (4.5.9) \]

Using (4.3.17) and (4.3.18) on the left hand side, this can be written as
\[ f(q^{18}, q^{30})f(q^2, q^{14}) + q^2f(q^8, q^{42})f(q^4, q^{10}) = \phi(q^2)\psi(q^{12}). \quad (4.5.10) \]

Replacing \( q^2 \) by \(-q\) in (4.5.10), we obtain
\[ f(-q^9, -q^{15})f(-q, -q^7) - qf(-q^3, -q^{21})f(-q^3, -q^5) = \phi(-q)\psi(q^8). \quad (4.5.11) \]

By use of (4.3.13), (4.3.14), (4.3.11), and (4.3.12) in (4.5.11), we easily arrive at (4.2.5) to complete the proof.

### 4.6 Proofs of (4.2.6) and (4.2.7):

**Proof of (4.2.6):** Using (2.2.7) in (2.3.8), we find that
\[ \frac{\phi^2(-q^{10})}{\phi^2(-q^2)} + q\left( \frac{\phi(q^5)\psi(q^{10})}{\phi(q)\psi(q^2)} - \frac{\phi(-q^5)\psi(q^{10})}{\phi(-q)\psi(q^2)} \right) = 1. \quad (4.6.1) \]

Employing (2.2.5), this can be written as
\[ \frac{q\psi(q^{10})}{\psi(q^2)} (\phi(-q)\phi(q^5) - \phi(q)\phi(-q^5)) = \phi^2(-q^2) - \phi^2(-q^{10}). \quad (4.6.2) \]

With the help of (2.3.9), we deduce from (4.6.2) that
\[ \frac{\psi(q^{10})}{\psi(q^2)} (4q^2f(-q^4)f(-q^{10})) = \phi^2(-q^{10}) - \phi^2(-q^2). \quad (4.6.3) \]

Replacing \( q^2 \) by \( q \) in (4.6.3), we find that
\[ \frac{\psi(q^5)}{\psi(q)} (4qf(-q^2)f(-q^{10})) = \phi^2(-q^5) - \phi^2(-q). \quad (4.6.4) \]
Replacing $q$ by $-q$ in (4.6.4), we obtain

$$-\frac{\psi(-q^5)}{\psi(-q)} (4q f(-q^2) f(-q^{10})) = \phi^2(q^5) - \phi^2(q). \quad (4.6.5)$$

Subtracting (4.6.5) from (4.6.4), and then using (2.2.6) and (4.3.3), we find that

$$\frac{f(-q^2) f(-q^{10})}{\phi(-q^2) \psi(q^2)} (\psi(q) \psi(-q^5) + \psi(-q) \psi(q^5)) = 2 (\psi^2(q^4) - q^4 \psi^2(q^{20})) \quad (4.6.6)$$

Replacing $q$ by $q^4$ in (2.3.16) and using in (4.6.6), we deduce that

$$\psi(q) \psi(-q^5) + \psi(-q) \psi(q^5) = 2 \frac{\phi(-q^2) \psi(q^2) \phi(-q^{20}) f(-q^{20})}{f(-q^2) f(-q^{10}) \chi(-q^2)}. \quad (4.6.7)$$

Employing (4.3.17) and (4.3.18) in (4.6.7), and then replacing $q^2$ by $-q$ in the resulting identity, we deduce that

$$f(-q^3, -q^5) f(-q^{15}, -q^{25}) + q^2 f(-q, -q^7) f(-q^5, -q^{35}) = \frac{\phi(q) \psi(-q) \phi(-q^{10}) f(-q^{10})}{f(q) f(q^2) \chi(-q^2)}. \quad (4.6.8)$$

Employing (4.3.13), (4.3.14), (4.3.11), and (4.3.12) in (4.6.8) we easily deduce (4.2.6) to complete the proof of (4.2.6).

