A function of a complex variable is said to be analytic at a point if its derivatives exists not only at that point but in some neighbourhood of that point. A single valued function which is defined and differentiable at each point of domain $D$ is said to be analytic in that domain. A single valued function of one complex variable which is analytic in the finite complex plane is called an integral (entire) function. On the other hand a single valued function of one complex variable which has no singularities other than poles in the finite complex plane is called a meromorphic function. For example $\exp z$, $\sin z$, $\cos z$ etc. are examples of entire functions where as $\frac{1}{z-1}$, $\frac{z}{(z-3)^2}$, $\frac{z^3}{(z-5)^2}$ etc. are meromorphic functions. In the Value Distribution theory one studies how an entire or meromorphic function assumes some values and the influence of assuming certain values in some specific manner on a function. In 1926 Rolf Nevanlinna initiated the value distribution theory of entire and meromorphic functions. This value distribution theory is a prominent branch of Complex Analysis and is the prime concern of the thesis. Perhaps the Fundamental Theorem of Classical Algebra which states that “If $f$ is a polynomial of degree $n$ with real or complex coefficients, then the equation $f(z) = 0$ has at least one root” is the most well known value distribution theorem.

The value distribution theory deals with various aspects of the behaviour of
entire and meromorphic functions one of which is the study of comparative growth properties. For any entire function \( f \), \( M(r, f) \), a function of \( r \) is defined as follows:

\[
M(r, f) = \max_{|z|=r} |f(z)|.
\]

Similarly for another entire function \( g \), \( M(r, g) \) is defined. The ratio \( \frac{M(r, f)}{M(r, g)} \) as \( r \to \infty \) is called the growth of \( f \) with respect to \( g \) in terms of their maximum moduli.

An entire function \( f \) has an everywhere convergent power series expansion as

\[
f = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n + \cdots
\]

But for meromorphic \( f \), \( M(r, f) \) is not defined. To overcome this situation, the following theory due to Rolf Nevanlinna may be considered:

A point \( z = \alpha \) is called an \( \alpha \)-point of \( f(z) \) if \( f(\alpha) = a \) where \( a \) is any complex number finite or infinite. Let \( f \) be a meromorphic function in the finite complex plane. Also let \( n(r, \alpha; f) \equiv n(r, \alpha) \) which is a nonnegative integer for each \( r \), denotes the number of \( \alpha \)-points of \( f \) in \( |z| \leq r \), counted with proper multiplicities, for a complex number \( \alpha \) finite or infinite. Obviously \( n(r, \infty; f) \equiv n(r, f) \) represents the number of poles of \( f \) in \( |z| \leq r \) counted with proper multiplicities. The function \( N(r, \alpha; f) = N(r, \alpha) \) is defined as follows:

\[
N(r, \alpha) = \int_0^r \frac{n(t, \alpha) - n(0, \alpha)}{t} dt + n(0, \alpha) \log r \text{ and }
\]

\[
N(r, \infty; f) = N(r, f) .
\]

The term \( N(r, f) \) is called the enumerative function of \( f \).

For distinct zeros of \( f \) the function \( \overline{N}(r, \alpha; f) = \overline{N}(r, \alpha) \) is defined by

\[
\overline{N}(r, \alpha) = \int_0^r \frac{n(t, \alpha) - n(0, \alpha)}{t} dt + n(0, \alpha) \log r \text{ and }
\]

\[
\overline{N}(r, \infty; f) = \overline{N}(r, f) .
\]

Next let us define

\[
\log^+ x = \begin{cases} 
\log x, & \text{if } x \geq 1 \\
0, & \text{if } 0 \leq x < 1 .
\end{cases}
\]

The following properties are then obvious
(i) \( \log^+ x \geq 0 \) if \( x \geq 0 \),

(ii) \( \log^+ x \geq \log x \) if \( x > 0 \),

(iii) \( \log^+ x \geq \log^+ y \) if \( x > y \),

(iv) \( \log x = \log^+ x - \log^+ \frac{1}{x} \) if \( x > 0 \).

The quantity \( m(r, f) \) is defined as follows:

\[
m(r, f) = \frac{1}{2\pi} \int_0^{2\pi} \log^+ |f(re^{i\theta})| \, d\theta.
\]

The term \( m(r, f) \) is called the proximity function of \( f \) and is a sort of average magnitude of \( \log |f(z)| \) on the arcs of \( |z| = r \) where \( |f(z)| \) is large.

