CHAPTER-3
EXACT SOLUTIONS OF RADIATING AND NON-RADIATING VISCOUS FLUID UNIVERSES COUPLED WITH AN ELECTROMAGNETIC FIELD IN GENERAL RELATIVITY.

3.1 INTRODUCTION.

Despite the fact, the cosmological solutions of the Einstein field equations given by Friedmann have successfully incorporated the expansion, homogeneity and isotropy of the universe, and though it is commonly accepted that the Friedmann models represent the present state of the universe quite accurately, and whatever deviations may exist are expected to be small, they do not explain the homogeneization and isotropization of the Universe, which is apparent at scales of the order of $10^8$ light-years. Besides this, statistical fluctuations in Friedmann models do not grow fast enough to explain the formation of galaxies, which implies the existence of real in homogeneities at all stages of the evolutions of the universe, and recent observations of voids pose a challenging problem to be explained by any responsible cosmological model. Thus here we consider a metric which can give more in right into the minute study of the models.

On the other hand, objects with large energy output, either in the forms of photons or neutrinos or both in some phases of either evolutions, are very much known to exist. A nonstatic distribution would be radiating energy, and so it would be surrounded by an ever expanding zone of radiation. It is widely recognized that in the distant past the universe was dominated by the radiation and the early universe was an undifferentiated soup of matter and radiation in a state of thermal equilibrium.

During the photon decoupling stage part of the electromagnetic radiation behaved as a perfect fluid co-moving with matter, while another part behaved like a unidirectional stream moving with fundamental velocity. And during the neutrino decoupling stage a similar situation arose in which apart from streaming neutrinos moving with fundamental velocity, there was a past behaving like a viscous fluid co-moving with matter. The discovery of quasi-steller objects and their huge energy requirements motivated the development of a theory of hob, convective, supermassive stars where general relativistic effects are important. It will, therefore be interesting to consider a radiating distribution in trying to explore new results and so that useful information's about the behavior of a realistic Universe can be obtained from such models.

In this we obtain four new solutions and try to study them from various angles. Even though some numerically computed solutions are available in the literature the efficiency of exact solutions for giving a clear understanding of the internal structure of a spherical star cannot be reached. With the help of the exact solutions obtained here we study the physical and dynamical properties of star models emitting and coupled with electromagnetic radiation ; and as special cases, models which have stopped emitting electromagnetic radiation are also discussed and examined. The effects of viscosity on such models are also investigated with great precision. The importance in studying the radiation models lies in the fact that nowadays radiation plays an important role in studying many scientific and astrophysical phenomena. Particularly investigations on electromagnetically charged models are important in connection with the study of the pulsars and the black holes. The models here are also taken to be composed of viscous fluid as so far from the evidences obtained, it is known that no astrophysical object is composed of a purely perfect fluid.

3.2 FIELD EQUATION AND THEIR SOLUTIONS.

The metric considered here is
\[ ds^2 = \exp(2\gamma)dt^2 - \exp(2\lambda)dr^2 - r^2d\theta^2 - r^2 \sin^2 \theta d\phi^2 \]
....................................................................................................................(1)
where $\gamma, \lambda$ and $\gamma$ are functions of $r$ and $t$.

The energy-momentum tensor $T_{\mu\nu}$ is here given by

$$T_{\mu\nu} = \rho u_{\mu} u_{\nu} + (p - \xi \theta) H_{\mu\nu} - 2\eta \sigma_{\mu\nu} + E_{\mu\nu},$$

where $p$ is the isotropic pressure; $\rho$, the fluid density; $\eta$ and $\xi$, the co-efficients of shear and bulk viscosities, respectively; and $u_{\mu}$, the four vector velocity of flow satisfying the relation.

$$g_{\mu\nu} u^\mu u^\nu = 1,$$

Here

$$E_{\mu\nu} = \frac{1}{4\pi G} \left[ \frac{1}{4} g_{\mu\nu} F_{\alpha\beta} F^{\alpha\beta} - F_{\mu\alpha} F^{\mu\alpha} \right],$$

is the energy-momentum tensor due to the electro-magnetic field where $F_{\alpha\beta}$ are electromagnetic field tensors satisfying the relations

$$F_{\cdot\cdot}^\alpha = -\sigma u^\alpha,$$

and

$$F_{[\mu;\nu]} = 0,$$

where $\sigma(\mu,t)$ is the charge density of the electro-magnetic field (a semi-colon followed by a subscript denotes covariant differentiation).
The $H_{\mu\nu}$ tensors are the projection tensors given by

$$H_{\mu\nu} = u_\mu u_\nu - g_{\mu\nu}$$

While

$$\sigma_{\mu\nu} = \frac{1}{2} \left( u_{\mu,\beta} H^\beta_{\nu} + u_{\nu,\beta} H^\beta_{\mu} \right) - \frac{1}{3} \partial H_{\mu\nu}$$

define the components of the shear tensor where

$$\theta = u^\alpha_{,\alpha} = \nabla u$$

is the expansion factor.