**Proof of (4.2.7):** Adding (4.6.4) and (4.6.5), and then using (2.2.6) and (4.3.4), we find that

$$2q f(-q^2) f(-q^{10}) \frac{\phi(-q^2) \psi(q^2)}{\phi(-q^2) \psi(q^2)} (\psi(q) \psi(-q^5) - \psi(-q) \psi(q^5)) = \phi^2(q^2) - \phi^2(q^{10}). \quad (4.6.9)$$

Using (4.3.12) in (2.3.1), we find that

$$\phi^2(q) - \phi^2(q^5) = 4q \chi(q) f(-q^5) f(-q^{20}). \quad (4.6.10)$$

Replacing $q$ by $q^2$ in (4.6.10) and using in (4.6.9), we find that

$$\psi(q) \psi(-q^5) - \psi(-q) \psi(q^5) = 2 \frac{\chi(q^2) \psi(q^2) \phi(-q^2) f(-q^{10})}{f(-q^2)}. \quad (4.6.11)$$
Employing (4.3.17) and (4.3.18) in (4.6.11), and then replacing \( q^2 \) by \(-q\) in the resulting identity, we obtain

\[
f(-q, -q^7)f(-q^{15}, -q^{25}) - q^2 f(-q^3, -q^5)f(-q^5, -q^{35}) = \frac{\chi(-q)\psi(-q)\phi(q)f(-q^{20})}{f(q)}. \quad (4.6.12)
\]

Employing (4.3.13), (4.3.14), (4.3.11), and (4.3.12), we easily deduce (4.2.7).

### 4.7 Proof of (4.2.8):

We recall from (4.3.18) that

\[
\psi(-q) = f(q^6, q^{10}) - qf(q^2, q^{14}). \quad (4.7.1)
\]

Using (4.3.8), we write (4.7.1) as

\[
\psi(-q) = \left\{ \frac{f(-q^{12}, -q^{20})}{f(-q^6, -q^{10})} - q \frac{f(-q^4, -q^{28})}{f(-q^2, -q^{14})} \right\} \phi(-q^{16}). \quad (4.7.2)
\]

Employing (4.3.13), (4.3.14), (4.3.11), and (4.3.12) we easily arrive at (4.2.8).

### 4.8 Proofs of (4.2.9)-(4.2.10):

**Proof of (4.2.9):** Using (3.2.2), with \( q \) replaced by \( q^3 \), in (3.2.4), we find that

\[
\psi(q) - q\psi(q^3) = \frac{\phi(-q^9)}{\chi(-q^3)}. \quad (4.8.1)
\]

This can also be written as

\[
\frac{\psi(q)}{q\psi(q^3)} = 1 + \frac{\phi(-q^9)}{\chi(q^3)}. \quad (4.8.2)
\]

Employing (3.2.1) in (4.8.2), we obtain

\[
\frac{\psi(q)}{q\psi(q^3)} = 1 + \frac{\phi(-q^9)}{q f(-q^3, -q^{15})}. \quad (4.8.3)
\]
Replacing $q$ by $-q$, we find that
\[ -\frac{\psi(-q)}{q\psi(-q^9)} = 1 - \frac{\phi(q^9)}{qf(q^3, q^{15})}. \tag{4.8.4} \]

Adding (4.8.3) and (4.8.4), and then using (1.1.8), we deduce that
\[ \frac{\psi(q)\psi(-q^9) - \psi(-q)\psi(q^9)}{\psi(q^9)\psi(-q^9)} = 2q + \frac{f(q^9, q^{15})f(-q^9, -q^9) - f(-q^3, -q^{15})f(q^3, q^9)}{f(q^3, q^{15})f(-q^3, -q^{15})}. \tag{4.8.5} \]

Employing (2.2.10) and (4.3.8), with $a = q^3$, $b = q^{15}$, $c = d = -q^9$, in (4.8.5), we deduce that
\[ \frac{\psi(q)\psi(-q^9) - \psi(-q)\psi(q^9)}{\psi(q^9)\psi(-q^9)} = 2q + 2q^3 \frac{f(-q^6, -q^{30})}{\phi(-q^{18})}. \tag{4.8.6} \]