Next let us write

\[
T(r, f) = m(r, f) + N(r, f).
\]

The function \( T(r, f) \) is called the Nevanlinna’s Characteristic function of \( f \) [p.4, [20]]. It plays an important role in the theory of meromorphic functions.

Since for any positive integer \( p \) and complex number \( a_\nu \),

\[
\log^+ \left| \prod_{\nu=1}^{p} a_\nu \right| \leq \sum_{\nu=1}^{p} \log^+ |a_\nu| \quad \text{and}
\]

\[
\log^+ \left| \sum_{\nu=1}^{p} a_\nu \right| \leq \log^+ \left( p \cdot \max_{\nu=1,2,\cdots,p} |a_\nu| \right) \leq \sum_{\nu=1}^{p} \log^+ |a_\nu| + \log p,
\]

it is easy to show that [p.5, [20]] for \( p \) meromorphic functions \( f_1, f_2, \cdots f_p \)

\[
m\left( r, \prod_{\nu=1}^{p} f_\nu \right) \leq \sum_{\nu=1}^{p} m(r, f_\nu) \quad \text{and}
\]

\[
m\left( r, \sum_{\nu=1}^{p} f_\nu \right) \leq \sum_{\nu=1}^{p} m(r, f_\nu) + \log p.
\]
Also one can easily verify that

\[
N \left( r, \prod_{\nu=1}^{p} f_{\nu} \right) \leq \sum_{\nu=1}^{p} N \left( r, f_{\nu} \right) \quad \text{and}
\]

\[
N \left( r, \sum_{\nu=1}^{p} f_{\nu} \right) \leq \sum_{\nu=1}^{p} N \left( r, f_{\nu} \right) .
\]

So

\[
T \left( r, \sum_{\nu=1}^{p} f_{\nu} \right) \leq \sum_{\nu=1}^{p} T \left( r, f_{\nu} \right) + \log p \quad \text{and}
\]

\[
T \left( r, \prod_{\nu=1}^{p} f_{\nu} \right) \leq \sum_{\nu=1}^{p} T \left( r, f_{\nu} \right) .
\]

If \( T(r, g) \) denotes the Nevanlinna’s Characteristic function (abbreviated as N.C.F.) of meromorphic \( g \), the ratio \( \frac{T(r, f)}{T(r, g)} \) as \( r \to \infty \) is called the growth of \( f \) with respect to \( g \) in terms of their N.C.Fs.

For any two entire functions \( f \) and \( g \), the composition \( f \circ g \) is defined as

\[
f \circ g (z) = f \left( g (z) \right)
\]

and the composite function will be entire. Similarly for meromorphic \( f \) and entire \( g \), the composition \( f \circ g \) is defined as follows:

\[
f \circ g (z) = f \left( g (z) \right)
\]

and the composite function will be meromorphic. But if the left factor is entire and the right factor is meromorphic the composition cannot be defined.

Now we state the Poisson-Jensen formula \{p.1, [20]\} in the form of the following theorem:

**Theorem 1.0.1** Suppose that \( f \) is meromorphic in \( |z| \leq R \) \((0 < R < \infty)\). Also let \( a_{\mu} \quad (\mu = 1, 2, \cdots M) \) and \( b_{\nu} \quad (\nu = 1, 2, \cdots N) \) denote the zeros and poles of \( f \) respectively in \( |z| < R \). Then if \( z = re^{i\theta} \quad (0 < r < R) \) and if
\( f(re^{i\theta}) \neq 0, \infty \) we have

\[
\log |f(z)| = \frac{1}{2\pi} \int_0^{2\pi} \log |f(Re^{i\phi})| \frac{R^2 - r^2}{R^2 - 2Rr \cos(\theta - \phi) + r^2} d\phi \\
+ \sum_{\mu=1}^M \log \left| \frac{R(z - a_\mu)}{R^2 - a_\mu z} \right| - \sum_{\nu=1}^N \log \left| \frac{R(z - b_\nu)}{R^2 - b_\nu z} \right|.
\]

The theorem holds good also when \( f \) has zeros and poles on \(|z| = R\).