Since the only non-zero component of the electro field is $F^{41}$ because of spherical symmetry, Maxwell's equations reduce to

$$(\epsilon^\nu \epsilon^\lambda y^2 F^{41})_4 = -4\pi y^2 \epsilon^\nu \epsilon^\lambda j^4 ..................(7)$$

$$(\epsilon^\nu \epsilon^\lambda y^2 F^{41})_4 = 4\pi y^2 \epsilon^\nu \epsilon^\lambda j^4 ..................(8)$$

Now, in the rest frame of the fluid, defined by $u_\mu = \delta^\nu_\mu$. Equations (7) and (8) may be integrated to obtain

$$F^{41} = - \epsilon^\nu \epsilon^\lambda y^2 Q(r,t) ..................(9)$$

Where $Q(r,t) = 4\pi \int_0^r j^4 y^2 e^{\nu + \lambda} dr ..................(10)$

Thus Einstein field equation

$$R_{\mu\nu} = \frac{1}{8\pi G} \left( T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T \right)$$

gives

$$- y^{-2} + 2y^{-1} e^{-24} \left( y'' - y' \lambda' + \frac{1}{2} y^{-1} y'^2 \right) - 2y^{-1} e^{-24} \left( y \lambda + \frac{1}{2} y^{-1} y'^2 \right)$$

$$= -8\pi \left( e + \frac{1}{8\pi} Q^3 y^{-4} \right) ..................(11)$$
Overhead dot and prime respectively denote partial differentiations with respect to 't' and 'r'. We shall now take up four cases.

CASE-1.

From above we see that there are only four equations [equations (11)-(14)] out of which six unknowns are to be solved.

Thus here we assume
\[ \dot{\lambda} = -y^{-1} \ddot{y}, \] .................................................................(15)

Making use of this relation in equation (12), we get
\[ y y = \frac{1}{a} e^y, \]..........................(16)

where 'a' is an arbitrary constant.

Now from equation (16) we can conveniently take

\[ y = rt, \]..........................(17)

\[ v = \log (a_1 \xi r^2 t), \]..........................(18)

Taking \( a = \xi a_1 \)

Also from equations (15) and (17), we have

\[ \lambda = \log \{ t^{-1} f(r) \}, \]..........................(19)

As a particular solution we can take

\[ \lambda = \log (b_1 \eta r t^{-1}), \]..........................(20)

where \( b_1 \) is an arbitrary constant.

Now making use of the relations (17), (18) and (20) we see that equations (13) and (14) respectively take the forms.

\[ -r^{-2}t^{-2} + 5b_1^{-2}r^{-4}t^2 + a_1^{-2}\xi^{-2}r^{-4}t^{-4} \]

\[ = 8np - Q^2r^{-4}t^{-4} + \frac{64}{3} \pi a_1^{-1}\xi^{-1}r^{-2}t^{-2} - 8\pi a_1^{-1}r^{-2}t^{-2}, \]..........................(21)

And

\[ b_1^{-2}\eta^{-2}r^{-4}t^2 + a_1^{-2}\xi^{-2}r^{-4}t^{-4} \]

\[ = 8np + Q^2r^{-4}t^{-4} - \frac{32}{3} \pi \eta a_1^{-1}\xi^{-1}r^{-2}t^{-2} - 8\pi a_1^{-1}r^{-2}t^{-2}, \]..........................(22)
Now from equation (21) and (22), we get

\[ Q^2 = 16\pi a_i^{-1}\xi^{-1}\eta r^2 t^2 + \frac{1}{2} r^2 t^2 - 2b_i^{-2}\eta^{-2}t^6, \] .........................................................(23)