Using (2.2.6), this can be written as
\[ \psi(q)\psi(-q^9) - \psi(-q)\psi(q^9) = 2q\psi(q^{18})\{\phi(-q^{18}) + q^2f(-q^9, -q^{30})\}. \tag{4.8.7} \]

Employing (3.2.3) in (4.8.7), we deduce that
\[ \psi(q)\psi(-q^9) - \psi(-q)\psi(q^9) = q\psi(q^{18})\{3\phi(-q^{18}) - \phi(-q^2)\}. \tag{4.8.8} \]

Now, using (3.2.1) in (3.2.3), we deduce that
\[ \phi(-q^9) - \phi(-q) = 2q\psi(q^9)\chi(-q^3). \tag{4.8.9} \]

Using (4.8.1), we rewrite this as
\[ \phi(-q^9) - \phi(-q) = 2\psi(q)\chi(-q^3) - 2\phi(-q^9). \tag{4.8.10} \]

Thus,
\[ 3\phi(-q^9) - \phi(-q) = 2\psi(q)\chi(-q^3). \tag{4.8.11} \]

Replacing $q$ by $q^2$ in (4.8.11), and then using it in (4.8.8), we find that
\[ \psi(q)\psi(-q^9) - \psi(-q)\psi(q^9) = 2q\psi(q^{18})\psi(q^2)\chi(-q^9). \tag{4.8.12} \]
Invoking (4.3.17) and (4.3.18), and then replacing \( q^2 \) by \(-q\), we deduce from (4.8.12) that

\[
f(-q^{27}, -q^{45})f(-q, -q^7) - q^4 f(-q^3, -q^5) f(-q^9, -q^{63}) = \psi(-q^9)\psi(-q)\chi(q^3). \tag{4.8.13}
\]

Employing (4.3.13), (4.3.14), and (4.3.12) in (4.8.13), we easily deduce (4.2.9).

**Proof of (4.2.10):** Subtracting (4.8.4) from (4.8.3), and then using (1.1.8), we obtain

\[
\frac{\psi(q)\psi(-q^9) + \psi(-q)\psi(q^9)}{\psi(q^9)\psi(-q^9)} = \frac{f(q^3, q^{15})f(-q^9, -q^9) + f(-q^3, -q^{15})f(q^9, q^9)}{f(q^3, q^{15})f(-q^3, -q^{15})}. \tag{4.8.14}
\]

Employing (2.2.9), with \( a = q^3, b = q^{15}, c = d = -q^9 \), and (3.2.1), in (4.8.14), we deduce that

\[
\psi(q)\psi(-q^9) + \psi(-q)\psi(q^9) = 2\frac{f^2(-q^{12})}{\chi(-q^6)}, \tag{4.8.15}
\]

where we have also used from [11, p. 39, Entry 24(iv)], that \( \chi(q)\chi(-q) = \chi(-q^2) \). Now, employing (4.3.17) and (4.3.18), and then replacing \( q^2 \) by \(-q\), we derive from (4.8.15) that

\[
f(-q^{27}, -q^{45})f(-q^3, -q^5) + q^6 f(-q, -q^7) f(-q^9, -q^{63}) = \frac{f^2(-q^6)}{\chi(q^3)}. \tag{4.8.16}
\]

Employing (4.3.13), (4.3.14), and (4.3.12) in (4.8.16), we easily arrive at (4.2.10).

**Remark:** The sixth of Ramanujan's 40 identities is given by

\[
G(q)G(q^9) + q^2 H(q) H(q^9) = \frac{f^2(-q^3)}{f(-q)f(-q^9)}, \tag{4.8.17}
\]

where \( G(q) \) and \( H(q) \) are as defined in (4.1.1) and (4.1.2).