When \( z = 0 \), we obtain Jensen’s formula

\[
\log |f(0)| = \frac{1}{2\pi} \int_0^{2\pi} \log |f(Re^{i\phi})| d\phi + \sum_{\mu=1}^M \log \frac{|a_\mu|}{R} - \sum_{\nu=1}^N \log \frac{|b_\nu|}{R},
\]

provided that \( f(0) \neq 0, \infty \).

If \( f \) has a zero of order \( \lambda \) or a pole of order \(-\lambda\) at \( z = 0 \) such that \( f = c_\lambda z^\lambda + \cdots \) then Jensen’s formula takes the form

\[
\log |c_\lambda| = \frac{1}{2\pi} \int_0^{2\pi} \log |f(Re^{i\phi})| d\phi + \sum_{\mu=1}^M \log \frac{|a_\mu|}{R} - \sum_{\nu=1}^N \log \frac{|b_\nu|}{R} - \lambda \log R.
\]

The complicated modification is one of the minor irritations of the theory.

Generally we shall assume that our function behave in such a way that the terms in the Jensen’s formula do not become infinite in our use of that formula knowing that exceptional cases can be treated.

When \( f \) has no \( a \) - points (i.e., the roots of the equation \( f = a \)) at \( z = 0 \), then it follows from Riemann-Stieltjes integral that

\[
\sum_{0 < a_\nu \leq r} \log \frac{r}{|a_\nu|} = \int_0^r \frac{n(t, a)}{t} dt,
\]

where \( a_\nu \) ’s are the \( a \) - points of \( f \) in \(|z| \leq r\).

Again since \( N(r, 0; f) = N\left(r, \frac{1}{f}\right) \), from Jensen’s formula we get that

\[
\log |f(0)| = m(R, f) - m\left(R, \frac{1}{f}\right) + N(R, f) - N\left(R, \frac{1}{f}\right)
\]

i.e., \( T(R, f) = T\left(R, \frac{1}{f}\right) + \log |f(0)| \).
For any finite complex number \( a \), let us denote by \( m(r, a) \) the function \( m\left(r, \frac{1}{f-a}\right) \) and \( m(r, \infty) = m(r, f) \). Now we express Nevanlinna’s First Fundamental theorem in the following form:

**Theorem 1.0.2** (\{p.6,\[20]\}) *Let* \( f \) *be a meromorphic function in* \( |z| < \infty \) *and* \( a \) *be any complex number finite or infinite, then*

\[
m(r, a) + N(r, a) = T(r, f) + O(1).
\]

This result shows that the remarkable symmetry exhibited by a meromorphic function in its behaviour relative to different complex number \( a \), finite or infinite. The sum \( m(r, a) + N(r, a) \) for different values of \( a \) maintains a total, given by the quantity \( T(r, f) \) which is invariant up to a bounded additive term involving \( r \).

One part of this invariant sum, the quantity \( N(r, a; f) \) hints how densely the roots of the equation \( f = a \) are distributed in the average in the disc \( |z| < r \). The larger the number of \( a - points \) the faster this counting function for \( a - points \) grows with \( r \).

The first term \( m(r, a) \) which is defined to be the mean value of \( \log^+ \left| \frac{1}{f-a} \right| \) (or \( \log^+ |f| \) if \( a = \infty \)) on the circle \( |z| = r \), receives a remarkable contribution only from those arcs on the circle where the functional values differ very little from the given value \( a \). The magnitude of the proximity function can thus be considered as a measure for the mean deviation on the circle \( |z| = r \) of the functional value \( f \) from the value \( a \).

If the \( a - points \) of a meromorphic function are relatively scarce for a certain \( a \), this fact finds expression analytically in the relatively slow growth of the function \( N(r, a) \) as \( r \to \infty \); in the extreme case where \( a \) is a Picard’s exceptional value of the function (so that \( f \neq a \) in \( |z| < \infty \)), \( N(r, a) \) is identically zero. But this fact on \( a - points \) finds a compensation:

The function deviates in the mean slightly from the value \( a \) in question, the corresponding proximity function \( m(r, a) \) will be relatively large, so that the sum \( m(r, a) + N(r, a) \) reaches the magnitude \( T(r, f) \), the characteristic function of \( f \).