And

\[ p = \frac{1}{16\pi} \left[ 6b_i^{-2}\eta^{-2}r^{-4}t^2 + 2a_i^{-2}\xi^{-2}r^{-4}t^4 + 16\pi a_i^{-1}r^{-2}t^2 - r^{-2}t^{-2} - \frac{32}{3} \pi a_i^{-1}\xi^{-1}\eta r^{-2}t^2 \right], \] ..............................................(24)

and equation (11) gives, making use of relation (23)

\[ e = \frac{1}{8\pi} \left[ -\frac{1}{2} r^{-2}t^{-2} + \left( 3b_i^{-2}\eta^{-2}t^6 - a_i^{-2}\xi^{-2} \right) r^{-4}t^{-4} - 16\pi a_i^{-2}\eta\xi^{-1}r^{-2}t^{-2} \right], \] ..............................................(25)

CASE-II

In this case we take \( u \) to be a function of time only. Then equation (2) gives

\[ y' = fe^\lambda, \] ........................................................................(26)

where \( f \) is an arbitrary function of \( r \).

Now we assume

\[ y' = fe^\lambda = f'g, \] ........................................................................(27)

where 'g' is an arbitrary function of time only

Then,

\[ y = fg, \] .................................................................(28)

and \[ \lambda = \log f' + \log g - \log f, \] .................................................................(29)
Therefore, equation (13) and (14) together give

\[ Q^2 = \tfrac{1}{2} r^4 g^2, \] ...............................................................

(30)

And \( p = \frac{1}{16\pi} \left[ 2g^{-2} - f^{-2} g^{-2} + 48\pi\xi \frac{\dot{g}}{g} e^{-\nu} + 2e^{-2\nu} \left( \frac{2\dot{g}}{g} \nu - 2 \frac{\dot{g}}{g} - \frac{\ddot{g}}{g^2} \right) \right] \) ...............................................(31)

From equation (11), we get

\[ \rho = \frac{1}{8\pi} \left[ 3e^{-2\nu} \frac{\dot{g}^2}{g^2} + \frac{1}{2} f^{-2} g^{-2} - 3g^{-2} \right] \] .......................................................(32)

As a particular example we may take

\[ f = r^2, \quad g = b^2, \quad \nu = 3 \log t, \] ...............................................................(33)

in which case we get

\[ Q^2 = \tfrac{1}{2} r^4 t^4, \] ...............................................................(34)

\[ p = \frac{1}{16\pi} \left[ 2r^{-4} + 8t^{-8} + 96\pi\xi t^{-4} - r^{-4} t^{-4} \right] \] .......................................................(35)

And \( \rho = \frac{1}{8\pi} \left[ 12r^{-4} - \left( \frac{1}{2} r^{-4} - 3 \right) t^{-4} \right] \) .......................................................(36)

CASE – III

Here we take up the case in which the metric assumes the form of the Robertson-Walker type as Robertson-Walker models are possibly most appropriate for a representation of the large-scale structure of the space-time. For that we take.
\[ u = 0, \ e^{2\lambda} = f^2(r) g^2(t), \ y = r.g(t), \] ..........................................................(37)

where \( f(r) \) is a function of \( r \) only and \( g(t) \) is a function of time only.

Now it is seen that relation (37) satisfy the equation (12) automatically. And equations (13), (14) and (11) give

\[ Q^2 = \frac{1}{2} \left[ f^{-2}r^2g^2(f-1) - r^3f^{-3}g^2f' \right] ..........................................................(38) \]

\[ p = \frac{1}{16\pi} \left[ \frac{6\xi g}{g} + r^{-2}f^{-2}g^{-2} - \frac{1}{r}g^{-2}f^{-3}f' - r^{-2}g^{-2} - 4 \frac{\xi}{g} - 2 \frac{g^2}{g^2} \right] ..........................................................(39) \]

\[ \rho = \frac{1}{8\pi} \left[ \frac{1}{2}r^{-2}g^{-2} + \frac{5}{2}r^{-1}g^{-2}f^{-3}f' + 3g^{-2}g^{-1} - \frac{1}{2}r^{-2}g^{-2} \right] ..........................................................(40) \]

As a particular case we study the solution when

\[ f = a_2\eta r^{-1}, \ g = b_2\xi t, ..........................................................(41) \]

where \( a_2 \) and \( b_2 \) are arbitrary constant.