For proofs of (4.8.17), see [32] and [16]. With the help of (4.8.12) and (4.8.15), we now present a new proof of this identity.
First, putting $\mu = 5$, $\nu = 4$, $A = 1$, and $B = -1$ in (4.3.20), and then employing Entry 18 (iv) [11, p. 34], it can be deduced that

$$
\phi(q)\phi(-q^9) - \phi(-q)\phi(q^9) = 4q \left\{ f(-q^8, -q^{12})f(-q^{72}, -q^{108}) + q^8 f(-q^4, -q^{16})f(-q^{36}, -q^{144}) \right\}.
$$

(4.8.18)

Using (2.2.7) this can be rewritten as

$$
\frac{\psi^2(q)\psi^2(-q^9) - \psi^2(-q)\psi^2(q^9)}{\psi(q^8)\psi(q^{18})} = 4q \left\{ f(-q^8, -q^{12})f(-q^{72}, -q^{108}) + q^8 f(-q^4, -q^{16})f(-q^{36}, -q^{144}) \right\}
$$

(4.8.19)

Employing (4.8.12) and (4.8.15) in (4.8.19), we arrive at

$$
f(-q^8, -q^{12})f(-q^{72}, -q^{108}) + q^8 f(-q^4, -q^{16})f(-q^{36}, -q^{144}) = f^2(-q^{12}).
$$

(4.8.20)

Using (3.4.39) in (4.8.20), we obtain

$$
G(q^4)G(q^{36}) + q^8 H(q^4)H(q^{36}) = \frac{f^2(-q^{12})}{f(-q^4)f(-q^{36})}.
$$

(4.8.21)

Replacing $q^4$ by $q$ in (4.8.21), we easily deduce (4.8.17).

**4.9 Proofs of (4.2.11)- (4.2.13):**

**Proof of (4.2.11):** From Entry 9(iv) [11, p. 377], we note that

$$
\psi(q)\psi(q^{15}) + \psi(-q)\psi(-q^{15}) = 2\psi(q^8)\psi(q^{10}).
$$

(4.9.1)

Using (4.3.17) and (4.3.18) in (4.9.1), and then replacing $q^2$ by $-q$. we obtain

$$
f(-q^3, -q^5)f(-q^{45}, -q^{75}) + q^8 f(-q, -q^7)f(-q^{15}, -q^{105}) = \psi(-q^3)\psi(-q^5).
$$

(4.9.2)

Employing (4.3.13), (4.3.14), and (4.3.12) in (4.9.2), we complete the proof of (4.2.11).
Proof of (4.2.12): From Entry 9(i) [11, p. 377], we have
\[ \psi(q^3)\psi(q^5) - \psi(-q^3)\psi(-q^5) = 2q^3\psi(q^2)\psi(q^{30}). \] (4.9.3)
The rest of the proof is same as the previous one.

Proof of (4.2.13): Putting $\mu = 4$, $\nu = 1$ in (4.3.22), we find that
\[ \psi(q^5)\psi(q^3) = \psi(q^8)\phi(q^{60}) + q^3\psi(q^2)\phi(q^{30}) + q^{14}\psi(q^{120})\phi(q^4). \] (4.9.4)
Replacing $q$ by $-q$ in (4.9.4), and then adding the resulting identity with (4.9.4), we deduce that
\[ \psi(q^5)\psi(q^3) + \psi(-q^5)\psi(-q^3) = 2\psi(q^8)\phi(q^{60}) + 2q^{14}\psi(q^{120})\phi(q^4). \] (4.9.5)
Using (4.3.1) and (4.3.2) on the right of (4.9.5), we deduce that
\[ \psi(q^5)\psi(q^3) + \psi(-q^5)\psi(-q^3) = \frac{1}{2q}\{\phi(q)\phi(q^{15}) - \phi(-q)\phi(-q^{15})}\}. \] (4.9.6)
Employing (2.2.7) we can rewrite this as
\[ \psi(q^5)\psi(q^3) + \psi(-q^5)\psi(-q^3) = \frac{\psi^2(q)\psi^2(q^{15}) - \psi^2(-q)\psi^2(-q^{15})}{2q\psi(q^2)\psi(q^{30})}. \] (4.9.7)
Thus,
\[ \frac{\psi(q^5)\psi(q^3) + \psi(-q^5)\psi(-q^3)}{\psi(q)\psi(q^{15}) - \psi(-q)\psi(-q^{15})} = \frac{\psi(q)\psi(q^{15}) + \psi(-q)\psi(-q^{15})}{2q\psi(q^2)\psi(q^{30})}. \] (4.9.8)
Employing (4.3.17) and (4.3.18) in (4.9.8), replacing $q^2$ by $-q$ in the resulting identity, and then using (4.3.13), (4.3.14) and (4.3.12), we easily deduce that
\[ \frac{S(q^5)S(q^3) + q^4T(q^5)T(q^3)}{S(q^{15})T(q) - q^7S(q)T(q^{15})} = \frac{\{S(q^{15})S(q) + q^8T(q^{15})T(q)\}f_2f_3f_5f_{12}f_{20}f_{30}}{f_1f_4f_6f_{10}f_{15}f_{60}}. \] (4.9.9)

