Abstract

The central theme of our work is to explore connections between values of hypergeometric functions and algebraic curves. The theory of classical hypergeometric series has been studied for centuries and their associations with counting points on algebraic curves have been fully explored. In 1980's, Greene introduced the notion of hypergeometric functions over finite fields analogous to classical hypergeometric series. Since then, connections between number of points on elliptic curves and hypergeometric functions over finite fields have been investigated by many mathematicians such as Ahlgren, Frechette, Koike, Ono, and Papanikolas.

Recently, Fuselier gave formulas for traces of Frobenius of certain families of elliptic curves in terms of Gaussian hypergeometric functions involving characters of orders 12 as parameters for primes p satisfying $p \equiv 1 \pmod{12}$. Following her approach, Lennon provided a general formula for the number of $\mathbb{F}_q$-points of an elliptic curve E with $j(E) \neq 0, 1728$ in terms of values of Gaussian hypergeometric series containing characters of order 12 for $q = p^e \equiv 1 \pmod{12}$. Following these, in this dissertation, we present some general formulas connecting the number of points on certain families of elliptic curves given by Weierstrass normal form over $\mathbb{F}_q$ with Gaussian hypergeometric series containing characters of order 6, 4, and 3, separately.

Most recently, Vega considered certain more general families of algebraic curves and expressed the number of $\mathbb{F}_q$-points on those families as a linear combination of ${}_2F_1$ hypergeometric functions. In our work, we have considered two families of algebraic curves, namely $y^\ell = x(x-1)(x-\lambda)$ and $y^\ell = (x-1)(x^2+\lambda)$; and give

explicit formulas for the number of $\mathbb{F}_q$-points on these families as sums of values of ${}_2F_1$ and ${}_3F_2$ Gaussian hypergeometric series, respectively. These formulas generalize certain known results on elliptic curves and Gaussian hypergeometric series. Further, we define period analogue for the algebraic curve $y^\ell = x(x-1)(x-\lambda)$, and obtain an expression for the period analogue in terms of ${}_2F_1$ classical hypergeometric series.

In all the known results connecting Gaussian hypergeometric series and algebraic curves, expressions are obtained in terms of ${}_2F_1$ and ${}_3F_2$ Gaussian hypergeometric series. Hence, the task remained to find similar results for ${}_{n+1}F_n$ Gaussian hypergeometric series for $n \geq 3$. Ahlgren and Ono studied this problem and deduced the value of ${}_4F_3$ hypergeometric series at 1 over $\mathbb{F}_p$ in terms of representations of $4p$ as a sum of four squares using the fact that the Calabi-Yau threefold is modular. For $n > 3$, the non-trivial values of ${}_{n+1}F_n$ Gaussian hypergeometric series are difficult to obtain, and this problem was also mentioned by Ono. We present explicitly the number of distinct zeros of the polynomial $x^d + ax + b$ over $\mathbb{F}_q$ in terms of the Gaussian hypergeometric functions ${}_dF_{d-1}$ and ${}_{d-1}F_{d-2}$ containing characters of orders d and $d-1$ as parameters.

Finally, we deduce certain special values of ${}_2F_1$ and ${}_3F_2$ Gaussian hypergeometric series containing higher order characters as parameters using our results.