For an entire function \( f \), \( N(r, f) = 0 \) and so \( T(r, f) = m(r, f) \), *i.e.*, in the case of an entire function, the *Nevanlinna’s characteristic* function and the proximity function are same.
Let us consider that \( f \) be an entire function, \( i.e. \), a function of a complex variable regular in the whole finite complex plane. By Taylor’s theorem such a function has an everywhere convergent power series expansion as
\[
f = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n + \cdots \tag{1.0.1}
\]
which forms a natural generalization of the polynomials.

The degree of a polynomial which is equal to its number of zeros estimates the rate of growth of the polynomial as the independent variable moves without bound. So the more zeros, the greater is the growth.

An analogous property that relate the set of zeros and the growth of a function can be developed for arbitrary entire functions.

Establishing relations between the distribution of the zeros of an entire function and its asymptotic behaviour as \( z \to \infty \) enriched most of the classical results of the theory of entire functions. The classical investigations of Borel, Hadamard and Lindelof are of this kind.

To characterise the growth of an entire function and the distribution of its zeros a special growth scale called maximum modulus function of \( f \) on \( |z| = r \) is introduced as \( M (r) \equiv M (r, f) = \max_{|z|=r} |f (z)| \).

It plays an important role in the theory of entire functions. Since by Liouville’s theorem a bounded entire function is constant, it follows that for non-constant \( f \) the maximum modulus function \( M (r) \) is unbounded.

The following theorem is due to Cauchy:

**Theorem 1.0.3** (**Theorem 1, p.5, [38])** The maximum of the modulus of a function \( f \), which is regular in a closed connected region \( D \), bounded by one or more curves \( C \), is attained on the boundary.

This theorem implies that when \( f \) is an entire function, \( M (r) \) is a non-decreasing function of \( r \) for all values of \( r \). Using the uniform continuity of \( f \) in any closed region and the above theorem, \( i.e. \), the value \( M (r) \) is attained by \( f \) on \( |z| = r \), it follows that \( M (r) \) is a continuous function of \( r \). Also \( M (r) \) is differentiable in adjacent intervals {Theorem 10, p.27, [38]}. In view of Hadamard’s theorem {Theorem 9, p.20, [38]} we know that \( \log M (r) \) is a continuous, convex and ultimately increasing function of \( \log r \).

For an entire function \( f \) the study of the comparative growth properties of \( T (r, f) \) and \( \log M (r, f) \) is a popular problem among the researchers.

Now we express a fundamental inequality relating \( T (r, f) \) and \( \log M (r, f) \).
Theorem 1.0.4 ({p. 18, [20]}) If \( f \) is regular for \( |z| \leq R \) then

\[
T(r, f) \leq \log^+ M(r, f) \leq \frac{R + r}{R - r} T(R, f), \quad 0 \leq r < R.
\]

In case of a transcendental entire function \( f \), \( M(r) \) grows faster than any positive power of \( r \). Thus in order to estimate the growth of transcendental entire functions we choose a comparison function \( e^{r^k} \), \( k > 0 \) that grows more rapidly than any positive power of \( r \).

More precisely \( f \) is said to be a function of finite order if there exists a positive constant \( k \) such that \( \log M(r) < r^k \) for all sufficiently large values of \( r \) (\( r > r_0(k) \), say). The infimum of such \( k \)'s is called the order of \( f \). If no such \( k \) exists, \( f \) is said to be of infinite order.

For example, the order of the function \( e^z \) is 1 i.e., finite but that of \( e^{e^z} \) is infinite.

Let \( \rho \) be the order of \( f \). It can be easily shown that the order \( \rho \) of \( f \) has the following alternative definition

\[
\rho = \limsup_{r \to \infty} \frac{\log \log M(r)}{\log r}.
\]

The lower order \( \lambda \) of \( f \) is defined as follows

\[
\lambda = \liminf_{r \to \infty} \frac{\log \log M(r)}{\log r}.
\]

Clearly \( \lambda \leq \rho \).

If in particular for a function \( f \), \( \lambda = \rho \), then \( f \) is said to be of regular growth.

For example, a polynomial or the functions \( e^z, \cos z \) etc. are of regular growth.