Now, an application of (4.2.11) easily yields (4.2.13).

4.10 Proofs of (4.2.14)-(4.2.16):

Proofs of (4.2.14)-(4.2.16) are similar in nature. So, we give details only for (4.2.14).
Proof of (4.2.14): In (4.3.22), we put \( \mu = 8, \nu = 5 \) to obtain (See [11, p. 75, (37.4)] for details)

\[
\psi(q^{13})\psi(q^3) - \psi(-q^{13})\psi(-q^3) = 2q^3 f(q^{390}, q^{234})(q^b, q^{10}) + 2q^{43} f(q^{78}, q^{546}) f(q^2, q^{14}). \tag{4.10.1}
\]

Using (4.3.17) and (4.3.18) in (4.10.1), replacing \( q^2 \) by \(-q\), we find that

\[
\frac{f(-q^{195}, -q^{117})(-q^3, -q^5) + q^{20} f(-q^{39}, -q^{273}) f(-q, -q^7)}{f(-q^{39}, -q^{35}) f(-q^3, -q^{21}) - q^6 f(-q^{15}, -q^{91}) f(-q^2, -q^{15})} = 1. \tag{4.10.2}
\]

Employing (4.3.13) and (4.3.14), we arrive at the desired result.

Proofs of (4.2.15) and (4.2.16): To prove (4.2.15) and (4.2.16), we put \((\mu, \nu) = (8, 3)\) and \((\mu, \nu) = (8, 1)\), respectively, in (4.3.22) and proceed as in the above proof.

4.11 Proofs of (4.2.17) and (4.2.18):

Proof of (4.2.17): Putting \( \mu = 5, \nu = 2, A = 1, B = -1 \) in (4.3.20), we find that

\[
\{\phi(q^7)\phi(-q^3) - \phi(q^3)\phi(-q^7)\} = -4q^3 \{f(-q^{252}, -q^{168})(-q^{16}, -q^4) - q^6 f(-q^{84}, -q^{336}) f(-q^8, -q^{12})\}. \tag{4.11.1}
\]

Using (2.2.7) this can be written as

\[
\{\psi(q^7)\psi(-q^3) + \psi(q^3)\psi(-q^7)\} \{\psi(q^7)\psi(-q^3) - \psi(q^3)\psi(-q^7)\}
= -4q^3 \psi(q^{14})\psi(q^6) \{f(-q^{252}, -q^{168})(-q^{16}, -q^4) - q^6 f(-q^{84}, -q^{336}) f(-q^8, -q^{12})\}. \tag{4.11.2}
\]

We now employ (4.3.17) and (4.3.18) on the left hand side of (4.11.2) and then replace \( q^2 \) by \(-q\). Then we use (4.3.13), (4.3.14), and (4.3.12), to obtain

\[
\{S(q^7)S(q^3) + q^5 T(q^7)T(q^3)\} \{S(q^7)T(q^3) - q^2 S(q^3)T(q^7)\} \frac{f_6 f_{14}}{f_3 f_7 f_{12} f_{28}} \{f(-q^{126}, -q^{44})(-q^8, -q^2) - q^8 f(-q^{42}, -q^{168}) f(-q^4, -q^6)\}. \tag{4.11.3}
\]

