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FEW RESULTS ON CAUCHY’S PROPER BOUND FOR THE ZEROS OF ENTIRE FUNCTIONS OF ORDER ZERO

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Abstract
The aim of this paper is to deduce the bounds for the moduli of zeros of entire functions of order zero. Some examples are provided to clear the notions.

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1 Introduction, Definitions and Notations.
Let

\[ P(z) = a_0 + a_1z + a_2z^2 + a_3z^3 + \ldots + a_{n-1}z^{n-1} + a_nz^n ; |a_n| \neq 0 \]


In this paper we intend to establish some sharper results concerning the theory of distribution of zeros of entire functions of order zero.

The following definitions are well known:

**Definition 1** The order \( \rho \) and lower order \( \lambda \) of a meromorphic function \( f \) are defined as

\[ \rho = \limsup_{r \to \infty} \frac{\log T(r,f)}{\log r} \quad \text{and} \quad \lambda = \liminf_{r \to \infty} \frac{\log T(r,f)}{\log r}. \]

If \( f \) is entire, one can easily verify that

\[ \rho = \limsup_{r \to \infty} \frac{\log [^2]M(r,f)}{\log r} \quad \text{and} \quad \lambda = \liminf_{r \to \infty} \frac{\log [^2]M(r,f)}{\log r}. \]
where \( \log^{[k]} x = \log(\log^{[k-1]} x) \) for \( k = 1, 2, 3, \ldots \) and \( \log^{[0]} x = x \).

If \( \rho < \infty \) then \( f \) is of finite order. Also \( \rho = 0 \) means that \( f \) is of order zero. In this connection Datta and Biswas [3] gave the following definition:

**Definition 2** Let \( f \) be a meromorphic function of order zero. Then the quantities \( \rho^* \) and \( \lambda^* \) of \( f \) are defined by:

\[
\rho^* = \limsup_{r \to \infty} \frac{T(r,f)}{\log r} \quad \text{and} \quad \lambda^* = \liminf_{r \to \infty} \frac{T(r,f)}{\log r}.
\]

If \( f \) is an entire function then clearly

\[
\rho^* = \limsup_{r \to \infty} \frac{\log M(r,f)}{\log r} \quad \text{and} \quad \lambda^* = \liminf_{r \to \infty} \frac{\log M(r,f)}{\log r}.
\]

**2 Lemmas.**

In this section we present a lemma which will be needed in the sequel.

**Lemma 1** If \( f(z) \) is an entire function of order \( \rho = 0 \), then for every \( \varepsilon > 0 \) the inequality

\[
N(r) \leq (\log r)^{\rho^* + \varepsilon}
\]

holds for all sufficiently large \( r \) where \( N(r) \) is the number of zeros of \( f(z) \) in \( |z| \leq \log r \).

**Proof.** Let us suppose that \( f(z) = 1 \). This supposition can be made without loss of generality because if \( f(z) \) has a zero of order \( m' \) at the origin then we may consider \( g(z) = c \frac{f(z)}{z^m} \) where \( c \) is so chosen that \( g(0) = 1 \). Since the function \( g(z) \) and \( f(z) \) have the same order therefore it will be unimportant for our investigations that the number of zeros of \( g(z) \) and \( f(z) \) differ by \( m \).

We further assume that \( f(z) \) has no zeros on \( |z| = \log 2r \) and the zeros \( z_i \)'s of \( f(z) \) in \( |z| < \log r \) are in non decreasing order of their moduli so that \( |z_i| \leq |z_{i+1}| \). Also let \( \rho^* \) suppose to be finite where \( \rho = 0 \) is the zero of order of \( f(z) \).

Now we shall make use of Jenson’s formula as state below

\[
\log|f(0)| = - \sum_{i=1}^{n} \log \frac{R}{|z_i|} + \frac{1}{2\pi} \int_{0}^{2\pi} \log|f(R e^{i\phi})| \, d\phi.
\]

Let us replace \( R \) by \( 2r \) and \( n \) by \( N(2r) \) in (1).

\[
\therefore \log|f(0)| = - \sum_{i=1}^{N(2r)} \log \frac{2r}{|z_i|} + \frac{1}{2\pi} \int_{0}^{2\pi} \log|f(2r e^{i\phi})| \, d\phi.
\]

Since \( f(0) = 1 \), \( \therefore \log|f(0)| = \log 1 = 0 \).

\[
\therefore \sum_{i=1}^{N(2r)} \log \frac{2r}{|z_i|} = \frac{1}{2\pi} \int_{0}^{2\pi} \log|f(2r e^{i\phi})| \, d\phi.
\]
\[
\text{L.H.S.} = \sum_{i=1}^{N(2r)} \log_{|z_i|}^{2r} \geq \sum_{i=1}^{N(r)} \log_{|z_i|}^{2r} \geq N(r) \log 2 \quad (3)
\]

because for large values of \( r \),

\[
\log_{|z_i|}^{2r} \geq \log 2 .
\]

\[
\text{R.H.S.} = \frac{1}{2\pi} 2\pi \int_0^{2\pi} \log|f(2r e^{i\phi})| \, d\phi
\]

\[
\leq \frac{1}{2\pi} 2\pi \int_0^{2\pi} \log M(2r) \, d\phi = \log M(2r) .\quad (4)
\]

Again by definition of order \( \rho^* \) of \( f(z) \) we have for every \( \varepsilon > 0 \),

\[
\log M(2r) \leq (\log(2r))^{\rho^*+\varepsilon/2} .\quad (5)
\]

Hence from (2) by the help of (3), (4) and (5) we have

\[
N(r) \log 2 \leq (\log 2r)^{\rho^*+\varepsilon/2}
\]

i.e.,

\[
N(r) \leq \frac{(\log 2)^{\rho^*+\varepsilon/2}}{\log 2} \cdot \frac{(\log r)^{\rho^*+\varepsilon}}{(\log r)^{\varepsilon/2}} \leq (\log r)^{\rho^*+\varepsilon} .
\]

This proves the lemma.

3 Theorems.

In this section we present the main results of the paper.

Theorem 1 Let \( P(z) \) be an entire function defined as

\[
P(z) = a_0 + a_1 z + \ldots + a_n z^n + \ldots
\]

whose order \( \rho = 0 \). Also for all sufficiently large \( r \) in the disc \( |z| \leq \log r \), \( a_0 \neq 0 \) and \( a_{N(r)} \neq 0 \). Also \( a_n \to 0 \) as \( n > N(r) \). Then all the zeros of \( P(z) \) lie in the ring shaped region

\[
\frac{1}{t_0} \leq |z| \leq t_0
\]

where \( t_0 \) and \( t_0' \) are the positive roots of the equations

\[
g(t) \equiv |a_{N(r)}| t^{N(r)} - |a_{N(r)-1}| t^{N(r)-1} - \ldots - |a_0| = 0
\]

and

\[
h(t) \equiv |a_0| t^{N(r)} - |a_1| t^{N(r)-1} - \ldots - |a_{N(r)}| = 0.
\]

respectively in \( |z| \leq \log r \) and \( N(r) \) denotes the number of zeros of \( P(z) \) in \( |z| \leq \log r \) for sufficiently large \( r \).
Proof. Since \( P(z) \) is an entire function of order \( \rho = 0 \), then from Lemma 1 we have for sufficiently large \( r \) in the disc \( |z| \leq \log r \),

\[
N(r) \leq (\log r)^{\rho+\varepsilon} \quad \text{for} \quad \varepsilon > 0.
\]

Also \( a_0 \neq 0 \) and \( a_{N(r)} \neq 0 \). Further \( a_n \to 0 \) as \( n > N(r) \).

Hence we have

\[
P(z) = a_0 + a_1 z + \cdots + a_n z^n + \cdots \\
\approx a_0 + a_1 z + \cdots + a_{N(r)} z^{N(r)}.
\]

Therefore

\[
|P(z)| \approx \left| a_0 + a_1 z + \cdots + a_{N(r)} z^{N(r)} \right| \\
\geq \left| a_{N(r)} \right||z|^{N(r)} - \left| a_{N(r)-1} \right||z|^{N(r)-1} - \cdots - |a_0|
\] (6)

in the disc \( |z| \leq \log r \) for sufficiently large \( r \). In fact (6) can be deduced in the following way

\[
|a_0 + \cdots + a_{N(r)-1} z^{N(r)-1}| \leq |a_0| + \cdots + |a_{N(r)-1}| |z|^{N(r)-1}
\]

i.e., \(-|a_0| - \cdots - |a_{N(r)-1}| |z|^{N(r)-1} \leq -|a_0| + \cdots + a_{N(r)-1} z^{N(r)-1}|.\)

Hence

\[
|a_{N(r)} z^{N(r)} + a_{N(r)-1} z^{N(r)-1} + \cdots + a_0| \geq |a_{N(r)}||z|^{N(r)} - \left| a_{N(r)-1} \right| |z|^{N(r)-1} - \cdots - |a_0|
\]

\[
\geq |a_{N(r)}||z|^{N(r)} - \left| a_{N(r)-1} \right| |z|^{N(r)-1} - \cdots - |a_0|.
\]

Now let us write

\[
g(t) \equiv |a_{N(r)}| t^{N(r)} - \left| a_{N(r)-1} \right| t^{N(r)-1} - \cdots - |a_0|.
\] (7)

Since (7) has one change of sign, by Descartes’ rule of sign, the maximum number of positive root of (7) is one. Moreover

\[
g(0) = -|a_0| < 0
\]

and \( g(\infty) \) is a positive quantity.

Clearly \( t > t_0 \) implies \( g(t) > 0 \).

If not, let for some \( t_1 > t_0, g(t_1) < 0 \).

Then \( g(t) = 0 \) has another positive root in \((t_1, \infty)\) which gives a contradiction. Hence \( g(t) > 0 \) for \( t > t_0 \).

Therefore \( |P(z)| > 0 \) for \( |z| > t_0 \). So \( P(z) \) does not vanish in \( |z| > t_0 \) and therefore all the zeros of \( P(z) \) lie in \( |z| \leq t_0 \) where \( t_0 \) is the positive root of

\[
g(t) \equiv |a_{N(r)}| t^{N(r)} - \left| a_{N(r)-1} \right| t^{N(r)-1} - \cdots - |a_0| = 0.
\]
Now we give the proof of the other part of the theorem.

Let us consider

$$Q(z) = z^{N(r)} P \left( \frac{1}{z} \right)$$

for sufficiently large $r$ in the disc $|z| \leq \log r$. Now

$$Q(z) = z^{N(r)} \left\{ a_0 + \frac{a_1}{z} + \ldots + a_{N(r)} \frac{1}{z^{N(r)}} \right\}$$

$$= a_0 z^{N(r)} + a_1 z^{N(r)-1} + \ldots + a_{N(r)}. \quad (8)$$

Again we have

$$|a_1 z^{N(r)-1} + \ldots + a_{N(r)}| \leq |a_1| |z|^{N(r)-1} + |a_2| |z|^{N(r)-2} + \ldots + |a_{N(r)}|$$

i.e., $-|a_1| |z|^{N(r)-1} - \ldots - |a_{N(r)}| \leq -|a_1 z^{N(r)-1} + \ldots + a_{N(r)}|.$

So we get that

$$|a_0 z^{N(r)} + \ldots + a_{N(r)}| \geq |a_0| |z|^{N(r)} - |a_1 z^{N(r)-1} + \ldots + a_{N(r)}|$$

$$\geq |a_0| |z|^{N(r)} - |a_1| |z|^{N(r)-1} - \ldots - |a_{N(r)}|. \quad (9)$$

Let us consider the equation

$$h(t) \equiv |a_0| t^{N(r)} - |a_1| t^{N(r)-1} - \ldots - |a_{N(r)}| = 0. \quad (11)$$

Since (11) has one change of sign, by Descartes’ rule of sign the maximum number of positive root of (11) is one. Moreover

$$h(0) = -|a_{N(r)}| < 0$$

and $h(\infty)$ is a positive quantity. So $h(t)$ has exactly one positive root.

Let $t_0'$ be the positive root of $h(t) = 0$. Clearly for $t > t_0'$ we get $h(t) > 0$. If not, let $t_1' > t_0'$. Then $h(t_1') < 0$. Hence $h(t) = 0$ has another positive root in $(t_1', \infty)$ which gives a contradiction.

Therefore $h(t) > 0$ for $t > t_0'$ and $|Q(z)| > 0$ for $|z| > t_0'$. So $Q(z)$ does not vanish in $|z| > t_0'$ and therefore all the zeros of $Q(z)$ lie in $|z| \leq t_0'$. Let $z = z_0$ be any zero of $P(z) = 0$. Clearly $z_0 \neq 0$ as $a_0 \neq 0$.

Putting $z = \frac{1}{z_0}$ in $Q(z)$ we get that

$$Q \left( \frac{1}{z_0} \right) = \left( \frac{1}{z_0} \right)^{N(r)} \cdot P(z_0) = \left( \frac{1}{z_0} \right)^{N(r)} \cdot 0 = 0.$$
So $\frac{1}{z_0}$ is a zero of Q(z). Therefore \( \left| \frac{1}{z_0} \right| \leq t'_0 \) i.e., \( |z_0| \geq \frac{1}{t_0} \). Since \( z_0 \) is any arbitrary zero of \( P(z) \), all the zeros of \( P(z) \) lie in \( |z| \geq \frac{1}{t_0} \).

Hence all the zeros of \( P(z) \) lie in the ring shaped region

\[
\frac{1}{t_0} \leq |z| \leq t_0
\]

where \( t_0 \) and \( t'_0 \) are the positive roots of

\[
g(t) \equiv \left| a_{N(r)} \right| t^{N(r)} - \left| a_{N(r)-1} \right| t^{N(r)-1} - \ldots - \left| a_0 \right| = 0
\]

and

\[
h(t) \equiv \left| a_0 \right| t^{N(r)} - \left| a_1 \right| t^{N(r)-1} - \ldots - \left| a_{N(r)} \right| = 0.
\]

respectively for sufficiently large \( r \) in the disc \( |z| \leq \log r \).

This proves the theorem.

**Remark 1** The limit in Theorem 1 is attained by \( P(z) = nz^2 + (n-1)z - 1 \) for any positive real number \( n \geq 2 \). It can be easily seen that \( M(r) = |n|r^2 = nr^2 \) for large \( r \) in \( |z| = r \).

So

\[
\rho = \limsup_{r \to \infty} \frac{\log \log M(r)}{\log r} = \limsup_{r \to \infty} \frac{\log \log nr^2}{\log r}
\]

\[
= \limsup_{r \to \infty} \frac{\frac{1}{\log(nr^2)} \cdot \frac{1}{nr^2} \cdot n.2r}{r}
\]

\[
= \limsup_{r \to \infty} \frac{2}{\log(nr^2)} = 0.
\]

Hence the order of the polynomial is 0. Also here \( \rho^* = 2 \) and \( N(r) = 2 \leq (\log r)^{2+\epsilon} \) for \( \epsilon > 0 \) and \( r \) be sufficiently large in \( |z| \leq \log r \). The zeros of \( P(z) \) is given by solving \( P(z) = 0 \) and \( a_{N(r)} = n \neq 0, a_{N(r)+1} = a_{N(r)+2} = \ldots = 0 \).

Now

\[
nz^2 + (n-1)z - 1 = 0
\]

i.e., if \( (nz - 1)(z + 1) = 0 \)

i.e., if \( z = \frac{1}{n}, -1 \).