With known order \( \rho (0 < \rho < \infty) \) the growth of an entire function can be characterised more precisely by the type of the function. The number \( \tau \) given by

\[
\tau = \limsup_{r \to \infty} \frac{\log M(r)}{r^\rho}, \quad 0 < \rho < \infty
\]

is called the type of \( f \).

Between two functions of same order one can be characterised to be of greater growth if its type is greater The quantities \( \rho, \lambda \) and \( \tau \) are extensively used to the study of growth properties of \( f \). At this stage we note the following definition:
**Definition 1.0.1** (**p. 16, [20]**) Let \( S \) be a real and non-negative function increasing for \( 0 < r_0 < r < \infty \), The order \( k \) and lower order \( \lambda \) of the function \( S(r) \) are defined as

\[
k = \limsup_{r \to \infty} \frac{\log S(r)}{\log r}
\]

and

\[
\lambda = \liminf_{r \to \infty} \frac{\log S(r)}{\log r}.
\]

Moreover if \( 0 < k < \infty \), we set

\[
c = \limsup_{r \to \infty} \frac{S(r)}{r^k}
\]

and distinguish the following possibilities:

(a) \( S(r) \) has maximal type if \( c = +\infty \);

(b) \( S(r) \) has mean type if \( 0 < c < +\infty \);

(c) \( S(r) \) has minimal type if \( c = 0 \);

(d) \( S(r) \) has convergence class if

\[
\int_{r_0}^{\infty} \frac{S(t)}{t^{k+1}} dt \text{ converges}.
\]

Now we state the following theorem:

**Theorem 1.0.5** (**p. 18, [20]**) If \( f \) is an entire function then the order \( k \) of the function \( S_1(r) = \log^+ M(r, f) \) and \( S_2(r) = T(r, f) \) is the same. Further if \( 0 < k < \infty \), \( S_1(r) \) and \( S_2(r) \) belong to the same classes (a), (b), (c) and (d).

Also we note that \( S_1(r) \) and \( S_2(r) \) have the same lower order.

A function \( f \) meromorphic in the plane is said to have order \( \rho \), lower order \( \lambda \) and maximal, minimal, mean type or convergence class if the function \( T(r, f) \) has this property. For entire functions these coincide by the above theorem.
with the corresponding definition in terms of $M(r,f)$ which is classical. The type of a meromorphic function $f$ is defined by

$$
\tau = \limsup_{r \to \infty} \frac{T(r,f)}{r^\rho}, \quad 0 < \rho < \infty.
$$

We know that the order and the lower order of an entire function $f$ and its derivative are equal. The same result holds for a meromorphic function also.

After revealing the important symmetry property of a meromorphic function $f$, which is expressed in the first fundamental theorem through the invariance of the sum $m(r,a) + N(r,a)$, it is natural to attempt for a more careful investigation of the relative strength of two terms in the sum, of the proximity component $m(r,a)$ and of the counting component $N(r,a)$. Individual results have been obtained in this direction:

1. Picard’s theorem shows that the counting function for a non constant meromorphic function in the finite complex plane vanish for at most two values of $a$.

2. For a meromorphic function of finite non-integral order there is at most one Picard’s exceptional value.

3. That the counting function $N(r,a)$ is in general i.e., for the great majority of the values of $a$, large in comparison with the proximity function.

We now state Nevanlinna’s Second Fundamental theorem.

**Theorem 1.0.6 (p. 31, [20])** Suppose that $f$ is a non-constant meromorphic function in $|z| \leq r$. Let $a_1, a_2, \cdots, a_q (q > 2)$ be distinct finite complex numbers, $\delta > 0$ and suppose that $|a_\mu - a_\nu| \geq \delta$ for $1 \leq \mu < \nu \leq q$. Then

$$
m(r, \infty) + \sum_{\nu=1}^{q} m(r,a_\nu) \leq 2T(r,f) - N_1(r) + S(r),
$$

where $N_1(r)$ is positive and is given by

$$
N_1(r) = N\left(r, \frac{1}{f'}\right) + 2N(r,f) - N(r,f') \quad \text{and}
$$

$$
S(r) = m\left(r, \frac{f'}{f}\right) + m\left\{ r, \sum_{\nu=1}^{q} \frac{f'}{f - a_\nu} \right\} + q \log \frac{3q}{\delta} + 2 \log + \log \frac{1}{|f'(0)|},
$$

with modifications if $f(0) = \infty$ or $f'(0) = 0$. 
The quantity $S(r)$ will in general play the role of an unimportant error term. The combination of this fact with the above theorem yields the second fundamental theorem.