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Employing (3.4.39) in (4.11.3), we obtain

\[ \{S(q^7)S(q^3) + q^5T(q^7)T(q^3)\} \{S(q^7)T(q^3) - q^2S(q^3)T(q^7)\} \]

\[ = \frac{f_{5,41}f_{2,42}}{f_{3,1,2,28}} \{G(q^{42})H(q^2) - q^8G(q^2)H(q^{42})\}. \quad (4.11.4) \]

Now replacing \( x \) by \( q^2 \) in the fifteenth of Ramanujan’s forty identities for the Rogers-Ramanujan functions [29, p. 236] (see also [26, (R.15) and (R.16)]), we note that

\[ G(q^{14})H(q^6) + q^4H(q^{14})H(q^6) = G(q^{42})H(q^2) - q^8G(q^2)H(q^{42}) \]

\[ = \frac{1}{2q} \left\{ \frac{f_{3,1,2,28,14,14,14,21}}{f_{3,1,2,14,14,14,14,21}} - \frac{f_{1,3,1,2,14,14,14,21}}{f_{3,1,2,14,14,14,14,21}} \right\}. \quad (4.11.5) \]

Using (4.11.5) in (4.11.4) we easily find (4.2.17) to complete the proof.

**Proof of (4.2.18):** Replacing \( q \) by \( q^{1/3} \) in (4.11.1), we find that

\[ \phi(q^{7/3})\phi(-q) - \phi(q)\phi(-q^{7/3}) \]

\[ = -4q\{f(-q^{84}, -q^{56})f(-q^{10/3}, -q^{4/3}) \]

\[ - q^{16/3} f(-q^{28}, -q^{112})f(-q^{8/3}, -q^4)\}. \quad (4.11.6) \]

Now, replacing \( q \) by \( \pm q^{7/3} \) in (3.2.3), we obtain

\[ \phi(\pm q^{7/3}) = \phi(\pm q^{21}) \pm 2q f(q^7, q^{35}). \quad (4.11.7) \]

Again, from Entry 10(iii) and 10(iv) [11, p. 379], we note that

\[ f(-q^{2/3}, -q) = f(-q^7, -q^8) - qf(-q^2, -q^{13}) - q^{2/3}f(-q^3, -q^{12}). \quad (4.11.8) \]

and

\[ f(-q^{1/3}, -q^{4/3}) = f(-q^6, -q^9) - q^{1/3}f(-q^4, -q^{11}) + qf(-q, -q^{14}). \quad (4.11.9) \]

Replacing \( q \) by \( q^4 \) in (4.11.8) and (4.11.9), we obtain

\[ f(-q^{8/3}, -q^4) = f(-q^{28}, -q^{32}) - q^4 f(-q, -q^{52}) - q^{8/3} f(-q^{12}, -q^{48}). \quad (4.11.10) \]

and

\[ f(-q^{4/3}, -q^{16/3}) = f(-q^{24}, -q^{36}) - q^{4/3} \{ f(-q^{16}, -q^{44}) + q^4 f(-q^4, -q^{56})\}, \quad (4.11.11) \]
respectively. Employing (4.11.7), (4.11.10), and (4.11.11) in (4.11.6), and then equating the rational parts from both sides of the resulting identity, we deduce that

\[
\phi(-q)\phi(q^{21}) - \phi(q)\phi(-q^{21}) \\
= -4q\{f(-q^{84}, -q^{56})f(-q^{24}, -q^{36}) + q^8 f(-q^{28}, -q^{112})f(-q^{12}, -q^{48})\}. \\
\]  

(4.11.12)

Using (2.2.7) this can be rewritten as

\[
\{\psi(-q)\psi(q^{21}) + \psi(q)\psi(-q^{21})\} \{\psi(-q)\psi(q^{21}) - \psi(q)\psi(-q^{21})\} \\
= -4q\psi(q^2)\psi(q^{42})\{f(-q^{84}, -q^{56})f(-q^{24}, -q^{36}) \\
+ q^8 f(-q^{28}, -q^{112})f(-q^{12}, -q^{48})\}. \\
\]  