Let \( z_1 = \frac{1}{n} \) and \( z_2 = -1 \). Then \( z_1 \) and \( z_2 \) are the zeros of

\[
P(z) = nz^2 + (n-1)z - 1 = 0.
\]

Here \( a_0 = 1, a_1 = n - 1 \) and \( a_2 = n \). Therefore \( |a_0| = 1, |a_1| = n - 1, |a_2| = n \) and so
\[ f(t) \equiv |a_1|^2 t^2 - |a_1| t - |a_0| = 0 \]

i.e., \[ nt^2 - (n - 1)t - 1 = 0. \]

Hence \( t = 1 \) and \( -\frac{1}{n} \).

Thus the positive root of \( f(t) = 0 \) is \( t = t_0 = 1 \).

Again to find the positive root of

\[ g(t) \equiv |a_0|^2 t^2 - |a_1| t - |a_2| = 0 \]

we get that

\[ t^2 - (n - 1)t - n = 0. \]

which implies \( t = n \) and \( t = -1 \). Therefore the positive root of \( g(t) = 0 \) is \( t_0' = n \).

Hence according to the Theorem 1 all the zeros of \( P(z) \) lie in

\[ \frac{1}{t_0} \leq |z| \leq t_0 \]

i.e., in \[ \frac{1}{n} \leq |z| \leq 1 \]

**Theorem 2** Let \( P(z) \) be an entire function defined by

\[ P(z) = a_0 + a_1 z + \ldots + a_n z^n + \ldots \]

whose order \( \rho = 0 \). Also for all sufficiently large \( r \) in the disc \[ |z| \leq \log r \], \( a_{N(r)} \neq 0 \) and \( a_0 \neq 0 \). Further \( a_n \to 0 \) as \( n > N(r) \). Then all the zeros of \( P(z) \) lie in the ring shaped region

\[ \frac{1}{1+M'} < |z| < 1 + M \]

where \( M = \max_{0 \leq k \leq N(r) - 1} \left| \frac{a_k}{a_{N(r)}} \right| \) and \( M' = \max_{0 \leq k \leq N(r) - 1} \left| \frac{a_k}{a_0} \right| \).

**Proof.** Since \( P(z) \) is an entire function of order \( \rho = 0 \), then by Lemma 1 for sufficiently large values of \( r \) in the disc \( |z| \leq \log r \) we have \( N(r) \leq (\log r)^{\rho + \varepsilon} \) for \( \varepsilon > 0 \). Also \( a_0 \neq 0 \), \( a_{N(r)} \neq 0 \) and \( a_n \to 0 \) as \( n > N(r) \). Hence we may write

\[ P(z) = a_0 + a_1 z + \ldots + a_n z^n + \ldots \]

\[ \approx a_0 + a_1 z + \ldots + a_{N(r)} z^{N(r)}. \]

Now

\[ |a_0 + a_1 z + \ldots + a_{N(r)-1} z^{N(r)-1}| \leq |a_0| + \ldots + |a_{N(r)-1}| |z|^{N(r)-1} \]

\[ = |a_{N(r)}| \left\{ \left| \frac{a_0}{a_{N(r)}} \right| + \ldots + \left| \frac{a_{N(r)-1}}{a_{N(r)}} \right| |z|^{N(r)-1} \right\} \]
\[ |a_0 + a_1 z + \cdots + a_{N(r)-1} z^{N(r)-1}| \leq |a_{N(r)}| |M| |z|^{N(r)-1} + |z|^{N(r)-2} + \cdots + 1 \]
\[ = |a_{N(r)}| |M| |z|^{N(r)} \left\{ \frac{1}{|z|} + \frac{1}{|z|^2} + \cdots + \frac{1}{|z|^{N(r)}} \right\} \]

where \(|z| \neq 0\). Therefore when \(|z| \neq 0\),
\[ -|a_0 + a_1 z + \cdots + a_{N(r)-1} z^{N(r)-1}| \geq -|a_{N(r)}| |M| |z|^{N(r)} \left\{ \frac{1}{|z|} + \frac{1}{|z|^2} + \cdots + \frac{1}{|z|^{N(r)}} \right\}. \]

So for \(|z| \neq 0\)
\[ |a_{N(r)}| |z|^{N(r)} - |a_0 + a_1 z + \cdots + a_{N(r)-1} z^{N(r)-1}| \]
\[ \geq |a_{N(r)}| |z|^{N(r)} - |a_0 + a_1 z + \cdots + a_{N(r)-1} z^{N(r)-1}|. \] (12)

Now
\[ |P(z)| \approx |a_0 + a_1 z + \cdots + a_{N(r)} z^{N(r)}| \]
\[ \geq |a_{N(r)}| |z|^{N(r)} - |a_0 + a_1 z + \cdots + a_{N(r)-1} z^{N(r)-1}|. \]

Using (12) we have
\[ |P(z)| \geq |a_{N(r)}| |z|^{N(r)} - |a_0 + a_1 z + \cdots + a_{N(r)-1} z^{N(r)-1}|. \]
\[ = |a_{N(r)}| |z|^{N(r)} \left\{ 1 - M \left( \frac{1}{|z|} + \frac{1}{|z|^2} + \cdots + \frac{1}{|z|^{N(r)}} \right) \right\} \text{ for } |z| \neq 0. \]

i.e., when \(|z| \neq 0\)
\[ |P(z)| > |a_{N(r)}| |z|^{N(r)} \left\{ 1 - M \sum_{j=1}^{\infty} \frac{1}{|z|^j} \right\} \text{ for } |z| \neq 0. \]

Therefore
\[ |P(z)| > |a_{N(r)}| |z|^{N(r)} \left\{ 1 - M \sum_{j=1}^{\infty} \frac{1}{|z|^j} \right\} \text{ for } |z| \neq 0. \]

Now the geometric series \( \sum_{j=1}^{\infty} \frac{1}{|z|^j} \) is convergent when \( \frac{1}{|z|} < 1 \) i.e., when \( |z| > 1 \)
and is equal to
\[ \frac{1}{|z|} \left( 1 - \frac{1}{|z|} \right) = \frac{1}{|z|-1}. \]

On \(|z| > 1\) we can write
\[ |P(z)| > |a_{N(r)}| |z|^{N(r)} \left( 1 - \frac{M}{|z|-1} \right). \]

Now on \(|z| > 1\),
\[ |P(z)| > 0 \text{ if } |a_{N(r)}||z|^N(r) \left(1 - \frac{M}{|z|-1}\right) \geq 0 \]

i.e., if \( \left(1 - \frac{M}{|z|-1}\right) \geq 0 \)

i.e., if \(|z| - 1 \geq M\)

i.e., if \(|z| \geq M + 1\).

Therefore

\[ |z| \geq M + 1 > 1 \text{ as } M > 0. \]

Hence

\[ |P(z)| > 0 \text{ if } |z| \geq M + 1. \]

Therefore all the zeros of \(P(z)\) lie in \(|z| < M + 1\).

Secondly, we give the proof of the lower bound. Let us consider

\[ Q(z) = z^{N(r)}P \left(\frac{1}{z}\right). \]

Therefore

\[ Q(z) = |z|^N(r) \left\{a_0 + \frac{a_1}{|z|} + \cdots + \frac{a_{N(r)}}{|z|^N(r)}\right\} \]

\[ = a_0|z|^N(r) + a_1|z|^{N(r)-1} + \cdots + a_{N(r)}. \]

Now

\[ |a_1| |z|^{N(r)-1} + \cdots + a_{N(r)}| \leq |a_1| |z|^{N(r)-1} + \cdots + |a_{N(r)}| \]

\[ = |a_0| \left\{\frac{|a_1|}{|a_0|} |z|^{N(r)-1} + \cdots + \frac{|a_{N(r)}}{|a_0|}\right\} \]

\[ \leq |a_0| M^r \left( |z|^{N(r)-1} + \cdots + 1 \right) \]

\[ = |a_0| M^r |z|^N(r) \left\{\frac{1}{|z|} + \cdots + \frac{1}{|z|^N(r)}\right\}. \]

Therefore

\[ -|a_1| |z|^{N(r)-1} + \cdots + a_{N(r)}| \geq -|a_0| M^r |z|^N(r) \left\{\frac{1}{|z|} + \cdots + \frac{1}{|z|^N(r)}\right\}. \]

So

\[ |Q(z)| \geq |a_0||z|^N(r) - |a_1| |z|^{N(r)-1} + \cdots + a_{N(r)}| \]

\[ \geq |a_0||z|^N(r) - |a_0| M^r |z|^N(r) \left\{\frac{1}{|z|} + \cdots + \frac{1}{|z|^N(r)}\right\}. \]
\[= |a_0||z|^{N(r)}\left\{1 - M' \left(\frac{1}{|z|} + \cdots + \frac{1}{|z|^{N(r)}}\right)\right\}\]
\[\geq |a_0||z|^{N(r)}\left\{1 - M' \left(\frac{1}{|z|} + \cdots + \frac{1}{|z|^{N(r)}} + \cdots\right)\right\}.
\]

Hence using above we get that
\[|Q(z)| > |a_0||z|^{N(r)}\left\{1 - M' \sum_{j = 1}^{\infty} \frac{1}{|z|^j}\right\}.
\]

Now the geometric series \[\sum_{j = 1}^{\infty} \frac{1}{|z|^j}\] is convergent when \[\frac{1}{|z|} < 1\] i.e., \[|z| > 1\] and is equal to
\[\frac{1}{|z| - 1} = \frac{1}{|z| - 1}.
\]

On \[|z| > 1\] we may write
\[|Q(z)| > |a_0||z|^{N(r)} \left(1 - \frac{M'}{|z| - 1}\right).
\]

Now for \[|z| > 1\],
\[|Q(z)| > 0 \text{ if } |a_0||z|^{N(r)} \left(1 - \frac{M'}{|z| - 1}\right) \geq 0 \]
\[\text{i.e., if } 1 - \frac{M'}{|z| - 1} \geq 0 \]
\[\text{i.e., if } |z| \geq 1 + M'.
\]

Therefore
\[|z| \geq 1 + M' > 1 \text{ as } M' > 0.
\]

Hence
\[|Q(z)| > 0 \text{ for } |z| \geq 1 + M'.
\]

So all the zeros of \(Q(z)\) lie in \(|z| < 1 + M'.\)

Let \(z = z_0\) be any zero of \(P(z)\). Therefore \(P(z_0) = 0\). Clearly \(z_0 \neq 0\) as \(a_0 \neq 0\). Putting \(z = \frac{1}{z_0}\) in \(Q(z)\) we have
\[Q\left(\frac{1}{z_0}\right) = \left(\frac{1}{z_0}\right)^n \cdot P\left(\frac{1}{z_0}\right) = \left(\frac{1}{z_0}\right)^n \cdot 0 = 0.
\]

Therefore \(z = \frac{1}{z_0}\) is a root of \(Q(z)\). So
\[\left|\frac{1}{z_0}\right| < 1 + M',
\]

which implies that
As $z_0$ is an arbitrary root of $P(z) = 0$, all the zeros of $P(z)$ lie in $|z| > \left| \frac{1}{1+M} \right|$. Hence all the zeros of $P(z)$ lie in the ring shaped region

$$\frac{1}{1+M'} < |z| < 1 + M.$$ 

This proves the theorem.

**Remark 2** Let us consider the polynomial

$$P(z) = nz^2 + (n-1)z - 1, \quad n \geq 2.$$ 

Here $a_0 = -1$, $a_1 = n-1$ and $a_2 = n$.

Therefore $|a_0| = 1$, $|a_1| = n-1$, $|a_2| = n$.

Also order $\rho$ of $P(z)$ is 0 and $\rho^* = 2$, so $N(r) = 2 \leq (\log r)^{2+\epsilon}$ for sufficiently large $r$.

Again

$$M = \max \left\{ \frac{|a_0|}{|a_1|} \right\} = \frac{n-1}{n}$$

and

$$M' = \max \left\{ \frac{|a_1|}{|a_0|} \right\} = \frac{n-1}{n} = n.$$ 

The roots of $P(z) = 0$ are $z_1 = \frac{1}{n}$ and $z_2 = -1$. So by Theorem 2 the roots of $P(z)$ lies in

$$\frac{1}{1+M'} < |z| < 1 + M$$

Hence

$$\frac{1}{1+n} < |z| < 1 + \frac{n-1}{n}$$

i.e.,

$$\frac{1}{1+n} < |z| < 2 - \frac{1}{n}.$$ 

**Theorem 3** Let $P(z)$ be an entire function defined by

$$P(z) = a_0 + a_1z + a_2z^2 + \ldots + a_nz^n + \ldots$$

whose order $\rho = 0$. Also for all sufficiently large $r$ in the disc $|z| \leq \log r$, $a_N(r) \neq 0$, $a_0 \neq 0$ and $a_n \to 0$ as $n > N(r)$. For any $p$, $q$ with $p > 1$, $q > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$, all the zeros of $P(z)$ lie in the annular region

$$\frac{1}{\left[ 1 + \left( \sum_{j=0}^{N(r)-1} \left| \frac{a_j}{a_N(r)} \right|^p \right)^{\frac{1}{p}} \right]^{\frac{q}{p}}} < |z| < \left[ 1 + \left( \sum_{j=0}^{N(r)-1} \left| \frac{a_j}{a_N(r)} \right|^p \right)^{\frac{1}{p}} \right]^{\frac{q}{p}}.$$
Proof. Given that \(a_0 \neq 0\), \(a_{N(r)} \neq 0\) and \(a_n \rightarrow 0\) as \(n > N(r)\). Therefore for sufficiently large \(r\) in the disc \(|z| \leq \log r\) the existence of \(N(r)\) implies that

\[
P(z) = a_0 + a_1 z + a_2 z^2 + \ldots + a_n z^n + \ldots
\]

\[
\approx a_0 + a_1 z + a_2 z^2 + \ldots + a_{N(r)} z^{N(r)}.
\]

Now

\[
|a_0 + a_1 z + a_2 z^2 + \ldots + a_{N(r)} z^{N(r)} - 1| \leq |a_0| + |a_1||z| + \ldots + |a_{N(r)} - 1||z|^{N(r) - 1}
\]

\[
= |a_{N(r)}| \left\{ \frac{|a_0|}{|a_{N(r)}|} + \ldots + \frac{|a_{N(r)} - 1|}{|a_{N(r)}|} |z|^{N(r) - 1}\right\}
\]

\[
= |a_{N(r)}| \sum_{j = 0}^{N(r) - 1} \left( \frac{a_j}{a_{N(r)}} \right) |z|^j.
\]

This implies

\[
|P(z)| \approx |a_0 + a_1 z + a_2 z^2 + \ldots + a_{N(r)} z^{N(r)}| 
\]

\[
\geq |a_{N(r)}||z|^{N(r)} - |a_0 + a_1 z + a_2 z^2 + \ldots + a_{N(r) - 1} z^{N(r) - 1}|
\]

\[
\geq |a_{N(r)}||z|^{N(r)} - |a_{N(r)}| \sum_{j = 0}^{N(r) - 1} \left( \frac{a_j}{a_{N(r)}} \right) |z|^j
\]

i.e.,

\[
|P(z)| \geq |a_{N(r)}| \left\{ |z|^{N(r)} - \sum_{j = 0}^{N(r) - 1} \left( \frac{a_j}{a_{N(r)}} \right) |z|^j \right\}.
\]