The following theorem gives an estimation of $S(r)$:

**Theorem 1.0.7** (p. 34, [20]) Let $f$ be a meromorphic function and not constant in $|z| < R_0 \leq \infty$ and that $S(r) \equiv S(r, f)$ is defined as in the above theorem. Then we have

(i) If $R_0 = +\infty$, $S(r, f) = O \{\log T(r, f)\} + O(\log r)$ as $r \to \infty$ through all values if $f$ has finite order and as $r \to \infty$ outside a set $E$ of finite linear measure otherwise.

(ii) If $0 < R_0 < \infty$, $S(r, f) = O \left\{ \log^+ T(r, f) + \log \frac{1}{(R_0 - r)} \right\}$ as $r \to R_0$ outside a set $E$ such that $\int_E \frac{dr}{R_0 - r} < +\infty$.

Further there is a point $r$ outside $E$ for which $\rho < r < \rho'$ provided that $0 < R - \rho' < e^{-2}(R - \rho)$.

Consequently we get the following theorem:

**Theorem 1.0.8** (p. 41, [20]) Let $f$ be meromorphic and non-constant in $|z| \leq R_0$. Then

$$\frac{S(r, f)}{T(r, f)} \to 0$$

(*)

as $r \to R_0$ with the following provisions:

(a) $(*)$ holds without restrictions if $R_0 = +\infty$ and $f$ is of finite order in the plane.

(b) If $f$ has infinite order in the plane, $(*)$ still holds as $r \to \infty$ outside a certain exceptional set $E_0$ of finite length.

Here $E_0$ depends only on $f$.

(c) If $R_0 < +\infty$ and $\limsup_{r \to \infty} \frac{T(r, f)}{\log (R_0 - r)} = +\infty$, then $(*)$ holds as $r \to R_0$ through a suitable sequence $r_n$, which depends on $f$ only.
This theorem points out why $S(r)$ plays the role of an unimportant error term.

Let $f$ be meromorphic and not constant in the plane. We shall denote by $S(r, f)$ any quantity $S(r, f) = o\{T(r, f)\}$ as $r \to \infty$ possibly outside a set $E$ of finite linear measure. Also we shall denote by $a, a_0, a_1$ etc. functions meromorphic in the plane and satisfying $T(r, a) = S(r, f)$ as $r \to \infty$. Now we introduce Milloux’s theorem which is important in studying the properties of the derivatives of meromorphic functions.

**Theorem 1.0.9 ([p. 55, [20]])** Let $l$ be a positive integer and $\psi = \sum_{\nu=0}^{l} a_{\nu} f^{(\nu)}$.

Then $m\left(r, \frac{\psi}{T}\right) = S(r, f)$ and $T(r, \psi) \leq (l + 1) T(r, f) + S(r, f)$.

Milloux showed that in the second fundamental theorem we can replace the counting functions for certain roots of $f = a$ by roots of the equation $\psi = b$, where $\psi$ is given as in the above theorem. In this connection we state the following theorem:

**Theorem 1.0.10 ([p. 57, [20]])** Let $f$ be meromorphic and non constant in the plane and $\psi = \sum_{\nu=0}^{l} a_{\nu} f^{(\nu)}$, where $l$ is a positive integer. If $\psi$ is non constant then

$$T(r, f) < \overline{N}(r, f) + N\left(r, \frac{1}{f}\right) + \overline{N}\left(r, \frac{1}{\psi - 1}\right) - N_0\left(r, \frac{1}{\psi'}\right) + S(r, f)$$

where in $N_0\left(r, \frac{1}{\psi}\right)$ only zeros of $\psi'$ not corresponding to the repeated roots of $\psi = 1$ are to be considered.