(4.11.13)

Employing (4.3.17) and (4.3.18) on the left hand side of (4.11.13), and then replacing \( q^2 \) by \(-q\) in the resulting identity, we arrive at

\[
f(-q^3, -q^5)f(-q^{63}, -q^{105}) + q^{11}f(-q, -q^7)f(-q^{21}, -q^{147}) \times \{f(-q, -q^7)f(-q^{63}, -q^{105}) - q^{10}f(-q^3, -q^5)f(-q^{21}, -q^{147})\} \\
= \psi(-q)\psi(-q^{21}) \{f(-q^{42}, -q^{28})f(-q^{12}, -q^{18}) \\
+ q^4 f(-q^{14}, -q^{56})f(-q^{6}, -q^{24})\}. \\
\]  

(4.11.14)

Using (4.3.13), (4.3.14), (4.3.12), and (3.4.39), we easily deduce that

\[
\{S(q^{21})S(q) + q^{11}T(q^{21})T(q)\} \{S(q^{21})T(q) - q^{10}S(q)T(q^{21})\} \\
= \frac{f_2 f_6 f_{14} f_{42}}{f_1 f_4 f_{21} f_{84}} \{G(q^{14})G(q^6) + q^4 H(q^{14}) H(q^6)\}. \\
\]  

(4.11.15)

Employing (4.11.5) in (4.11.15), we arrive at (4.2.18). Hence the proof is complete.

**4.12 Proofs of (4.2.19) - (4.2.21):**

**Proof of (4.2.19):** Putting \( a = 1, b = -q^4, c = q \) and \( d = -q^3 \) in (2.2.9), we obtain

\[
f(1, -q^4)f(q, -q^3) = 2f(q, q^7)f(-q^3, -q^5). \\
\]  

(4.12.1)

Replacing \( q \) by \(-q\), we have

\[
f(1, -q^4)f(-q, q^3) = 2f(-q, -q^7)f(q^3, q^5). \\
\]  

(4.12.2)
Adding (4.12.1) and (4.12.2), we find that
\[f(1, -q^4)\{f(q, -q^3) + f(-q, q^3)\} = 2\{f(q, q^7)f(-q^3, -q^5) + f(-q, -q^7)f(q^3, q^5)\}. \tag{4.12.3}\]

Using (4.3.6) in (4.12.3), we obtain
\[f(1, -q^4)f(-q^6, -q^{10}) = \{f(q, q^7)f(-q^3, -q^5) + f(-q, -q^7)f(q^3, q^5)\}. \tag{4.12.4}\]

Employing (1.1.9) in (4.12.4), we deduce that
\[\{f(q, q^7)f(-q^3, -q^5) + f(-q, -q^7)f(q^3, q^5)\} = 2\psi(-q^4)f(-q^6, -q^{14}). \tag{4.12.5}\]

Using (4.3.13), (4.3.14) and (4.3.12), we deduce (4.2.19).

**Proof of (4.2.20):** Subtracting (4.12.9) from (4.12.1), we obtain
\[f(1, -q^4)\{f(q, -q^3) - f(-q, q^3)\} = 2\{f(q, q^7)f(-q^3, -q^5) - f(-q, -q^7)f(q^3, q^5)\}. \tag{4.12.6}\]

Using (4.3.7) in (4.12.6), then using (1.1.9), we find that
\[\{f(q, q^7)f(-q^3, -q^5) - f(-q, -q^7)f(q^3, q^5)\} = 2q\psi(-q^4)\phi(-q^6). \tag{4.12.7}\]

Invoking (4.3.13), (4.3.14) and (4.3.12), we easily deduce (4.2.20). Thus, we complete the proof.