By Holder’s inequality we have

\[
\sum_{j = 0}^{N(r) - 1} \left( \frac{a_j}{a_{N(r)}} \right) |z|^j \leq \left( \sum_{j = 0}^{N(r) - 1} \left( \frac{a_j}{a_{N(r)}} \right)^p \right)^{\frac{1}{p}} \left( \sum_{j = 0}^{N(r) - 1} (|z|^q)^q \right)^{\frac{1}{q}}.
\]

(14)

In view of (14) we obtain that

\[
|P(z)| \geq |a_{N(r)}| \left\{ |z|^{N(r)} - \left( \sum_{j = 0}^{N(r) - 1} \left( \frac{a_j}{a_{N(r)}} \right)^p \right)^{\frac{1}{p}} \left( \sum_{j = 0}^{N(r) - 1} (|z|^q)^q \right)^{\frac{1}{q}} \right\}
\]

\[
= |a_{N(r)}| \left\{ |z|^{N(r)} - |z|^{N(r)} \left( \sum_{j = 0}^{N(r) - 1} \left( \frac{a_j}{a_{N(r)}} \right)^p \right)^{\frac{1}{p}} \left( \sum_{j = 0}^{N(r) - 1} (|z|^q)^q \right)^{\frac{1}{q}} \right\}
\]
\[
= |a_{N(r)}||z|^{N(r)} \left\{ 1 - \left( \sum_{j=0}^{N(r) - 1} \left| \frac{a_j}{a_{N(r)}} \right|^p \right)^{\frac{1}{p}} \left( \sum_{j=0}^{N(r) - 1} \left( \frac{1}{|z|^q} \right)^j \right)^{\frac{1}{q}} \right\}
\]

\[
= |a_{N(r)}||z|^{N(r)} \left\{ 1 - \left( \sum_{j=0}^{N(r) - 1} \left| \frac{a_j}{a_{N(r)}} \right|^p \right)^{\frac{1}{p}} \left( \sum_{j=1}^{N(r)} \left( \frac{1}{|z|^q} \right)^j \right)^{\frac{1}{q}} \right\}.
\]

Now the geometric series \( \sum_{j=1}^{\infty} \left( \frac{1}{|z|^q} \right)^j \) is convergent for

\[
\left| \frac{1}{|z|^q} \right| < 1
\]

i.e., for \(|z|^q > 1\)

i.e., for \(|z| > 1\)

and is convergent to

\[
\left| \frac{1}{|z|^q} \right| \cdot \frac{1}{1 - \frac{1}{|z|^q}} = \frac{1}{|z|^q - 1}.
\]

So

\[
\left( \sum_{j=1}^{\infty} \left( \frac{1}{|z|^q} \right)^j \right)^{\frac{1}{q}} \text{ converges to } \left( \frac{1}{|z|^q - 1} \right)^{\frac{1}{q}} \text{ for } |z| > 1.
\]

Therefore on \(|z| > 1\)

\[
|P(z)| > |a_{N(r)}||z|^{N(r)} \left\{ 1 - \left( \sum_{j=0}^{N(r) - 1} \left| \frac{a_j}{a_{N(r)}} \right|^p \right)^{\frac{1}{p}} \left( \frac{1}{|z|^q - 1} \right)^{\frac{1}{q}} \right\}.
\]

Now if \(|P(z)| > 0\) then we have

\[
|a_{N(r)}||z|^{N(r)} \left\{ 1 - \left( \sum_{j=0}^{N(r) - 1} \left| \frac{a_j}{a_{N(r)}} \right|^p \right)^{\frac{1}{p}} \left( \frac{1}{|z|^q - 1} \right)^{\frac{1}{q}} \right\} > 0
\]

i.e.,

\[
1 - \left( \sum_{j=0}^{N(r) - 1} \left| \frac{a_j}{a_{N(r)}} \right|^p \right)^{\frac{1}{p}} \left( \frac{1}{|z|^q - 1} \right)^{\frac{1}{q}} \geq 0
\]
\[
\left( \frac{N(r) - 1}{a_{N(r)}} \right)^{\frac{1}{p}} \geq \left( \sum_{j=0}^{1} \left| \frac{a_j}{a_{N(r)}} \right|^{p} \right)^{\frac{1}{q}}
\]

i.e.,

\[
\left( \frac{N(r) - 1}{a_{N(r)}} \right)^{\frac{1}{p}} \geq \left( \sum_{j=0}^{1} \left| \frac{a_j}{a_{N(r)}} \right|^{p} \right)^{\frac{1}{q}}
\]

i.e.,

\[
\left( \frac{N(r) - 1}{a_{N(r)}} \right)^{\frac{1}{p}} \geq \left( \sum_{j=0}^{1} \left| \frac{a_j}{a_{N(r)}} \right|^{p} \right)^{\frac{1}{q}}
\]

i.e.,

\[
\left( \frac{N(r) - 1}{a_{N(r)}} \right)^{\frac{1}{p}} \geq \left( \sum_{j=0}^{1} \left| \frac{a_j}{a_{N(r)}} \right|^{p} \right)^{\frac{1}{q}}
\]

Clearly

\[
\left[ 1 + \left( \sum_{j=0}^{1} \left| \frac{a_j}{a_{N(r)}} \right|^{p} \right)^{\frac{1}{q}} \right] > 1.
\]

Therefore \(|P(z)| > 0\) for

\[
\left| z \right| \geq \left[ 1 + \left( \sum_{j=0}^{1} \left| \frac{a_j}{a_{N(r)}} \right|^{p} \right)^{\frac{1}{q}} \right] > 1.
\]

Therefore all the zeros of \(P(z)\) lie in

\[
\left| z \right| < \left[ 1 + \left( \sum_{j=0}^{1} \left| \frac{a_j}{a_{N(r)}} \right|^{p} \right)^{\frac{1}{q}} \right].
\]

(15)

For the lower bound let us take \(Q(z) = z^{N(r)} \left( \frac{1}{z} \right)^{\frac{1}{p}} \). Therefore

\[
Q(z) = z^{N(r)} \left( \frac{1}{z} \right)^{\frac{1}{p}}
\]

\[
\approx z^{N(r)} \left\{ a_0 + \frac{a_1}{z} + \cdots + \frac{a_{N(r)}}{z^{N(r)}} \right\}
\]

\[
= a_0 z^{N(r)} + a_1 z^{N(r)-1} + \cdots + a_{N(r)}.
\]

Therefore

\[
|Q(z)| \approx \left| a_0 z^{N(r)} + a_1 z^{N(r)-1} + \cdots + a_{N(r)} \right|.
\]

Now
\[ |a_1 z^{N(r)-1} + \cdots + a_{N(r)}| \leq |a_0| |z|^{N(r)-1} + \cdots + |a_{N(r)}| \]

\[ = |a_0| \left( \frac{|a_1|}{|a_0|} |z|^{N(r)-1} + \cdots + \frac{|a_{N(r)}|}{|a_0|} \right) \]

\[ = |a_0| \sum_{j=0}^{N(r)-1} \frac{a_{N(r)-j}}{a_0} |z|^j. \quad (16) \]

Therefore using (16) we get that

\[ |Q(z)| \geq |a_0| |z|^{N(r)} - |a_1 z^{N(r)-1} + \cdots + a_{N(r)}| \]

\[ \geq |a_0| |z|^{N(r)} - |a_0| \sum_{j=0}^{N(r)-1} \frac{a_{N(r)-j}}{a_0} |z|^j \]

\[ = |a_0| \left( |z|^{N(r)} - \sum_{j=0}^{N(r)-1} \frac{a_{N(r)-j}}{a_0} |z|^j \right). \]

Now by Holdar’s inequality we have

\[ \sum_{j=0}^{N(r)-1} \frac{a_{N(r)-j}}{a_0} |z|^j \leq \left( \sum_{j=0}^{N(r)-1} \left( \frac{a_{N(r)-j}}{a_0} \right)^p \right)^{\frac{1}{p}} \left( \sum_{j=0}^{N(r)-1} |z|^j \right)^{\frac{1}{q}}. \quad (17) \]

Using (17) we obtain from above that

\[ |Q(z)| \geq |a_0| \left( |z|^{N(r)} - \left( \sum_{j=0}^{N(r)-1} \left( \frac{a_{N(r)-j}}{a_0} \right)^p \right)^{\frac{1}{p}} \left( \sum_{j=0}^{N(r)-1} (|z|^j)^q \right)^{\frac{1}{q}} \right) \]

\[ = |a_0| \left( |z|^{N(r)} - |z|^{N(r)} \left( \sum_{j=0}^{N(r)-1} \left( \frac{a_{N(r)-j}}{a_0} \right)^p \right)^{\frac{1}{p}} \left( \sum_{j=0}^{N(r)-1} (|z|^j)^q \right)^{\frac{1}{q}} \right) \]

\[ = |a_0| \left( 1 - \left( \sum_{j=0}^{N(r)-1} \left( \frac{a_{N(r)-j}}{a_0} \right)^p \right)^{\frac{1}{p}} \left( \sum_{j=0}^{N(r)-1} (|z|^j)^q \right)^{\frac{1}{q}} \right) \]

\[ = |a_0| |z|^{N(r)} \left( 1 - \left( \sum_{j=0}^{N(r)-1} \left( \frac{a_{N(r)-j}}{a_0} \right)^p \right)^{\frac{1}{p}} \left( \sum_{j=0}^{N(r)-1} \frac{(|z|^j)^q}{|z|^{N(r)-j}} \right)^{\frac{1}{q}} \right) \]
\[ a_0 \| z \|^{N(r)} \left\{ 1 - \left( \sum_{j=0}^{N(r)-1} \frac{a_{N(r)-j}}{a_0} \right)^{\frac{1}{p}} \left( \sum_{j=0}^{N(r)-1} \left( \frac{1}{\| z \|^q} \right)^j \right)^{\frac{1}{q}} \right\} \]

\[ = a_0 \| z \|^{N(r)} \left\{ 1 - \left( \sum_{j=0}^{N(r)-1} \frac{a_{N(r)-j}}{a_0} \right)^{\frac{1}{p}} \left( \sum_{j=0}^{N(r)-1} \left( \frac{1}{\| z \|^q} \right)^j \right)^{\frac{1}{q}} \right\} \]

Therefore

\[ |Q(z)| > |a_0\| z \|^{N(r)} \left\{ 1 - \left( \sum_{j=0}^{N(r)-1} \frac{a_{N(r)-j}}{a_0} \right)^{\frac{1}{p}} \left( \sum_{j=0}^{N(r)-1} \left( \frac{1}{\| z \|^q} \right)^j \right)^{\frac{1}{q}} \right\} \]

Now the geometric series \[ \sum_{j=1}^{\infty} \left( \frac{1}{\| z \|^q} \right)^j \] is convergent for

\[ \left| \frac{1}{\| z \|^q} \right| < 1 \]

i.e., for \[ \| z \|^q > 1. \]

Therefore for \[ \| z \| > 1 \] and the series is convergent to

\[ \left( \sum_{j=1}^{\infty} \left( \frac{1}{\| z \|^q} \right)^j \right)^{\frac{1}{q}} \]

So

\[ \left( \sum_{j=1}^{\infty} \left( \frac{1}{\| z \|^q} \right)^j \right)^{\frac{1}{q}} \] is convergent to \( \left( \frac{1}{\| z \|^q-1} \right)^{\frac{1}{q}} \) for \[ \| z \| > 1. \]

Therefore on \[ \| z \| > 1, \]

\[ |Q(z)| > |a_0\| z \|^{N(r)} \left\{ 1 - \left( \sum_{j=0}^{N(r)-1} \frac{a_{N(r)-j}}{a_0} \right)^{\frac{1}{p}} \left( \sum_{j=0}^{N(r)-1} \left( \frac{1}{\| z \|^q-1} \right)^j \right)^{\frac{1}{q}} \right\} \]

Now if \[ |Q(z)| > 0 \] then
\(|a_0|z|^{N(r)}\left\{1 - \left(\sum_{j=0}^{N(r)-1} \left|\frac{a_{N(r)-j}}{a_0}\right|^p\right)^{\frac{1}{p}}\right\} \geq 0 \tag{1}\)

i.e., \(1 - \left(\sum_{j=0}^{N(r)-1} \left|\frac{a_{N(r)-j}}{a_0}\right|^p\right)^{\frac{1}{p}} \geq 0\)

i.e., \(1 \geq \left(\sum_{j=0}^{N(r)-1} \left|\frac{a_{N(r)-j}}{a_0}\right|^p\right)^{\frac{1}{p}}\) \(\text{or} \quad (|z|^q - 1)^{\frac{1}{q}} \geq \left(\sum_{j=0}^{N(r)-1} \left|\frac{a_{N(r)-j}}{a_0}\right|^p\right)^{\frac{1}{p}}\)

i.e., \(|z|^q - 1 \geq \left(\sum_{j=0}^{N(r)-1} \left|\frac{a_{N(r)-j}}{a_0}\right|^p\right)^{\frac{1}{p}}\)

i.e., \(|z| \geq 1 + \left(\sum_{j=0}^{N(r)-1} \left|\frac{a_{N(r)-j}}{a_0}\right|^p\right)^{\frac{1}{p}}\) \(\text{or} \quad |z| \geq 1 + \left(\sum_{j=0}^{N(r)-1} \left|\frac{a_{N(r)-j}}{a_0}\right|^p\right)^{\frac{q}{P}}\)

Clearly

\[1 + \left(\sum_{j=0}^{N(r)-1} \left|\frac{a_{N(r)-j}}{a_0}\right|^p\right)^{\frac{q}{P}} \geq 1.\]

Therefore \(|Q(z)| > 0\) if

\[|z| \geq 1 + \left(\sum_{j=0}^{N(r)-1} \left|\frac{a_{N(r)-j}}{a_0}\right|^p\right)^{\frac{q}{P}}\]

Therefore all the zeros of \(Q(z)\) lie in

\[|z| < 1 + \left(\sum_{j=0}^{N(r)-1} \left|\frac{a_{N(r)-j}}{a_0}\right|^p\right)^{\frac{q}{P}}\]

Let \(z = z_0\) be any other zero of \(P(z)\). Therefore \(P(z_0) = 0\). Clearly \(z_0 \neq 0\) as \(a_0 \neq 0\). Putting \(z = \frac{1}{z_0}\) in \(Q(z)\) we have
\[ Q \left( \frac{1}{z_0} \right) = \left( \frac{1}{z_0} \right)^n \cdot P(z_0) = \left( \frac{1}{z_0} \right)^n \cdot 0 = 0. \]

Therefore \( z = \frac{1}{z_0} \) is a zero of \( Q(z) \). So

\[
\left| \frac{1}{z_0} \right| < \left[ 1 + \left( \sum_{j=0}^{N(r) - 1} \frac{|a_{N(r) - j}|^p}{a_0} \right)^{\frac{1}{p}} \right].
\]

i.e., \( |z_0| > \frac{1}{\left[ 1 + \left( \sum_{j=0}^{N(r) - 1} \frac{|a_{N(r) - j}|^p}{a_0} \right)^{\frac{1}{p}} \right]} \).