And here,

\[ Q = \frac{1}{2} \left( b_2\xi r t \right)^2, ..........................................................(42) \]

\[ p = \frac{1}{16\pi} \left[ 6\xi t^{-1} + 2\left(a_2b_2\eta \xi t \right)^{-2} - 2t^{-2} - (b_15rt)^{-2} \right] ..........................................................(43) \]

And

\[ \rho = \frac{1}{8\pi} \left[ \frac{1}{2} \left( b_2\xi rt \right)^{-2} + 3t^{-2} - 3\left(a_2b_2\eta \xi t \right)^{-2} \right] ..........................................................(44) \]
CASE IV:

In this case we take up radiating fluid for which

$$\rho = 3 \rho,$$ ...........................................(45)

Here we assume $\nu$ to be a function of time only and take

$$\dot{\lambda} = -\frac{\nu}{y},$$ ...........................................(46)

Then from equations (12), (13) and (14), we obtain

$$2y^{-2} + e^{-2\nu} \left( \frac{\nu}{y} + 2 \frac{\nu^2}{y^2} - \frac{2y \nu}{y} \right) = 24\pi e^{-\nu} \frac{y}{y},$$ ...........................................(47)

A particular solution of equation (47) is

$$y = \eta t,$$ ...........................................(48)

$$\nu = \log (12\pi \xi \eta^2 c^2 t),$$ ...........................................(49)

where $c$ is an arbitrary constant.

Thus Equation (46) gives

$$\lambda = \log (d/t),$$ ...........................................(50)

where $d$ is an arbitrary constant or at most an arbitrary function of $r$.

Therefore making use of the relation (50) in equation (11) (13) and (14), we get

$$p = \frac{1}{48\pi} c^{-2} \xi^{-1} \eta^{-1} \left( \xi - \frac{8}{3} \eta \right)^{-2},$$ ...........................................(51)
3.3 DISCUSSION OF THE RESULTS

In case I, the fluid pressure as well as the fluid density is found to be a decreasing function of the time and the radial distance both. In this case, viscosity has the tendency to decrease the pressure. On the other hand the bulk viscosity has a tendency to enhance the density while the shear viscosity acts the other way. The charge field on the other hand is an increasing function of the time and the radial distance both; however the shear viscosity and the bulk viscosity both have tendencies to decrease the strength of the electric charge. Thus the models here will gradually lose the potentiality of emitting radiation and come to that of a dust era, thus modeling a good example which demonstrates the evolution of the Universe. This model will be able to explain the early Universe in some ways.

In case II, if we take \( f(r) \) and \( g(t) \) to be decreasing functions of \( r \) and \( t \) respectively, then the pressure and the density are found to be increasing functions of the time and the radial distance both whereas the electric charge is found to be a decreasing function of both. The viscosity has an effect to increase the pressure, while both the density and the electric charge are both unaffected by the viscosity. In the special case obtained when \( f(r) \) is an increasing function of the radial distance, and both \( g \) and \( v \) are increasing functions of the time, we see that both the fluid pressure and the fluid density are decreasing functions of \( 'r' \) and \( 't' \) whereas the electric charge is found to be an increasing function of the time and the radial distance both. In this special case also the viscosity has the tendency to increase the fluid pressure, whereas the electric charge and the fluid density are both unaffected by viscosity. Concerning

\[
Q = \left[ \frac{1}{2} c^2 \left( 1 + \frac{8}{3} \xi^{-1} \eta \right) \eta^2 t^2 - 12 \pi \xi \right]^{\frac{1}{2}}, \quad \text{..........(52)}
\]

\[
\rho = \frac{1}{16\pi} \xi^{-1} \eta^{-2} \left( \xi - \frac{8}{3} \eta \right)^{t^{-2}}, \quad \text{..........(53)}
\]
the electric charge there is a singularity at the origin of the epoch and also at the center of the model, and at these instances this model will cease to radiate.

Regarding case III, we see that if we take 'f' and 'g' to be respectively increasing functions of 'r' and 't' then the fluid pressure is found to be a decreasing function of the time and the radial distance both. The fluid density also here behaves in the same way. The electric charge is an increasing function of the time.

In the particular case where 'f' and 'g' are given by the relations (41), the fluid pressure is found to be a decreasing function of the time but an increasing function of the radial distance. Here the viscosity has a tendency to decrease the pressure and the density both. The density also is a decreasing function of the time and the radial distance both. But on the other hand the electric charge is an increasing function of 'r' and 't' both, and the viscosity accelerates the radiation effect. At t = 0 that is at the origin of the epoch the model is seen to have a singularity.

In case IV, we see that the fluid pressure and the fluid density are decreasing functions of the time t and the radial distance r both, whereas the electric charge is an increasing function of the time. Here the viscosity decelerates the pressure and the density both, but it has a tendency to accelerate the electric charge and thereby the radiation. We also see that the relation between the co-efficient of the bulk and shear viscosities are such that 35 cannot be equal to 87; for in that case the charge density become imaginary, and both the pressure and the density cannot exist.