**Proof of (4.2.21):** Putting \(a = q, b = -q^3, c = q^2\) and \(d = -q^2\), in (2.2.9) and (2.2.10), we obtain
\[f(q, -q^3)f(q^3, -q^5) + f(-q, q^3)f(-q^2, q^5) = 2f(q^3, q^5)f(-q^3, -q^5). \tag{4.12.8}\]
\[f(q, -q^3)f(q^2, -q^2) - f(-q, q^3)f(-q^2, q^2) = 2qf(-q, -q^7)f(q, q^7). \tag{4.12.9}\]

Subtracting (4.12.9) from (4.12.9), we find that
\[f(-q, q^3)f(q^2, -q^2) = f(q^3, q^5)f(-q^3, -q^5) - qf(-q, -q^7)f(q, q^7). \tag{4.12.10}\]

Using (4.3.5), we obtain
\[f(q^2, -q^2) = f(-q^8, -q^8) = \phi(-q^8). \tag{4.12.11}\]

Employing (4.12.11) in (4.12.10), and then using (4.3.13), (4.3.14) and (4.3.12) we complete the proof.
4.13 New modular relations for the Göllnitz Gordon functions:

Making different choices for \( \mu, \nu, A, \) and \( B \) in the Schroter’s formulas (4.3.19)-(4.3.24), and then using the methods as in the previous nine sections, one can find many other relations for \( S(q) \) and \( T(q) \). In the following, we give some examples.

\[
S(q)S(q^{11}) + q^6T(q)T(q^{11}) = \frac{f_2f_6f_{12}f_{22}f_{33}f_{33}}{f_1f_4f_6f_{11}f_{44}f_{44}} - q^3 \frac{f_6f_6f_{96}f_{96}}{f_1f_1f_{12}f_{33}f_{33}}.
\]

\[
T(q)S(q^{11}) - q^5S(q)T(q^{11}) = \frac{f_6f_6f_{66}f_{66}}{f_1f_4f_{11}f_{132}} - q^7 \frac{f_2f_2f_{12}f_{12}f_{12}f_{132}f_{132}}{f_1f_1f_4f_6f_{11}f_{44}f_{44}}.
\]

\[
\{S(q^7)S(q^5) + q^6T(q^7)T(q^5)\} \{S(q^7)T(q^5) - qT(q^7)S(q^5)\}
= \frac{1}{q} \left\{ \frac{f_2f_2f_{10}f_{14}f_{170}f_{170}f_{70}}{f_1f_4f_5f_{17}f_{10}f_{25}f_{35}f_{140}} - 1 \right\}.
\]

\[
\{S(q^{35})S(q) + q^{18}T(q)T(q^{35})\} \{S(q^{35})T(q) - q^{17}T(q^{35})S(q)\}
= \frac{f_2f_2f_{10}f_{10}f_{14}f_{14}f_{170}}{f_1f_4f_5f_{17}f_{25}f_{35}f_{140}} + q^4.
\]

\[
\{S(q)S(q^{27}) + q^{14}T(q)T(q^{27})\} \{T(q)S(q^{27}) - q^{13}S(q)T(q^{27})\}
= \frac{f_2f_6f_6f_{18}f_{18}f_{18}f_{18}f_{18}f_{541}}{f_1f_3f_4f_9f_{12}f_{27}f_{36}f_{108}} - q^3.
\]

**Proofs of (4.13.1)-(4.13.5):** To prove (4.13.1) and (4.13.2), we set \( \mu = 6 \) and \( \nu = 5 \) in (4.3.22), and then proceed as in the proofs of (4.2.4) and (4.2.5).

To prove (4.13.3), (4.13.4) and (4.13.5), we use Entry 17(ii), (i) [11, p. 417] and Entry 4(iv) [11, p. 359], respectively, and proceed as in the proof of (4.2.17). It is worthwhile to note that Berndt [11] used Schröter’s formulas (4.3.19) and (4.3.20), and also other theta-function identities of Ramanujan to establish these entries. One can also get many other analogous identities from (4.3.19)-(4.3.24).

**Remark:** I am grateful to my mathematical brother Nipen Saikia for giving the idea of proofs of identities (4.2.19)-(4.2.21).