As \( z_0 \) is an arbitrary zero of \( P(z) \) so all the zeros of \( P(z) \) lie in

\[
|z| > \frac{1}{\left[ 1 + \left( \sum_{j=0}^{N(r) - 1} \frac{|a_{N(r) - j}|^p}{a_0} \right)^{\frac{1}{p}} \right]}.
\] (18)

Hence combining (15) and (18) we may say that all the zeros of \( P(z) \) lie in the ring shaped region

\[
\frac{1}{\left[ 1 + \left( \sum_{j=0}^{N(r) - 1} \frac{|a_{N(r) - j}|^p}{a_0} \right)^{\frac{1}{p}} \right]} < |z| < \left[ 1 + \left( \sum_{j=0}^{N(r) - 1} \frac{|a_{N(r) - j}|^p}{a_0} \right)^{\frac{1}{p}} \right].
\]

This proves the theorem.

**Corollary 1** In particular if we take \( p = 2 \), \( q = 2 \) in Theorem 3 then we get that all the zeros of the polynomial

\[ P(z) = a_0 + a_1 z + a_2 z^2 + \ldots + a_n z^n \]

lie in the ring shaped region

\[
\frac{1}{\left[ 1 + \left( \sum_{j=0}^{N(r) - 1} \frac{|a_{N(r) - j}|^2}{a_0} \right)^{\frac{1}{2}} \right]} < |z| < \left[ 1 + \left( \sum_{j=0}^{N(r) - 1} \frac{|a_{N(r) - j}|^2}{a_0} \right)^{\frac{1}{2}} \right].
\]

**References**
FURTHER RESULTS ON THE BASIS OF CAUCHY’S PROPER BOUND FOR THE ZEROS OF ENTIRE FUNCTIONS OF ORDER ZERO

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ABSTRACT: A single valued function of one complex variable which is analytic in the finite complex plane is called an entire function. The purpose of this paper is to establish the bounds for the moduli of zeros of entire functions of order zero.

Some examples are provided to clear the notions.

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I. INTRODUCTION, DEFINITIONS AND NOTATIONS

Let, \( P(z) = a_0 + a_1z + a_2z^2 + \ldots + a_nz^n + a_nz^n \); \( |a_n| \neq 0 \)

Be a polynomial of degree \( n \). Datt and Govil [2]; Govil and Rahaman [5]; Marden [9]; Mohammad [10]; Chattopadhyay, Das, Jain and Konwar [1]; Joyal, Labelle and Rahaman [6]; Jain [7], [8]; Sun and Hsieh [11]; Zilovic, Roytman, Combettes and Swamy [13]; Das and Datta [4] etc. worked in the theory of the distribution of the zeros of polynomials and obtained some newly developed results.

In this paper we intend to establish some of sharper results concerning the theory of distribution of zeros of entire functions of order zero.

The following definitions are well known:

Definition 1 The order \( \rho \) and lower order \( \lambda \) of a meromorphic function \( f \) are defined as

\[
\rho = \limsup_{r \to \infty} \frac{\log T(r,f)}{\log r}
\]

If \( f \) is entire, one can easily verify that

\[
\rho = \limsup_{r \to \infty} \frac{\log M(r,f)}{\log r}
\]

where \( \log^{[k]}(x) = \log(\log^{[k-1]}(x)) \) for \( k = 1, 2, 3, \ldots \) and \( \log^{[0]}(x) = x \).

If \( \rho < \infty \) then \( f \) is of finite order. Also \( \rho = 0 \) means that \( f \) is of order zero. In this connection Datta and Biswas [3] gave the following definition:

Definition 2 Let \( f \) be a meromorphic function of order zero. Then the quantities \( \rho^* \) and \( \lambda^* \) of \( f \) are defined by:

\[
\rho^* = \limsup_{r \to \infty} \frac{\log M(r,f)}{\log r}
\]

If \( f \) is an entire function then clearly

\[
\rho^* = \limsup_{r \to \infty} \frac{\log M(r,f)}{\log r}
\]

II. LEMMAS

In this section we present a lemma which will be needed in the sequel.

Lemma 1 If \( f(z) \) is an entire function of order \( \rho = 0 \), then for every \( \varepsilon > 0 \) the inequality \( N(r) \leq (\log r)^{\rho^*+\varepsilon} \)

holds for all sufficiently large \( r \) for which \( N(r) \) is the number of zeros of \( f(z) \) in \( |z| \leq r \).

Proof. Let us suppose that \( f(z) = 1 \). This supposition can be made without loss of generality because if \( f(z) \) has a zero of order \( m' \) at the origin then we may consider \( g(z) = c \cdot f(z) \) where \( c \) is so chosen that \( g(0) = 1 \).

Since the function \( g(z) \) and \( f(z) \) have the same assumption it will be unimportant for our investigations that the number of zeros of \( g(z) \) and \( f(z) \) differ by \( m \).

We further assume that \( f(z) \) has no zeros on \( |z| = 2r \) and the zeros \( n \)’s of \( f(z) \) in \( |z| < 2r \) are in non decreasing order of their moduli so that \( |z_n| \leq |z_{n+1}| \). Also let \( \rho^* \) suppose to be finite where \( \rho = 0 \) is the zero of order of \( f(z) \).

Now we shall make use of Jenson’s formula as state below

\[
|\log f(0)| = -\sum_{i=1}^{n} \frac{\log R}{|z_i|} + \frac{2\pi}{2\pi} \int_{0}^{2\pi} \log |f(RE^{i\phi})| \, d\phi.
\]

Let us replace \( R \) by \( 2r \) and \( n \) by \( N(2r) \) in (1).

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Again by definition of order \( \rho \) of \( f(z) \) we have for every \( \epsilon > 0 \), 
\[
\log M(2r) \leq (\log(2r))^{\rho^*+\epsilon/2}.
\]
Hence from (2) by the help of (3), (4) and (5) we have 
\[
N(r) \log 2 \leq (\log(2r))^{\rho^*+\epsilon/2}
\]
\[
\text{i.e., } N(r) \leq \frac{(\log 2)^{\rho^*+\epsilon}}{\log(2r)^{\rho^*+\epsilon}} \leq (\log r)^{\rho^*+\epsilon}.
\]
This proves the lemma.

III. THEOREMS

In this section we present the main results of the paper.

**Theorem 1** Let \( P(z) \) be an entire function having order \( \rho = 0 \) in the disc \( |z| \leq \log r \) for sufficiently large \( r \). Also let the Taylor’s series expansion of \( P(z) \) be given by 
\[
P(z) = a_0 + a_{p_1} z^{p_1} + \ldots + a_{p_m} z^{p_m} + a_{N(r)} z^{N(r)}, \quad a_0 \neq 0, \quad a_{N(r)} \neq 0
\]
with \( 1 \leq p_1 < p_2 < \ldots < p_m \leq N(r) - 1 \), \( p_i \)'s are integers such that for some \( \rho^* > 0 \), 
\[
|a_0(\rho^*)^{N(r)}| \geq |a_{p_1}(\rho^*)^{N(r)-p_1}| \geq \ldots \geq |a_{p_m}(\rho^*)^{N(r)-p_m}| \geq |a_{N(r)}|.
\]
Then all the zeros of \( P(z) \) lie in the ring shaped region 
\[
\left| \varphi \right| \leq \frac{1}{\log(1+|z|^{\rho^*})} < |z| < \frac{1}{\log(1+|z|^{\rho^*})} - \frac{|a_0(\rho^*)^{N(r)}|}{\log(1+|z|^{\rho^*})}.
\]

**Proof.** Given that 
\[
P(z) = a_0 + a_{p_1} z^{p_1} + \ldots + a_{p_m} z^{p_m} + a_{N(r)} z^{N(r)}
\]
where \( p_i \)'s are integers and \( 1 \leq p_1 < p_2 < \ldots < p_m \leq N(r) - 1 \). Then for some \( \rho^* > 0 \), 
\[
|a_0(\rho^*)^{N(r)}| \geq |a_{p_1}(\rho^*)^{N(r)-p_1}| \geq \ldots \geq |a_{p_m}(\rho^*)^{N(r)-p_m}| \geq |a_{N(r)}|.
\]
Let us consider 
\[
Q(z) = (\rho^*)^{N(r)} p \left( \frac{z}{\rho^*} \right)
\]
\[
= (\rho^*)^{N(r)} \left\{ a_0 + a_{p_1} \frac{z^{p_1}}{\rho^*(p_1)} + \ldots + a_{p_m} \frac{z^{p_m}}{\rho^*(p_m)} + a_{N(r)} \frac{z^{N(r)}}{\rho^*(N(r))} \right\}
\]
\[
= a_0(\rho^*)^{N(r)} + a_{p_1}(\rho^*)^{N(r)-p_1} z^{p_1} + \ldots + a_{p_m}(\rho^*)^{N(r)-p_m} z^{p_m} + a_{N(r)} z^{N(r)}.
\]
Therefore 
\[
|Q(z)| \geq |a_0(\rho^*)^{N(r)}| - |a_0(\rho^*)^{N(r)}| + 1 + \sum_{n=1}^{\infty} \left| \frac{1}{|z|^{\rho^*}} \frac{1}{n!} \right| \left| z^{n_{N(r)}} \right|\]
\[
\geq |a_0(\rho^*)^{N(r)}| - |a_0(\rho^*)^{N(r)}| + 1 + \sum_{n=1}^{\infty} \left| \frac{1}{|z|^{\rho^*}} \frac{1}{n!} \right| \left| z^{n_{N(r)}} \right|.
\]
Now using the given condition of Theorem 1 we obtain that 
\[
|a_0(\rho^*)^{N(r)}| + a_{p_1}(\rho^*)^{N(r)-p_1} z^{p_1} + \ldots + a_{p_m}(\rho^*)^{N(r)-p_m} z^{p_m} \leq 1 + \sum_{n=1}^{\infty} \left| \frac{1}{|z|^{\rho^*}} \right| \left| z^{n_{N(r)}} \right|\]
\[
\leq |a_0(\rho^*)^{N(r)}| - |a_0(\rho^*)^{N(r)}| + 1 + \sum_{n=1}^{\infty} \left| \frac{1}{|z|^{\rho^*}} \right| \left| z^{n_{N(r)}} \right|\]
\[
\text{for } |z| \neq 0.
\]
Using (6) we get for \( |z| \neq 0 \) that 
\[
|Q(z)| \geq |a_0(\rho^*)^{N(r)}| - |a_0(\rho^*)^{N(r)}| + 1 + \sum_{n=1}^{\infty} \left| \frac{1}{|z|^{\rho^*}} \right| \left| z^{n_{N(r)}} \right|\]
\[
> |a_0(\rho^*)^{N(r)}| - |a_0(\rho^*)^{N(r)}| + 1 + \sum_{n=1}^{\infty} \left| \frac{1}{|z|^{\rho^*}} \right| \left| z^{n_{N(r)}} \right|.
\]
Also, now and converges to

Using (7) we get from above that for

So all the zeros of 

Putting

Since

So

\[ |Q(z)| \geq |a_N(\rho)| |z|^N(r) - |a_0(\rho)^N(r)| |z|^N(r) \left( \frac{1}{|z|-1} \right) \]

Now for \(|z| > 1\),

\[ |Q(z)| \geq 0 \text{ if } |a_N(\rho)| - \frac{|a_0(\rho)^N(r)|}{|z|-1} \geq 0 \]

i.e., if \(|z| - 1 \geq \frac{|a_0(\rho)^N(r)}{|a_N(\rho)|}\)

i.e., if \(|z| \geq 1 + \frac{|a_0(\rho)^N(r)}{|a_N(\rho)|}\)

Therefore \(|Q(z)| \geq 0\) if

So all the zeros of \(Q(z)\) lie in

Let \(z = z_0\) be any zero of \(P(z)\). Therefore \(P(z_0) = 0\). Clearly \(z_0 \neq 0\) as \(a_0 \neq 0\).

Putting \(z = \rho^* z_0\) in \(Q(z)\) we get that

\(Q(\rho^* z_0) = (\rho^*)^N(r) P(z_0) = (\rho^*)^N(r) 0 = 0\).

So \(z = \rho^* z_0\) is a zero of \(Q(z)\). Hence

\[ |\rho^* z_0| < 1 + \frac{|a_0(\rho)^N(r)|}{|a_N(\rho)|} \]

i.e., \(|z_0| < \frac{1}{\rho^*} \left( 1 + \frac{|a_0(\rho)^N(r)|}{|a_N(\rho)|} \right)\).

Since \(z_0\) is an arbitrary zero of \(P(z)\), therefore all the zeros of \(Q(z)\) lie in

Again let us consider

\[ R(z) = (\rho^*)^N(r) z^N(r) P \left( \frac{1}{\rho^* z_0} \right) \]

Therefore

\[ R(z) = (\rho^*)^N(r) z^N(r) \left( a_0 + a_1(\rho^*)^N(r-1) z^N(r-1) + \ldots + a_{m-1}(\rho^*)^N(r-m) z^N(r-m) + a_N(\rho^*)^N(r) z^N(r) \right) \]

\[ \begin{align*} 
&= a_0(\rho^*)^N(r) z^N(r) + a_1(\rho^*)^N(r-1) z^N(r-1) + \ldots + a_{m-1}(\rho^*)^N(r-m) z^N(r-m) + a_N(\rho^*)^N(r) \left( a_N(\rho^*)^N(r) z^N(r) \right) \end{align*} \]

Now

\[ |R(z)| \geq |a_0(\rho^*)^N(r) z^N(r)| - |a_1(\rho^*)^N(r-1) z^N(r-1)| - \ldots - |a_{m-1}(\rho^*)^N(r-m) z^N(r-m)| + |a_N(\rho^*)^N(r) z^N(r)| \]

Also

\[ |a_0(\rho^*)^N(r) z^N(r)| \leq |a_0(\rho^*)^N(r) z^N(r)| + |a_1(\rho^*)^N(r-1) z^N(r-1)| + \ldots + |a_{m-1}(\rho^*)^N(r-m) z^N(r-m)| + |a_N(\rho^*)^N(r) z^N(r)| \]
Now the geometric series
\[ \sum_{k=1}^{\infty} \frac{1}{|z|^k} \]

is convergent for
\[ \frac{1}{|z|} < 1 \]

i.e., for \(|z| > 1\)

And converges to
\[ \frac{1}{|z|} - \frac{1}{|z| - 1} \]

So
\[ \sum_{k=1}^{\infty} \frac{1}{|z|^k} = \frac{1}{|z| - 1} \quad \text{for} \quad |z| > 1. \]

Therefore for \(|z| > 1\),
\[ |R(z)| > |a_0| (\rho^*)^{N(r)} |z|^{N(r)} - |a_{p_1}| (\rho^*)^{N(r)-p_1} |z|^{N(r)} \left( \frac{1}{|z|-1} \right) \]

i.e., for \(|z| > 1\)
\[ |R(z)| > |z|^{N(r)} (\rho^*)^{N(r)-p_1} \left( |a_0| (\rho^*)^{p_1} - \frac{|a_{p_1}|}{|z|-1} \right). \]

Now
\[ R(z) > 0 \quad \text{if} \quad \left| a_0 \right| (\rho^*)^{p_1} - \frac{|a_{p_1}|}{|z|-1} \geq 0 \]

i.e., if \(|a_0| (\rho^*)^{p_1} \geq \frac{|a_{p_1}|}{|z|-1} \)

i.e., if \(|z| - 1 \geq \frac{|a_{p_1}|}{|a_0| (\rho^*)^{p_1}} \)

i.e., if \(|z| \geq 1 + \frac{|a_{p_1}|}{|a_0| (\rho^*)^{p_1}} > 1. \]

Therefore
\[ R(z) > 0 \quad \text{if} \quad |z| \geq 1 + \frac{|a_{p_1}|}{|a_0| (\rho^*)^{p_1}}. \]

Since \(R(z)\) does not vanish in
\[ |z| \geq 1 + \frac{|a_{p_1}|}{|a_0| (\rho^*)^{p_1}}, \]

all the zeros of \(R(z)\) lie in
\[ |z| < 1 + \frac{|a_{p_1}|}{|a_0| (\rho^*)^{p_1}}. \]

Let \(z = z_0\) be any zero of \(P(z)\). Therefore \(P(z_0) = 0\). Clearly \(z_0 \neq 0\) as \(a_0 \neq 0\).