Now we set

$$\delta(a) = \delta(a; f) = \liminf_{r \to \infty} \frac{m(r, a)}{T(r, f)} = 1 - \limsup_{r \to \infty} \frac{N(r, a)}{T(r, f)},$$

$$\Theta(a) = \Theta(a; f) = 1 - \limsup_{r \to \infty} \frac{\overline{N}(r, a)}{T(r, f)},$$

where $\overline{N}(r, a, f) \equiv \overline{N}(r, a)$ is the counting function for distinct $a - points$, 

$$\theta(a) = \theta(a; f) = \liminf_{r \to \infty} \frac{N(r, a) - \overline{N}(r, a)}{T(r, f)}.$$
Evidently, given \( \varepsilon ( > 0 ) \), we have for sufficiently large values of \( r \),

\[
N ( r, a ) - \overline{N} ( r, a ) > \{ \theta ( a ) - \varepsilon \} T ( r, f ) ,
\]

\[
N ( r, a ) < \{ 1 - \delta ( a ) + \varepsilon \} T ( r, f )
\]

and hence

\[
\overline{N} ( r, a ) < \{ 1 - \delta ( a ) - \theta ( a ) + 2 \varepsilon \} T ( r, f )
\]

so that

\[
\Theta ( a ) \geq \delta ( a ) + \theta ( a ) .
\]

The quantity \( \delta ( a ) \) is known as the deficiency of the value \( a \) and \( \theta ( a ) \) is called the index of multiplicity. Evidently \( \delta ( a ) \) is positive only if there are relatively few roots of the equation \( f = a \), while \( \theta ( a ) \) is positive if there are relatively many multiple roots.

Let us now state a fundamental theorem called Nevanlinna’s theorem on deficient values.

**Theorem 1.0.11** ([p.43, [20]]) Let \( f \) be a non constant meromorphic function defined on the plane. Then the set of values \( a \) for which \( \Theta ( a ) > 0 \) is countable and we have on summing over all such values \( a \)

\[
\sum_a \{ \delta ( a ) + \theta ( a ) \} \leq \sum_a \Theta ( a ) \leq 2 .
\]

The magnitude of the deficiency \( \delta ( a ) \) lies in the closed unit interval \([0, 1]\) and it gives us a very accurate measure for the relative density of the points where the function \( f \) assumes the value \( a \) in question. The larger the deficiency is, the more rare are latter points. The deficiency reaches its maximum value 1 when the latter have been very sparsely distributed, as for example, in the extreme case where the value \( a \) is a Picard exceptional value i.e., a complex number which is not assumed by the function \( f \). We shall call every value of vanishing deficiency \( \delta ( a ) \), a normal value in contrast to the deficient values for which \( \delta ( a ) \) is positive.

We know from Picard’s theorem that a meromorphic function can have at most two Picard exceptional values. This theorem follows easily from Nevanlinna’s theorem on deficient values because as we have stated before that for a Picard exceptional value \( a \), \( \delta ( a ) = 1 \).

The quantity \( \Delta ( a ; f ) = 1 - \liminf_{r \to \infty} \frac{N(r,a)}{T(r,f)} = \limsup_{r \to \infty} \frac{m(r,a)}{T(r,f)} \) gives another measure of deficiency and is called the Valiron deficiency.
Clearly $0 \leq \delta (a; f) \leq \Delta (a; f) \leq 1$.

The value distribution theory due to R. Nevanlinna has immense applications in Number Theory, Probability and Statistics, Theoretical Physics, Complex Dynamics, Factorization of meromorphic functions, Complex Differential Equations etc. In the thesis we have not deeply investigated the above applications of the Nevanlinna theory. For details, one may see [4].

Apart from **Chapter 1**, the thesis consists of **Eight** chapters.

- **Chapter 2** is concerned with the comparative growth properties of the composite entire or meromorphic functions and differential polynomials generated by one of the factors improving some earlier results. The results of this chapter have been published in the *International Mathematical Forum*, see [13].

- In **Chapter 3** we study some growth rates of composite entire and meromorphic functions and special type of differential polynomials generated by one of the factors improving some earlier results. The results of this chapter have been published in the *International Journal of Mathematical Analysis*, see [14].

- In **Chapter 4** we study the comparative growth properties of a special type of differential polynomial generated by entire or meromorphic functions on the basis of $L - (p, q)$th order ($L - (p, q)$th lower order) and $L^* - (p, q)$th order ($L^* - (p, q)$th lower order) where $L \equiv L(r)$ is a slowly changing function and $p, q$ are positive integers with $p > q$. The results of this chapter have been published in the *Proceedings of the UGC Sponsored National Seminar on Recent Advances in the Application of Mathematical Analysis and Computational Techniques in Applied Sciences* held at the Department of Mathematics, Siliguri College, P.O. Siliguri, Dist. Darjeeling, West Bengal, India during December 2nd - 4th, 2011, see [12].