Putting \(z = \rho^* z_0\) in \(R(z)\) we obtain that
\[ R \left( \frac{1}{\rho^* z_0} \right) = (\rho^*)^{N(r)} \left( \frac{1}{\rho^* z_0} \right)^{N(r)} \cdot P(z_0) \]

\[ = \left( \frac{1}{z_0} \right)^{N(r)} \cdot 0 = 0. \]

So
\[ \left| \frac{1}{\rho^* z_0} \right| < 1 + \frac{|a_{p_1}|}{|a_0| (\rho^*)^{p_1}} \]

i.e.,
\[ \frac{1}{z_0} < \rho^* \left( 1 + \frac{|a_{p_1}|}{|a_0| (\rho^*)^{p_1}} \right) \]
Let us consider series expansion of \( P(z) \). From (8) and (12) we may conclude that all the zeros of \( P(z) \) lie in the proper ring shaped region
\[
\frac{1}{\rho \left( 1 + \frac{1}{\rho} \right)} < |z| < \frac{1}{\rho} \left( 1 + \frac{|a_0|}{|a_n|} \rho^{N(r)} \right).
\]

This proves the theorem.

**Theorem 2**  
Let \( P(z) \) be an entire function having order \( \rho = 0 \). For sufficiently large \( r \) in the disc \( |z| \leq \log r \), the Taylor's series expansion of \( P(z) \) be given by \( P(z) = a_0 + a_1 z + \ldots + a_N(z)^N(r) \), \( a_0 \neq 0 \). Further for some \( \rho^* > 0 \),
\[
|a_0| |(\rho^*)^{N(r)}| \geq |a_1| |(\rho^*)^{N(r)-1}| \geq \ldots \geq |a_N|.
\]
Then all the zeros of \( P(z) \) lie in the ring shaped region
\[
\frac{1}{\rho^{N(r)}} \leq |z| \leq \frac{1}{\rho^*} t_0
\]
on putting \( \rho^* = 1 \) in Theorem 1.

**Corollary 1**  
In view of Theorem 1 we may conclude that all the zeros of
\[ P(z) = a_0 + a_1 z + \ldots + a_n z^n \]
of degree \( n \) with \( 1 \leq p_1 < p_2 < \ldots < p_m \leq n - 1 \), \( p_i \)'s are integers such that for some \( \rho^* > 0 \),
\[
|a_0| |(\rho^*)^{N(r)}| \geq |a_1| |(\rho^*)^{N(r)-1}| \geq \ldots \geq |a_n|.
\]

Let us consider
\[
Q(z) = (\rho^*)^{N(r)} p(z)^{\frac{1}{\rho^*}}.
\]
Now
\[
|Q(z)| \geq \frac{|a_{N(r)}| |z|^N(r) - |a_0| |(\rho^*)^{N(r)}| |z|^N(r)-1}{|z|^N(r)-1}.
\]
Therefore it follows from above that
\[
|Q(z)| > 0 \text{ if } |a_{N(r)}| |z|^N(r) - |a_0| |(\rho^*)^{N(r)}| |z|^N(r)-1 > 0.
\]

The above can be written as
\[
\text{i.e., if } |a_{N(r)}| |z|^N(r) > |a_0| |(\rho^*)^{N(r)}| |z|^N(r)-1
\]
\[
\text{i.e., if } |a_{N(r)}| |z|^N(r)(|z| - 1) > |a_0| |(\rho^*)^{N(r)}| |z|^N(r)-1
\]
\[
\text{i.e., if } |a_{N(r)}| |z|^N(r)-1 > (|a_{N(r)}| + |a_0| |(\rho^*)^{N(r)}|) |z|^N(r) + |a_0| |(\rho^*)^{N(r)}| > 0.
\]

Let us consider
\[
g(t) \equiv \frac{|a_{N(r)}| |z|^N(r)+1 - (|a_{N(r)}| + |a_0| |(\rho^*)^{N(r)}|) |z|^N(r) + |a_0| |(\rho^*)^{N(r)}|}{|z|^N(r)+1}.
\]

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In order to prove the lower bound of Theorem 2 let us consider

\[ R(z) = (\rho^*)^{N(r)} z^{N(r)} P \left( \frac{1}{\rho^*} \right). \]

Then

\[
R(z) = (\rho^*)^{N(r)} z^{N(r)} \left( a_0 + \frac{a_1}{\rho^* z} + \cdots + a_{N(r)} \frac{1}{(\rho^*)^{N(r)} z^{N(r)}} \right) = a_0 \rho^* z^{N(r)} + a_1 (\rho^*)^{N(r)} z^{N(r)-1} + \cdots + a_{N(r)}. \]

Now

\[ |R(z)| \geq |a_0| (\rho^*)^{N(r)} |z|^{N(r)} - |a_1| (\rho^*)^{N(r)} |z|^{N(r)-1} - \cdots - |a_{N(r)}|. \]

So applying the condition \( |a_0| \rho^* |z|^{N(r)} \geq |a_1| (\rho^*)^{N(r)} |z|^{N(r)-1} \) we get from above that

\[ -|a_1| (\rho^*)^{N(r)} |z|^{N(r)-1} - \cdots - |a_{N(r)}| \geq -|a_1| (\rho^*)^{N(r)} |z|^{N(r)-1} - \cdots - |a_{N(r)}|. \]

Using (15) we get for \( |z| \neq 1 \)

\[ |R(z)| \geq (\rho^*)^{N(r)-1} \left( |a_0| \rho^* |z|^{N(r)} - |a_1| \cdot \frac{|z|^{N(r)-1}}{|z|^{N(r)}-1} \right). \]

Now

\[ R(z) > 0 \text{ if } (\rho^*)^{N(r)-1} \left( |a_0| \rho^* |z|^{N(r)} - |a_1| \cdot \frac{|z|^{N(r)-1}}{|z|^{N(r)}-1} \right) > 0 \]

i.e., \( |a_0| \rho^* |z|^{N(r)} - |a_1| \cdot \frac{|z|^{N(r)-1}}{|z|^{N(r)}-1} > 0 \)

i.e., \( |a_0| \rho^* |z|^{N(r)} > |a_1| \cdot \frac{|z|^{N(r)-1}}{|z|^{N(r)}-1} \)

i.e., \( |a_0| \rho^* |z|^{N(r)} - |a_1| \left( \frac{|z|^{N(r)-1}}{|z|^{N(r)}-1} \right) > 0 \)

i.e., \( |a_0| \rho^* |z|^{N(r)} - |a_1| \left( \frac{|z|^{N(r)-1}}{|z|^{N(r)}-1} \right) > 0 \).

Let us consider

\[ f(t) \equiv |a_0| \rho^* t^{N(r)-1} - |(a_0 | a_1| t^{N(r)} + |a_1|. \]

Clearly \( f(t) = 0 \) has two positive roots, because the number of changes of sign of \( f(t) \) is two. If it is less, less by two. Also \( t = 1 \) is the one of the positive roots of \( f(t) = 0 \). Let us suppose that \( t = t_0 \) be the other positive root. Also let \( t_0 = \max \{1, t_2\} \) and so \( t_0 \geq 1 \). Now \( t > t_0 \) implies \( f(t) > 0 \). If not then there exists some \( t_0 < t_0 \) such that \( f(t_0) < 0 \).

Also \( f(\infty) > 0 \). Therefore there exists another positive root in \( (t_0, \infty) \) which is a contradiction.

So \( |R(z)| > 0 \text{ if } |z| > t_0 \). Thus \( R(z) \) does not vanish in \( |z| > t_0 \). In other words all the zeros of \( R(z) \) lie in \( |z| \leq t_0 \).

Let \( z = z_0 \) be any zero of \( P(z) \). So \( P(z_0) = 0 \). Clearly \( z_0 \neq 0 \) as \( a_0 \neq 0 \).

Putting \( z = \frac{1}{\rho^* z_0} \) in \( R(z) \) we get that

\[ R \left( \frac{1}{\rho^* z_0} \right) = (\rho^*)^{N(r)} \cdot \left( \frac{1}{\rho^* z_0} \right)^{N(r)} \cdot P(z_0) = \left( \frac{1}{\rho^* z_0} \right)^{N(r)} \cdot 0 = 0. \]

Therefore \( \frac{1}{\rho^* z_0} \) is a root of \( R(z) \). So

\[ \frac{1}{\rho^* z_0} \leq t_0 \quad \text{implies } |z_0| \geq \frac{1}{\rho^* z_0}. \]

As \( z_0 \) is an arbitrary zero of \( P(z) = 0 \), all the zeros of \( P(z) \) lie in \( |z| \geq \frac{1}{\rho^* z_0} \).

From (14) and (17) we have all the zeros of \( P(z) \) lie in the ring shaped region given by

\[ \frac{1}{\rho^* z_0} \leq |z| \leq \frac{1}{\rho^* z_0}. \]

where \( t_0 \) and \( t_0 \) are the greatest positive roots of \( g(t) = 0 \) and \( f(t) = 0 \) respectively.

This proves the theorem.
Corollary 2 From Theorem 2 we can easily conclude that all the zeros of 
\[ P(x) = a_0 + a_1 x + \ldots + a_n x^n \]
of degree \( n \) with property \( |a_0| \geq |a_1| \geq \ldots \geq |a_n| \) lie in the ring shaped region 
\[ \frac{1}{\rho^* t_0} \leq |x| \leq \frac{1}{\rho^* t_0} \]
where \( t_0 \) and \( t'_0 \) are the greatest positive roots of 
\[ g(t) \equiv |a_n| t^{n+1} - (|a_n| + |a_0|) t^n + |a_0| = 0 \]
and 
\[ f(t) \equiv |a_0| t^{n+1} - (|a_0| + |a_1|) t^n + |a_1| = 0 \]
respectively by putting \( \rho^* = 1 \).

Remark 1 The limit of Theorem 2 is attained by \( P(x) = a^2 x^2 - a x - 1, \) \( a > 0 \). Here \( P(x) = a^2 x^2 - a x - 1, \) \( a_0 = -1, \)
\( a_1 = -a, \) \( a_2 = a^2 \). Therefore \( |a_0| = 1, |a_1| = a, |a_2| = a^2 \). Let \( \rho^* = a \). So \( |a_0| (\rho^*)^2 \geq |a_1| \rho^* \geq |a_2| \) holds. Hence 
\[ g(t) \equiv |a_2| t^3 - (|a_2| + |a_0|) t^2 + |a_0| a^2 = 0 \]
i.e., 
\[ a^2 (t^3 - 2t^2 + 1) = 0. \]
Now \( g(t) = 0 \) has two positive roots which are \( t_1 = 1 \) and 
\[ t_2 = \frac{\sqrt{5} + 1}{2}. \]
So \( t_0 = \max (t_1, t_2) = \frac{\sqrt{5} + 1}{2} \).

Again 
\[ t'_0 = \max \text{ (positive roots of } f(t) = 0) = \frac{\sqrt{5} + 1}{2}. \]

Hence by Theorem 2, all the zeros lie in 
\[ \frac{1}{\rho^* t_0} \leq |x| \leq \frac{1}{\rho^* t_0} \]
i.e., 
\[ \frac{\sqrt{5} - 1}{2a} \leq |x| \leq \frac{\sqrt{5} + 1}{2a}. \]

Now 
\[ P(x) = 0 \]
i.e., 
\[ a^2 x^2 - a x - 1 = 0 \]
i.e., 
\[ x = \frac{1 \pm \sqrt{5}}{2a}. \]

Let 
\[ z_1 = \frac{1 + \sqrt{5}}{2a} \quad \text{and} \quad z_2 = \frac{1 - \sqrt{5}}{2a}. \]
Clearly \( z_1 \) lie on the upper bound and \( z_2 \) lie on the lower bound of the boundary. Also here the order \( \rho = 0 \) because 
\[ M(r) = |a|^r \quad \text{for large } r \text{ in the circle } |x| = r. \]
Therefore 
\[ \rho = \limsup_{r \to \infty} \frac{\log \log M(r)}{\log r} = \limsup_{r \to \infty} \frac{\log \log a^r r^2}{\log r} = \limsup_{r \to \infty} \frac{1}{\log a + \frac{1}{2} \log r + 2 \log a^r} = \limsup_{r \to \infty} \frac{2}{\log a + \frac{1}{2} \log r + 2 \log a^r} = 0. \]

Also \( \rho^* = 2 \) and \( N(r) = 2 \leq (\log r)^{2 + \epsilon} \) for \( \epsilon > 0 \) and sufficiently large \( r \) in \( |x| \leq \log r \) and \( a_n = 0 \) for \( n > N(r) \).

Corollary 3 Under the conditions of Theorem 2 and 
\[ P(x) = a_0 + a_{p_1} z^{p_1} + \ldots + a_{p_m} z^{p_m} + a_{N(r)} z^{N(r)} \]
with 
\[ 1 \leq p_1 \leq p_2 \leq \ldots \leq p_m \leq N(r) - 1, \]
where \( p_i \)'s are integers \( a_0, a_{p_1}, \ldots, a_{N(r)} \) are non vanishing coefficients with 
\[ |a_0| (\rho^*)^{N(r)} \geq |a_{p_1}| (\rho^*)^{N(r)-p_1} \geq \ldots \geq |a_{p_m}| (\rho^*)^{N(r)-p_m} \geq |a_{N(r)}| \]
then we can show that all the zeros of \( P(x) \) lie in 
\[ \frac{1}{\rho^* t_0} \leq |x| \leq \frac{1}{\rho^* t_0} \]
where \( t_0 \) and \( t'_0 \) are the greatest positive roots of 
\[ g(t) \equiv |a_{N(r)}| t^{N(r)+1} - (|a_{N(r)}| + |a_0| (\rho^*)^{N(r)}) t^{N(r)} + |a_0| (\rho^*)^{N(r)} = 0 \]
and 
\[ f(t) \equiv |a_0| (\rho^*)^{p_1} t^{N(r)+1} - (|a_0| (\rho^*)^{p_1} + |a_{p_1}|) t^{N(r)} - |a_{p_1}| = 0 \]
respectively.

Corollary 4 If we put \( \rho^* = 1 \) in Corollary 3 then all the zeros of 
\[ P(x) = a_0 + a_{p_1} z^{p_1} + \ldots + a_{p_m} z^{p_m} + a_n z^n \]
lie in the ring shaped region.
where $t_0$ and $t'_0$ are the greatest positive roots of
\[ g(t) \equiv |\alpha_0| t^{n+1} - (|\alpha_n| + |\alpha_0|) t^n + |\alpha_0| = 0 \]
and
\[ f(t) \equiv |\alpha_0| t^{n+1} - (|\alpha_0| + |\alpha_{p_1}|) t^n - |\alpha_{p_1}| = 0 \]
respectively

provided
\[ |\alpha_0| \geq |\alpha_{p_1}| \geq \ldots \geq |\alpha_{p_n}| \geq |\alpha_n|. \]

REFERENCES


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SOME RESULTS RELATED TO CAUCHY’S PROPER BOUND FOR THE ZEROS OF ENTIRE FUNCTIONS OF ORDER ZERO

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Abstract

A single valued function of one complex variable which is analytic in the finite complex plane is called an entire function. In this paper we would like to establish the bounds for the moduli of zeros of entire functions of order zero. Some examples are provided to clear the notions.

AMS Subject Classification 2010: Primary 30C15, 30C10, Secondary 26C10.

Key words and phrases: Zeros of entire functions of order zero, Cauchy’s bound, Proper ring shaped region.
1 Introduction, Definitions and Notations.

Let 

\[ P(z) = a_0 + a_1 z + a_2 z^2 + a_3 z^3 + \ldots + a_{n-1} z^{n-1} + a_n z^n; |a_n| \neq 0 \]

be a polynomial of degree \( n \). Datt and Govil[2]; Govil and Rahaman[5]; Marden[9]; Mohammad[10]; Chattopadhyay, Das, Jain and Konwer[1]; Joyal, Labelle and Rahaman[6]; Jain{[7],[8]}; Sun and Hsieh[11]; Zilovic, Roytman, Combettes and Swamy[13]; Das and Datta[4] etc. worked in the theory of the distribution of the zeros of polynomials and obtained some newly developed results.

In this paper we intend to establish some sharper results concerning the theory of distribution of zeros of entire functions of order zero.

The following definitions are well known:

**Definition 1** The order \( \rho \) and lower order \( \lambda \) of a meromorphic function \( f \) are defined as

\[ \rho = \limsup_{r \to \infty} \frac{\log T(r,f)}{\log r} \quad \text{and} \quad \lambda = \liminf_{r \to \infty} \frac{\log T(r,f)}{\log r}. \]

If \( f \) is entire, one can easily verify that

\[ \rho = \limsup_{r \to \infty} \frac{\log^2 M(r,f)}{\log r} \quad \text{and} \quad \lambda = \liminf_{r \to \infty} \frac{\log^2 M(r,f)}{\log r}, \]

where \( \log^k x = \log(\log^{k-1} x) \) for \( k = 1, 2, 3, \ldots \) and \( \log^0 x = x \).

If \( \rho < \infty \) then \( f \) is of finite order. Also \( \rho = 0 \) means that \( f \) is of order zero. In this connection Datta and Biswas [3] gave the following definition:

**Definition 2** Let \( f \) be a meromorphic function of order zero. Then the quantities \( \rho^* \) and \( \lambda^* \) of \( f \) are defined by:

\[ \rho^* = \limsup_{r \to \infty} \frac{T(r,f)}{\log r} \quad \text{and} \quad \lambda^* = \liminf_{r \to \infty} \frac{T(r,f)}{\log r}. \]

If \( f \) is an entire function then clearly

\[ \rho^* = \limsup_{r \to \infty} \frac{\log M(r,f)}{\log r} \quad \text{and} \quad \lambda^* = \liminf_{r \to \infty} \frac{\log M(r,f)}{\log r}. \]
2 Lemmas.

In this section we present a lemma which will be needed in the sequel.

**Lemma 1** If \( f(z) \) is an entire function of order \( \rho = 0 \), then for every \( \varepsilon > 0 \) the inequality

\[
N(r) \leq (\log r)^{\rho^* + \varepsilon}
\]

holds for all sufficiently large \( r \) where \( N(r) \) is the number of zeros of \( f(z) \) in \( |z| \leq \log r \).

**Proof.** Let us suppose that \( f(z) = 1 \). This supposition can be made without loss of generality because if \( f(z) \) has a zero of order \( 'm' \) at the origin then we may consider \( g(z) = c \cdot f(z)^{1/m} \) where \( c \) is so chosen that \( g(0) = 1 \). Since the function \( g(z) \) and \( f(z) \) have the same order therefore it will be unimportant for our investigations that the number of zeros of \( g(z) \) and \( f(z) \) differ by \( m \).

We further assume that \( f(z) \) has no zeros on \( |z| = \log 2r \) and the zeros \( z_i \)'s of \( f(z) \) in \( |z| < \log r \) are in non decreasing order of their moduli so that \( |z_i| \leq |z_{i+1}| \). Also let \( \rho^* \) suppose to be finite where \( \rho = 0 \) is the zero of order of \( f(z) \).

Now we shall make use of Jenson's formula as state below

\[
\log |f(0)| = - \sum_{i=1}^{n} \log \frac{R}{|z_i|} + \frac{1}{2\pi} \int_{0}^{2\pi} \log |f(R e^{i\phi})| \, d\phi. \tag{1}
\]

Let us replace \( R \) by \( 2r \) and \( n \) by \( N(2r) \) in (1).

\[
\therefore \log |f(0)| = - \sum_{i=1}^{N(2r)} \log \frac{2r}{|z_i|} + \frac{1}{2\pi} \int_{0}^{2\pi} \log |f(2r e^{i\phi})| \, d\phi. \tag{2}
\]

Since \( f(0) = 1 \).. \( \log |f(0)| = \log 1 = 0. \)

\[
\therefore \sum_{i=1}^{N(2r)} \log \frac{2r}{|z_i|} = \frac{1}{2\pi} \int_{0}^{2\pi} \log |f(2r e^{i\phi})| \, d\phi. \tag{3}
\]

L.H.S. = \( \sum_{i=1}^{N(2r)} \log \frac{2r}{|z_i|} \geq \sum_{i=1}^{N(2r)} \log \frac{2r}{|z_i|} \geq N(r) \log 2 \)  \( (3) \)
because for large values of \( r \),

\[
\log \frac{2r}{|z_i|} \geq \log 2.
\]

\[
\text{R.H.S} = \frac{1}{2\pi} \int_0^{2\pi} \log |f(2re^{i\phi})| \, d\phi
\]

\[
\leq \frac{1}{2\pi} \int_0^{2\pi} \log M(2r) \, d\phi = \log M(2r). \tag{4}
\]

Again by definition of order \( \rho^* \) of \( f(z) \) we have for every \( \varepsilon > 0 \),

\[
\log M(2r) \leq \{\log (2r)\}^{\rho^*+\varepsilon/2}. \tag{5}
\]

Hence from (2) by the help of (3), (4) and (5) we have

\[
N(r) \log 2 \leq (\log 2r)^{\rho^*+\varepsilon/2}
\]

i.e., \( N(r) \leq \frac{(\log 2)^{\rho^*+\varepsilon/2}}{\log 2} \cdot \frac{(\log r)^{\rho^*+\varepsilon}}{(\log r)^{\varepsilon/2}} \leq (\log r)^{\rho^*+\varepsilon} \).

This proves the lemma.

## 3 Theorems.

In this section we present the main results of the paper.

**Theorem 1** Let \( P(z) \) be an entire function defined by

\[
P(z) = a_0 + a_1z + a_2z^2 + \ldots + a_nz^n + \ldots
\]

whose order \( \rho = 0 \). Also for all sufficiently large \( r \) in the disc \( |z| \leq \log r, |a_{N(r)}| \neq 0, |a_0| \neq 0 \). and also \( a_n \to 0 \) as \( n > N(r) \). Then all the zeros of \( P(z) \) lie in the ring shaped region

\[
\frac{1}{t_0^\varepsilon} \leq |z| \leq t_0
\]
where \( t_0 \) is the greatest positive root of

\[
g(t) \equiv \left|a_{N(r)}\right| t^{N(r)+1} - \left|a_{N(r)}\right| \left( t^{N(r)} + M \right) = 0
\]

and \( t'_0 \) is the greatest positive root of

\[
f(t) \equiv \left|a_0\right| t^{N(r)+1} - \left|a_0\right| t^{N(r)} + M' = 0
\]

where \( M = \max \left\{ |a_0|, |a_1|, \ldots, |a_{N(r)-1}| \right\} \)

and \( M' = \max \left\{ |a_1|, |a_2|, \ldots, |a_{N(r)}| \right\} \).

Proof. Now

\[
P(z) \approx a_0 + a_1 z + a_2 z^2 + \ldots + a_{N(r)} z^{N(r)}
\]

because \( N(r) \) exists for \(|z| \leq \log r\); \( r \) is sufficiently large and \( a_n \to 0 \) as \( n > N(r) \). Then all the zeros of \( P(z) \) lie in the ring shaped region given in Theorem 1 which we are to prove.

Now

\[
|P(z)| \approx \left| a_0 + a_1 z + a_2 z^2 + \ldots + a_{N(r)} z^{N(r)} \right|
\]

\[
\geq \left| a_{N(r)} \right| |z|^{N(r)} - \left| a_0 + a_1 z + a_2 z^2 + \ldots + a_{N(r)-1} z^{N(r)-1} \right|
\]

Also

\[
\left| a_0 + a_1 z + a_2 z^2 + \ldots + a_{N(r)-1} z^{N(r)-1} \right|
\]

\[
\leq |a_0| + \ldots + |a_{N(r)-1}| |z|^{N(r)-1}
\]

\[
\leq M \left( 1 + |z| + \ldots + |z|^{N(r)-1} \right)
\]

\[
= M \frac{|z|^{N(r)} - 1}{|z| - 1} \quad \text{if } |z| \neq 1. \tag{6}
\]

Therefore using (6) we obtain that

\[
|P(z)| \geq |a_{N(r)}| |z|^{N(r)} - \left| a_0 + a_1 z + a_2 z^2 + \ldots + a_{N(r)-1} z^{N(r)-1} \right|
\]

\[
\geq |a_{N(r)}| |z|^{N(r)} - M \frac{|z|^{N(r)} - 1}{|z| - 1}
\]

Hence

\[
|P(z)| \geq 0 \text{ if } |a_{N(r)}| |z|^{N(r)} - M \frac{|z|^{N(r)} - 1}{|z| - 1} > 0
\]
\[ |a_{N(r)}| \cdot |z|^{N(r)} > M \left| \frac{|z|^{N(r)} - 1}{|z| - 1} \right| \]

i.e., if \[ |a_{N(r)}| \cdot |z|^{N(r) + 1} - |a_{N(r)}| \cdot |z|^{N(r)} > M \left( |z|^{N(r)} - 1 \right) \]

i.e., if \[ |a_{N(r)}| \cdot |z|^{N(r) + 1} - |a_{N(r)}| \cdot |z|^{N(r)} - M \cdot |z|^{N(r)} + M > 0 \]

i.e., if \[ |a_{N(r)}| \cdot |z|^{N(r) + 1} - (|a_{N(r)}| + M) \cdot |z|^{N(r)} + M > 0. \]

Therefore on \[ |z| \neq 1, \]

\[ |P(z)| \geq 0 \text{ if } |a_{N(r)}| \cdot |z|^{N(r) + 1} - (|a_{N(r)}| + M) \cdot |z|^{N(r)} + M > 0. \]

Now let us consider

\[ g(t) \equiv |a_{N(r)}| \cdot t^{N(r) + 1} - (|a_{N(r)}| + M) \cdot t^{N(r)} + M = 0. \quad (7) \]

Clearly the maximum number of changes in sign in (7) is two. So the maximum number of positive roots of \( g(t) = 0 \) is two and by Descartes' rule of sign if it is less, less by two. Clearly \( t = 1 \) is one positive root of (7). So \( g(t) = 0 \) must have another positive root \( t_1 \) (say).

Let us take \( t_0 = \max \{1, t_1\} \). Clearly for \( t > t_0 \), \( g(t) > 0 \). If not, for some \( t = t_2 > t_0, g(t_2) < 0 \).

Now \( g(t_2) < 0 \) and \( g(\infty) > 0 \) imply that \( g(t) = 0 \) has another positive root in \((t_2, \infty)\) which gives a contradiction.

Therefore for \( t > t_0, g(t) > 0 \) and so \( t_0 > 1 \).

Hence \[ |P(z)| \geq 0 \text{ for } |z| > t_0. \]

Therefore all the zeros of \( P(z) \) lie in the disc \[ |z| \leq t_0. \] (8)

Again let us consider

\[ Q(z) = z^{N(r)}P\left( \frac{1}{z} \right) \]

\[ \approx z^{N(r)} \left\{ a_0 + \frac{a_1}{z} + \ldots + \frac{a_{N(r)}}{z^{N(r)}} \right\} \]

\[ = a_0z^{N(r)} + a_1z^{N(r)-1} + \ldots + a_{N(r)} \]

i.e., \[ |Q(z)| \geq |a_0| \cdot |z|^{N(r)} - |a_1z^{N(r)-1} + \ldots + a_{N(r)}| \] for \[ |z| \neq 1. \]

Now \[ |a_1z^{N(r)-1} + \ldots + a_{N(r)}| \leq |a_1| \cdot |z|^{N(r)-1} + \ldots + |a_{N(r)}| \]
\[
M' \left( |z|^{N(r)-1} + \ldots + 1 \right) \\
= M' \left( \frac{|z|^{N(r)} - 1}{|z| - 1} \right) \text{ for } |z| \neq 1. \quad (9)
\]

Using (9) we get that
\[
|Q(z)| \geq |a_0| |z|^{N(r)} - |a_1 z^{N(r)-1} + \ldots + a_N(r)| \\
\geq |a_0| |z|^{N(r)} - M' \left( \frac{|z|^{N(r)} - 1}{|z| - 1} \right) \text{ for } |z| \neq 1.
\]

Therefore for $|z| \neq 1$,
\[
|Q(z)| \geq 0 \text{ if } |a_0| |z|^{N(r)} - M' \left( \frac{|z|^{N(r)} - 1}{|z| - 1} \right) > 0 \\
\text{i.e., if } |a_0| |z|^{N(r)} > M' \left( \frac{|z|^{N(r)} - 1}{|z| - 1} \right)
\]
\[
\text{i.e., if } |a_0| |z|^{N(r)+1} - |a_0| |z|^{N(r)} - M' |z|^{N(r)} + M' > 0 \\
\text{i.e., if } |a_0| |z|^{N(r)+1} - (|a_0| + M') |z|^{N(r)} + M' > 0.
\]

So for $|z| \neq 1$,
\[
|Q(z)| \geq 0 \text{ if } |a_0| |z|^{N(r)+1} - (|a_0| + M') |z|^{N(r)} + M' > 0.
\]

Let us consider
\[
f(t) \equiv |a_0| t^{N(r)+1} - (|a_0| + M') t^{N(r)} + M' = 0.
\]

Since the maximum number of changes of sign in $f(t)$ is two, the maximum number of positive roots of $f(t) = 0$ is two and by Descartes’ rule of sign if it is less, less by two. Clearly $t = 1$ is one positive root of $f(t) = 0$. So $f(t) = 0$ must have another positive root.

Let us take $t'_0 = \max \{1, t_2\}$. Clearly for $t > t'_0$, $f(t) > 0$. If not, for some $t_3 > t'_0$, $f(t_3) < 0$. Now $f(t_3) < 0$ and $f(\infty) > 0$ implies that $f(t) = 0$ have another positive root in the interval $(t_3, \infty)$ which is a contradiction. Therefore for $t > t'_0$, $f(t) > 0$.  

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Also $t'_0 \geq 1$. So $|Q(z)| \geq 0$ for $|z| > t'_0$.
Therefore $Q(z)$ does not vanish in $|z| > t'_0$.

Hence all the zeros of $Q(z)$ lie in $|z| \leq t'_0$.