- In **Chapter 5** we compare the relative Valiron defect with the relative Nevanlinna defect of a meromorphic function or a transcendental meromorphic function on the basis of sharing of values of them. The results of this chapter have been published in the *International Journal of Contemporary Mathematical Sciences*, see [15].
• **Chapter 6** deals with the comparison of the relative \((k, n)\) Valiron defect with relative Nevanlinna defect of a meromorphic function where \(k\) and \(n\) are both non-negative integers. The results of this chapter have been published in the *International Journal of Advanced Scientific Research and Technology*, see [16].

• In **Chapter 7** we consider several meromorphic functions having common roots and find some new relations involving their relative defects. The results of this chapter have been published in the *International Journal of Advanced Scientific and Technical Research*, See[17].

• In **Chapter 8** we compare the relative Valiron defect with the relative Nevanlinna defect of special type of differential polynomials generated by transcendental meromorphic functions. The results of this chapter have been published in the *International Journal of Mathematical Archive*, see[18].

• In **Chapter 9** we wish to compare the relative Valiron defect with the relative Nevanlinna defect of wronskians generated by transcendental meromorphic functions. The results of this chapter have been published in the *International Journal of Mathematical Manuscripts*, see [19].

**Future Prospects of the research work presented in the thesis:**

In 1982, in search for special algebras, Corrado Segre [32] published a paper in which he treated an infinite family of algebras whose elements are commutative generalization of complex numbers called bicomplex numbers, tricomplex numbers etc.

Segre[32] defined a bicomplex number as

\[
\xi = x_0 + i_1x_1 + i_2x_2 + i_1i_2x_3,
\]

where \(x_0, x_1, x_2, x_3\) are real numbers, \(i_1^2 = i_2^2 = -1\) and \(i_1i_2 = i_2i_1\). The set of bicomplex numbers is denoted as \(C_2\). In the theory of bicomplex numbers, the set of real numbers and complex numbers are denoted as \(C_0\) and \(C_1\) respectively. Thus

\[
C_2 = \{\xi : \xi = a_0 + i_1a_1 + i_2a_2 + i_1i_2a_3, \ a_0, a_1, a_2, a_3 \in C_0\}
\]
or

\[ C_2 = \{ \xi : \xi = z_1 + i_2 z_2, \ z_1, z_2 \in C_1 \}. \]

There are two non-trivial idempotent elements in \( C_2 \), denoted by \( e_1 \) and \( e_2 \) and are defined as

\[
e_1 = \frac{1 + i_1 i_2}{2}, \ e_2 = \frac{1 - i_1 i_2}{2}; \ e_1 + e_2 = 1 \text{ and } e_1 e_2 = e_2 e_1 = 0.
\]

Every element of \( C_2 \) can be uniquely expressed as a complex combination of \( e_1 \) and \( e_2 \), viz.

\[
\xi = z_1 + i_2 z_2 = (z_1 - i_1 z_2) e_1 + (z_1 + i_1 z_2) e_2.
\]

This representation of a bicomplex number is known as the Idempotent Representation of \( \xi \). Further, the complex coefficients \((z_1 - i_1 z_2)\) and \((z_1 + i_1 z_2)\) are called the Idempotent Components of bicomplex number \( \xi = z_1 + i_2 z_2 \).

The value distribution theory of entire and meromorphic functions especially comparative growth properties of differential monomials, differential polynomials generated by entire or meromorphic functions as well as the estimation of different kinds of relative deficiencies of meromorphic functions presented in the thesis may be treated through the analysis of Bicomplex numbers as stated above and is still a further area of research.

When we write Theorem \( a.b.c \) (or Corollary \( a.b.c \) or Equation \( a.b.c \) etc.) where \( a, b \) and \( c \) are positive integers, we mean the \( c \) th theorem (or \( c \) th corollary or \( c \) th equation etc.) in the \( b \) th section of the \( a \) th chapter. Individual chapters have been presented in such a way that they are more or less independent of the other chapters. The references to books and journals have been classified as bibliography and are given at the end of the thesis.

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