Let $z = z_0$ be a zero of $P(z)$. Therefore $P(z_0) = 0$. Clearly $z_0 \neq 0$ as $a_0 \neq 0$.
Putting $z = \frac{1}{z_0}$ in $Q(z)$ we get that

$$Q\left(\frac{1}{z_0}\right) = \left(\frac{1}{z_0}\right)^{N(r)}P(z_0) = \left(\frac{1}{z_0}\right)^{N(r)} . 0 = 0.$$ 

Therefore $Q\left(\frac{1}{z_0}\right) = 0$. So $z = \frac{1}{z_0}$ is a root of $Q(z) = 0$. Hence \(\left|\frac{1}{z_0}\right| \leq t'_0\)
implies that $|z_0| \geq \frac{1}{t'_0}$.
As $z_0$ is an arbitrary root of $P(z) = 0$.

Therefore all the zeros of $P(z)$ lie in $|z| \geq \frac{1}{t'_0}$. \hspace{1cm} (10)

From (8) and (10) we get that all the zeros of $P(z)$ lie in the proper ring shaped region

$$\frac{1}{t'_0} \leq |z| \leq t_0$$

where $t_0$ and $t'_0$ are the greatest positive roots of the equations

$$g(t) \equiv |a_{N(r)}| t^{N(r)+1} - (|a_{N(r)}| + M) t^{N(r)} + M = 0$$

and

$$f(t) \equiv |a_0| t^{N(r)+1} - (|a_0| + M') t^{N(r)} + M' = 0$$

where $M$ and $M'$ are given in the statement of Theorem 1.

This proves the theorem.

**Remark 1** The limit in Theorem 1 is attained by $P(z) = z^2 - z - 1$. Here $\rho = 0$, $\rho^* = 2$ and $N(r) = 2 \leq (\log r)^{2+\epsilon}$. For $\epsilon > 0$ and sufficiently large $r$, all $a_n = 0$, $n \geq 2$. Also $a_0 = -1$, $a_1 = -1$, $a_2 = 1$.

Therefore

$$M = \max \{|a_0|, |a_1|\} = 1 \text{ and } M' = \max \{|a_1|, |a_2|\} = 1$$

and

$$g(t) \equiv |a_2| t^3 - (|a_2| + M) t^2 + M = 0$$

i.e., $g(t) \equiv t^3 - (1 + 1)t^2 + 1 = 0$

i.e., $g(t) \equiv t^3 - 2t^2 + 1 = 0$. 
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Again

\[ f(t) \equiv |a_0| t^3 - (|a_0| + M') t^2 + M' = 0 \]

i.e., \[ f(t) \equiv 1. t^3 - (1 + 1) t^2 + 1 = 0 \]

i.e., \[ f(t) \equiv t^3 - 2t^2 + 1 = 0. \]

So \( f(t) = 0 \) and \( g(t) = 0 \) represent the same equation. Maximum number of positive roots of \( f(t) = 0 \) and \( g(t) = 0 \) are same. Now

\[ g(t) = 0 \]

implies that \[ t^3 - 2t^2 + 1 = 0 \]

i.e., \[ (t - 1) (t^2 - t - 1) = 0. \]

Therefore

\[ t = 1 \text{ and } t = \frac{1 \pm \sqrt{(-1)^2 - 4.1.(-1)}}{2.1} = \frac{1 \pm \sqrt{5}}{2}. \]

Hence the positive roots of \( g(t) = 0 \) are 1 and \( \frac{1 + \sqrt{5}}{2} \). So

\[ t_0 = \max \left\{ 1, \frac{1 + \sqrt{5}}{2} \right\} = \frac{1 + \sqrt{5}}{2}. \]

Also the maximum positive root of \( f(t) = 0 \) is

\[ t_0' = \max \left\{ 1, \frac{1 + \sqrt{5}}{2} \right\} = \frac{1 + \sqrt{5}}{2}. \]

So in view of Theorem 1 all the zeros of \( P(z) \) lie in

\[ \frac{1}{t_0'} \leq |z| \leq t_0 \]

i.e., \[ \frac{1}{\frac{1 + \sqrt{5}}{2}} \leq |z| \leq \frac{1 + \sqrt{5}}{2} \]

i.e., \[ \frac{\sqrt{5} - 1}{2} \leq |z| \leq \frac{1 + \sqrt{5}}{2}. \]
Now the zeros of $P(z)$ are given by solving $z^2 - z - 1 = 0$. Therefore $z = \frac{1\pm\sqrt{5}}{2}$. Let us denote the zeros of $P(z)$ by $z_1 = \frac{1+\sqrt{5}}{2}$ and $z_2 = \frac{1-\sqrt{5}}{2}$. Clearly $z_1$ lies on the upper boundary and $z_2$ lies on the lower boundary. So the best possible result is given by $P(z) = z^2 - z - 1$.

**Theorem 2** Let $P(z)$ be an entire function defined by

$$P(z) = a_0 + a_1 z + a_2 z^2 + \ldots + a_n z^n + \ldots\ldots$$

with order $\rho = 0$, $a_{N(r)} \neq 0$, $a_0 \neq 0$ and also $a_n \to 0$ for $n > N(r)$ for the disc $|z| \leq \log r$ when $r$ is sufficiently large. Further for some $\rho^* > 0$,

$$|a_0| (\rho^*)^{N(r)} \geq |a_1| (\rho^*)^{N(r)-1} \geq \ldots \geq |a_{N(r)-1}| \rho^* \geq |a_{N(r)}|.$$  

Then all the zeros of $P(z)$ lie in the ring shaped region

$$\frac{1}{\rho^* \left(1 + \frac{|a_1|}{|a_0| \rho^*} \right)} < |z| < \frac{1}{\rho^* \left(1 + \frac{|a_0|}{|a_{N(r)}|} (\rho^*)^{N(r)} \right)}.$$  

**Proof.** For the given entire function

$$P(z) = a_0 + a_1 z + a_2 z^2 + \ldots + a_n z^n + \ldots\ldots$$

with $a_n \to 0$ as $n > N(r)$, where $r$ is sufficiently large, $N(r)$ exists and $N(r) \leq (\log r)^\rho^* + \varepsilon$.

Therefore

$$P(z) \approx a_0 + a_1 z + a_2 z^2 + \ldots + a_{N(r)} z^{N(r)}$$

as $a_0 \neq 0, a_{N(r)} \neq 0$ and $a_n \to 0$ for $n > N(r)$.

Let us consider

$$R(z) = (\rho^*)^{N(r)} P \left( \frac{z}{\rho^*} \right)$$

$$\approx (\rho^*)^{N(r)} \left( a_0 + \frac{a_1 z}{\rho^*} + \frac{a_2 z^2}{(\rho^*)^2} + \ldots + a_{N(r)} \frac{z^{N(r)}}{(\rho^*)^{N(r)}} \right)$$

$$= (a_0 (\rho^*)^{N(r)} + a_1 (\rho^*)^{N(r)-1} z + \ldots + a_{N(r)} z^{N(r)}) \cdot$$

Therefore

$$|R(z)| \geq \left| a_{N(r)} \right| |z|^{N(r)}$$

$$\geq - \left| a_0 (\rho^*)^{N(r)} + a_1 (\rho^*)^{N(r)-1} z + \ldots + a_{N(r)-1} \rho^* z^{N(r)-1} \right| . (11)$$
Now by the given condition $|a_0| (\rho^*)^{N(r)} \geq |a_1| (\rho^*)^{N(r)-1} \geq \ldots$ provided $|z| \neq 0$, we obtain that

$$
|a_0(\rho^*)^{N(r)} + a_1(\rho^*)^{N(r)-1} z + \ldots + a_{N(r)-1} \rho^* z^{N(r)-1}| \\
\leq |a_0| (\rho^*)^{N(r)} + \ldots + |a_{N(r)-1}| \rho^* |z|^{N(r)-1} \\
\leq |a_0| (\rho^*)^{N(r)} |z|^{N(r)} \left( \frac{1}{|z|} + \ldots + \frac{1}{|z|^{N(r)}} \right).
$$

Therefore on $|z| \neq 0$,

$$
- |a_0(\rho^*)^{N(r)} + a_1(\rho^*)^{N(r)-1} z + \ldots + a_{N(r)-1} \rho^* z^{N(r)-1}| \\
\geq - |a_0| (\rho^*)^{N(r)} |z|^{N(r)} \left( \frac{1}{|z|} + \ldots + \frac{1}{|z|^{N(r)}} \right).
$$

Therefore using (12) we get from (11) that

$$
|R(z)| \geq |a_{N(r)}| |z|^{N(r)} - |a_0| (\rho^*)^{N(r)} |z|^{N(r)} \left( \frac{1}{|z|} + \ldots + \frac{1}{|z|^{N(r)}} \right) \\
\geq |a_{N(r)}| |z|^{N(r)} - |a_0| (\rho^*)^{N(r)} |z|^{N(r)} \left( \frac{1}{|z|} + \ldots + \frac{1}{|z|^{N(r)}} + \ldots \right) \\
= |z|^{N(r)} \left[ |a_{N(r)}| - |a_0| (\rho^*)^{N(r)} \left( \sum_{k=1}^{\infty} \frac{1}{|z|^k} \right) \right].
$$

Clearly $\sum_{k=1}^{\infty} \frac{1}{|z|^k}$ is a geometric series which is convergent for $\frac{1}{|z|} < 1$ i.e., for $|z| > 1$ and converges to

$$
\frac{1}{|z|} \frac{1}{1 - \frac{1}{|z|}} = \frac{1}{|z| - 1}.
$$

Therefore

$$
\sum_{k=1}^{\infty} \frac{1}{|z|^k} = \frac{1}{|z| - 1} \text{ if } |z| > 1.
$$

Hence we get from above that for $|z| > 1$

$$
|R(z)| > |z|^{N(r)} \left( |a_{N(r)}| - (\rho^*)^{N(r)} |a_0| \frac{1}{|z| - 1} \right).
$$
Now for \(|z| > 1\),

\[ |R(z)| > 0 \text{ if } |z|^N(r) \left( |a_{N(r)}| - (\rho^*)^N(r) |a_0| \frac{1}{|z| - 1} \right) \geq 0 \]

i.e., if \(|a_{N(r)}| - (\rho^*)^N(r) |a_0| \frac{1}{|z| - 1} \geq 0\)

i.e., if \(|a_{N(r)}| \geq (\rho^*)^N(r) \frac{|a_0|}{|z| - 1}\)

i.e., if \(|z| - 1 \geq (\rho^*)^N(r) \frac{|a_0|}{a_{N(r)}}\)

i.e., if \(|z| \geq 1 + (\rho^*)^N(r) \frac{|a_0|}{a_{N(r)}} > 1\).

Therefore

\[ |R(z)| > 0 \text{ if } |z| \geq 1 + (\rho^*)^N(r) \frac{|a_0|}{a_{N(r)}}. \]

So all the zeros of \(R(z)\) lie in

\[ |z| < 1 + \frac{|a_0|}{a_{N(r)}} (\rho^*)^N(r). \]

Let \(z_0\) be an arbitrary zero of \(P(z)\). Therefore \(P(z_0) = 0\). Clearly \(z_0 \neq 0\) as \(z_0 \neq 0\). Putting \(z = \rho^* z_0\) in \(R(z)\) we have

\[ R(\rho^* z_0) = (\rho^*)^N(r) P(z_0) = (\rho^*)^N(r).0 = 0. \]

Hence \(z = \rho^* z_0\) is a zero of \(R(z)\). Therefore

\[ |\rho^* z_0| < 1 + \frac{|a_0|}{a_{N(r)}} (\rho^*)^N(r) \]

i.e., \(|z_0| < \frac{1}{\rho^*} \left( 1 + \frac{|a_0|}{a_{N(r)}} (\rho^*)^N(r) \right)\).

Since \(z_0\) is any zero of \(P(z)\) therefore all the zeros of \(P(z)\) lie in

\[ |z| < \frac{1}{\rho^*} \left( 1 + \frac{|a_0|}{a_{N(r)}} (\rho^*)^N(r) \right). \quad (13) \]
Again let us consider

\[ F(z) = (\rho^*)^N(r) z^N(r) P \left( \frac{1}{\rho^* z} \right). \]

Now

\[
F(z) = (\rho^*)^N(r) z^N(r) P \left( \frac{1}{\rho^* z} \right) \\
\approx (\rho^*)^N(r) z^N(r) \left\{ a_0 + \frac{a_1}{\rho^* z} + \ldots + \frac{a_N(r)}{(\rho^* z)^N(r)} \right\} \\
= a_0 (\rho^*)^N(r) z^N(r) + a_1 (\rho^*)^N(r-1) z^{N(r)-1} + \ldots + a_N(r).
\]

Therefore

\[
|F(z)| \geq |a_0| (\rho^*)^N(r) |z|^N(r) - |a_1 (\rho^*)^{N(r)-1} z^{N(r)-1} + \ldots + a_N(r)|.
\]

Again

\[
|a_1 (\rho^*)^{N(r)-1} z^{N(r)-1} + \ldots + a_N(r)| \\
\leq |a_1| (\rho^*)^{N(r)-1} |z|^{N(r)-1} + \ldots + |a_N(r)| \\
\leq |a_1| (\rho^*)^{N(r)-1} \left( |z|^{N(r)-1} + \ldots + |z| + 1 \right)
\]

provided \(|z| \neq 0\). So

\[
a_1 (\rho^*)^{N(r)-1} z^{N(r)-1} + \ldots + a_N(r) \\
\leq |a_1| (\rho^*)^{N(r)-1} |z|^{N(r)} \left( \frac{1}{|z|} + \ldots + \frac{1}{|z|^{N(r)}} \right).
\]

So for \(|z| \neq 0\),

\[
|F(z)| \geq |a_0| (\rho^*)^{N(r)} |z|^{N(r)} - |a_1| (\rho^*)^{N(r)-1} |z|^{N(r)} \left( \frac{1}{|z|} + \ldots + \frac{1}{|z|^{N(r)}} \right) \\
= (\rho^*)^{N(r)-1} |z|^{N(r)} \left| a_0 |\rho^* - a_1 | \right( \frac{1}{|z|} + \ldots + \frac{1}{|z|^{N(r)}} \right).
\]
Therefore for $|z| \neq 0$,

$$|F(z)| > (\rho^*)^{N(r)-1} |z|^{N(r)} \left[ |a_0| \rho^* - |a_1| \sum_{k=1}^{\infty} \frac{1}{|z|^k} \right].$$ \hspace{1cm} (14)

The geometric series $\sum_{k=1}^{\infty} \frac{1}{|z|^k}$ is convergent for

$$\frac{1}{|z|} < 1$$

i.e., for $|z| > 1$

and converges to

$$\frac{1}{|z| - 1}$$

Therefore

$$\sum_{k=1}^{\infty} \frac{1}{|z|^k} = \frac{1}{|z| - 1} \quad \text{if} \quad |z| > 1.$$ \hspace{1cm} (15)

Using (14) and (15) we have for $|z| > 1$,

$$|F(z)| > (\rho^*)^{N(r)-1} |z|^{N(r)} \left[ |a_0| \rho^* - \frac{|a_1|}{|z| - 1} \right].$$

Hence for $|z| > 1$,

$$|F(z)| > 0 \quad \text{if} \quad |z|^{N(r)} (\rho^*)^{N(r)-1} \left[ |a_0| \rho^* - \frac{|a_1|}{|z| - 1} \right] \geq 0$$

i.e., if $|a_0| \rho^* - \frac{|a_1|}{|z| - 1} \geq 0$

i.e., if $|a_0| \rho^* \geq \frac{|a_1|}{|z| - 1}$

i.e., if $|z| \geq 1 + \frac{|a_1|}{|a_0| \rho^*} > 1$.

Therefore

$$|F(z)| > 0 \quad \text{for} \quad |z| > 1 + \frac{|a_1|}{|a_0| \rho^*}.$$
So \( F(z) \) does not vanish in
\[
|z| \geq 1 + \frac{|a_1|}{|a_0| \rho^*}.
\]
Equivalently all the zeros of \( F(z) \) lie in
\[
|z| < 1 + \frac{|a_1|}{|a_0| \rho^*}.
\]
Let \( z = z_0 \) be any zero of \( P(z) \). Therefore \( P(z_0) = 0 \). Clearly \( a_0 \neq 0 \) and \( z_0 \neq 0 \).

Now let us put \( z = \frac{1}{\rho^* z_0} \) in \( F(z) \). So we have
\[
F \left( \frac{1}{\rho^* z_0} \right) = (\rho^*)^{N(r)} \left( \frac{1}{\rho^* z_0} \right)^{N(r)} P(z_0)
\]
\[
= \left( \frac{1}{z_0} \right)^{N(r)} P(z_0)
\]
\[
= 0.
\]
Therefore \( z = \frac{1}{\rho^* z_0} \) is a root of \( F(z) \).

Hence
\[
\left| \frac{1}{\rho^* z_0} \right| < 1 + \frac{|a_1|}{|a_0| \rho^*}
\]
i.e.,
\[
\left| \frac{1}{z_0} \right| < \rho^* \left( 1 + \frac{|a_1|}{|a_0| \rho^*} \right)
\]
i.e.,
\[
\left| z_0 \right| > \frac{1}{\rho^* \left( 1 + \frac{|a_1|}{|a_0| \rho^*} \right)}.
\]

As \( z_0 \) is an arbitrary zero of \( P(z) \), all the zeros of \( P(z) \) lie on
\[
|z| > \frac{1}{\rho^* \left( 1 + \frac{|a_1|}{|a_0| \rho^*} \right)}.
\]
(16)

From (13) and (16) we get that all the zeros of \( P(z) \) lie on the proper ring shaped region
\[
\frac{1}{\rho^* \left( 1 + \frac{|a_1|}{|a_0| \rho^*} \right)} < |z| < \rho^* \left( 1 + \frac{|a_0|}{|a_N(r)| (\rho^*)^{N(r)}} \right)
\]
where
\[ |a_0| (\rho^*)^{N(r)} \geq |a_1| (\rho^*)^{N(r)-1} \geq \ldots \geq |a_{N(r)}| \]
for some \( \rho^* > 0 \).

This proves the theorem.

**Corollary 1** From Theorem 2 we can easily conclude that all the zeros of
\[ P(z) = a_0 + a_1 z + \ldots + a_n z^n \]
of degree \( n \), \( a_n \neq 0 \) with the property \( |a_0| \geq |a_1| \geq \ldots \geq |a_n| \) lie in the proper ring shaped region
\[ \frac{1}{1 + \frac{|a_1|}{|a_0|}} < |z| < \left( 1 + \frac{|a_0|}{|a_n|} \right) \]
just on putting \( \rho^* = 1 \).

**Theorem 3** Let \( P(z) \) be an entire function with order \( \rho = 0 \). For sufficiently large values of \( r \) in the disk \( |z| \leq \log r \), the Taylor’s series expansion of \( P(z) \)
\[ P(z) = a_0 + a_{p_1} z^{p_1} + a_{p_2} z^{p_2} + \ldots + a_{p_m} z^{p_m} + a_{N(r)} z^{N(r)} \], \( a_0 \neq 0 \)
be such that \( 1 \leq p_1 < p_2 \ldots \ldots < p_m \leq N(r) - 1 \), \( p_i \)'s are integers and for some \( \rho^* > 0 \),
\[ |a_0| (\rho^*)^{N(r)} \geq |a_{p_1}| (\rho^*)^{N(r)-p_1} \geq \ldots \geq |a_{p_m}| (\rho^*)^{N(r)-p_m}. \]
Then all the zeros of \( P(z) \) lie in the proper ring shaped region
\[ \frac{1}{\rho^* t_0'} < |z| < \frac{1}{\rho^* t_0} \]
where \( t_0 \) and \( t_0' \) are the unique positive roots of the equations
\[ g(t) \equiv a_{N(r)} t^{N(r)-p_m} - a_{N(r)} t^{N(r)-p_m-1} - |a_0| (\rho^*)^{N(r)} = 0 \text{ and } \]
\[ f(t) \equiv |a_0| (\rho^*)^{p_1} t^{p_1} - |a_0| (\rho^*)^{p_1} t^{p_1-1} - |a_{p_1}| = 0 \]
respectively.
Proof. Let
\[ P(z) = a_0 + a_{p_1}z^{p_1} + \ldots + a_{p_m}z^{p_m} + a_{N(r)}z^{N(r)}, \quad |a_{N(r)}| \neq 0. \]  
(17)

Also for some \( \rho^* > 0 \),
\[ |a_0| (\rho^*)^{N(r)} \geq |a_{p_1}| (\rho^*)^{N(r)-p_1} \geq \ldots \geq |a_{N(r)}|. \]

Let us consider
\[ R(z) = (\rho^*)^{N(r)} P \left( \frac{z}{\rho^*} \right) \]
\[ = (\rho^*)^{N(r)} \left\{ a_0 + a_{p_1} \frac{z^{p_1}}{\rho^*} + \ldots + a_{p_m} \frac{z^{p_m}}{(\rho^*)^{p_m}} + a_{N(r)} \frac{z^{N(r)}}{(\rho^*)^{N(r)}} \right\} \]
\[ = a_0 (\rho^*)^{N(r)} + a_{p_1} (\rho^*)^{N(r)-p_1} z^{p_1} + \ldots + a_{p_m} (\rho^*)^{N(r)-p_m} z^{p_m} + a_{N(r)} z^{N(r)}. \]
Therefore
\[ |R(z)| \geq |a_{N(r)} z^{N(r)}| - |a_0 (\rho^*)^{N(r)} + a_{p_1} (\rho^*)^{N(r)-p_1} z^{p_1} + \ldots + a_{p_m} (\rho^*)^{N(r)-p_m} z^{p_m}| \]  
(18)

Now for \( |z| \neq 0 \),
\[ |a_0 (\rho^*)^{N(r)} + a_{p_1} (\rho^*)^{N(r)-p_1} z^{p_1} + \ldots + a_{p_m} (\rho^*)^{N(r)-p_m} z^{p_m}| \]
\[ \leq |a_0| (\rho^*)^{N(r)} + |a_{p_1}| (\rho^*)^{N(r)-p_1} |z|^{p_1} + \ldots + |a_{p_m}| (\rho^*)^{N(r)-p_m} |z|^{p_m} \]
\[ \leq |a_0| (\rho^*)^{N(r)} (1 + |z|^{p_1} + \ldots + |z|^{p_m}) \]
\[ = |a_0| (\rho^*)^{N(r)} |z|^{p_m+1} \left( \frac{1}{|z|} + \ldots + \frac{1}{|z|^{p_m+1-p_2}} + \frac{1}{|z|^{p_m+1-p_1}} + \frac{1}{|z|^{p_m+1}} \right). \]
(19)

Using (18) and (19), we have for \( |z| \neq 0 \)
\[ |R(z)| \]
\[ \geq |a_{N(r)}| |z|^{N(r)} - |a_0| (\rho^*)^{N(r)} |z|^{p_m+1} \left( \frac{1}{|z|} + \ldots + \frac{1}{|z|^{p_m+1-p_1}} + \frac{1}{|z|^{p_m+1}} \right) \]
\[ > |a_{N(r)}| |z|^{N(r)} - |a_0| (\rho^*)^{N(r)} |z|^{p_m+1} \left( \frac{1}{|z|} + \ldots + \frac{1}{|z|^{p_m+1-p_1}} + \frac{1}{|z|^{p_m+1}} + \ldots \right) \]
\[ = |a_{N(r)}| |z|^{N(r)} - |a_0| (\rho^*)^{N(r)} |z|^{p_m+1} \sum_{k=1}^{\infty} \frac{1}{|z|^k}. \]  
(20)
The geometric series $\sum_{k=1}^{\infty} \frac{1}{|z|^k}$ is convergent for
\[ \frac{1}{|z|} < 1 \]
i.e., for $|z| > 1$
and converges to
\[ \frac{1}{|z| - 1} = \frac{1}{|z| - 1}. \]
Therefore
\[ \sum_{k=1}^{\infty} \frac{1}{|z|^k} = \frac{1}{|z| - 1} \text{ for } |z| > 1. \]
So on $|z| > 1$,
\[ |R(z)| > 0 \text{ if } |a_{N(r)}| |z|^N(r) - \frac{|a_0| (\rho^*)^N(r) |z|^{p_m+1}}{|z| - 1} \geq 0 \]
i.e., if $|a_{N(r)}| |z|^N(r) \geq \frac{|a_0| (\rho^*)^N(r) |z|^{p_m+1}}{|z| - 1}$
i.e., if $|a_{N(r)}| |z|^{N(r)+1} - |a_{N(r)}| |z|^N(r) \geq |a_0| (\rho^*)^N(r) |z|^{p_m+1}$
i.e., if $|z|^{p_m+1} \left( |a_{N(r)}| |z|^N(r)-p_m - |a_{N(r)}| |z|^{N(r)-p_m-1} - |a_0| (\rho^*)^N(r) \right) \geq 0$.

Let us consider
\[ g(t) \equiv |a_{N(r)}| |t|^{N(r)-p_m} - |a_{N(r)}| |t|^{N(r)-p_m-1} - |a_0| (\rho^*)^N(r) = 0. \]
Clearly $g(t) = 0$ has one positive root because the maximum number of changes in sign in $g(t)$ is one and $g(0) = -|a_0| \rho^N(r)$ is $-ve$, $g(\infty)$ is $+ve$.
Let $t_0$ be the positive root of $g(t) = 0$ and $t_0 > 1$. Clearly for $t > t_0$, $g(t) \geq 0$.
If not for some $t_1 > t_0$, $g(t_1) < 0$.
Then $g(t_1) < 0$ and $g(\infty) > 0$. Therefore $g(t) = 0$ must have another positive root in $(t_1, \infty)$ which gives a contradiction.
Hence for $t \geq t_0$, $g(t) \geq 0$ and $t_0 > 1$. So $|R(z)| > 0$ for $|z| \geq t_0$.
Thus $R(z)$ does not vanish in $|z| \geq t_0$.
Hence all the zeros of $R(z)$ lie in $|z| < t_0$.

Let $z = z_0$ be any zero of $P(z)$. So $P(z_0) = 0$. Clearly $z_0 \neq 0$ as $a_0 \neq 0$.
Putting $z = \rho^* z_0$ in $R(z)$ we have
\[ R(\rho^* z_0) = (\rho^*)^N(r).P(z_0) = (\rho^*)^N(r).0 = 0. \]
Therefore \( R(\rho^*z_0) = 0 \) and so \( z = \rho^*z_0 \) is a zero of \( R(z) \) and consequently \( |\rho^*z_0| < t_0 \) which implies \( |z_0| < \frac{t_0}{\rho^*} \). As \( z_0 \) is an arbitrary zero of \( P(z) \),

all the zeros of \( P(z) \) lie in \( |z| < \frac{t_0}{\rho^*} \). \hspace{1cm} (21)

Again let us consider

\[
F(z) = (\rho^*)^{N(r)}z^{N(r)}P \left( \frac{1}{\rho^*z} \right).
\]

Now

\[
F(z) = \left(\rho^*\right)^{N(r)}z^{N(r)}
\]

\[
= a_0 + a_{p_1} \frac{1}{(\rho^*)^{p_1}z^{p_1}} + \ldots + a_{p_m} \frac{1}{(\rho^*)^{p_m}z^{p_m}} + a_{N(r)} \frac{1}{(\rho^*)^{N(r)}z^{N(r)}}
\]

\[
= a_0 (\rho^*)^{N(r)}z^{N(r)} + a_{p_1} (\rho^*)^{N(r) - p_1}z^{N(r) - p_1} + \ldots + a_{p_m} (\rho^*)^{N(r) - p_m}z^{N(r) - p_m} + a_{N(r)}.
\]

Also

\[
|a_{p_1} (\rho^*)^{N(r) - p_1}z^{N(r) - p_1}| + \ldots + |a_{p_m} (\rho^*)^{N(r) - p_m}z^{N(r) - p_m} + a_{N(r)}|
\]

\[
\leq |a_{p_1}| (\rho^*)^{N(r) - p_1} |z|^{N(r) - p_1} + \ldots + |a_{p_m}| (\rho^*)^{N(r) - p_m} |z|^{N(r) - p_m} + |a_{N(r)}|
\]

\[
\leq |a_{p_1}| (\rho^*)^{N(r) - p_1} \left( |z|^{N(r) - p_1} + |z|^{N(r) - p_2} + \ldots + |z|^{N(r) - p_m} + 1 \right).
\]

So for \( |z| \neq 0 \),

\[
|F(z)|
\]

\[
\geq |a_0| (\rho^*)^{N(r)} |z|^{N(r)}
\]

\[
- |a_{p_1}| (\rho^*)^{N(r) - p_1} |z|^{N(r) - p_1} \ldots + |a_{p_m}| (\rho^*)^{N(r) - p_m} |z|^{N(r) - p_m} + a_{N(r)}|
\]

\[
\geq |a_0| (\rho^*)^{N(r)} |z|^{N(r)}
\]

\[
- |a_{p_1}| (\rho^*)^{N(r) - p_1} \left( |z|^{N(r) - p_1} + |z|^{N(r) - p_2} + \ldots + |z|^{N(r) - p_m} + 1 \right)
\]

\[
= |a_0| (\rho^*)^{N(r)} |z|^{N(r)}
\]

\[
- |a_{p_1}| (\rho^*)^{N(r) - p_1} |z|^{N(r) - p_1 + 1} \left( \frac{1}{|z|} + \frac{1}{|z|^{p_2 - p_1 + 1}} + \ldots + \frac{1}{|z|^{N(r) - p_1 + 1}} \right)
\]

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i.e., on $|z| \neq 0$, 

$$|F(z)| > |a_0| \left( \rho^* \right)^{N(r)} |z|^{N(r)} - |a_{p_1}| \left( \rho^* \right)^{N(r) - p_1} |z|^{N(r) - p_1 + 1} \left( \sum_{k=1}^{\infty} \frac{1}{|z|^k} \right).$$

The geometric series $\sum_{k=1}^{\infty} \frac{1}{|z|^k}$ is convergent for

$$\frac{1}{|z|} < 1$$

i.e., for $|z| > 1$

and converges to

$$\frac{1}{|z| - 1} = \frac{1}{|z| - 1}.$$

Therefore

$$\sum_{k=1}^{\infty} \frac{1}{|z|^k} = \frac{1}{|z| - 1} \quad \text{for} \quad |z| > 1.$$

Therefore for $|z| > 1$

$$|F(z)| > |a_0| \left( \rho^* \right)^{N(r)} |z|^{N(r)} - |a_{p_1}| \left( \rho^* \right)^{N(r) - p_1} |z|^{N(r) - p_1 + 1} \left( \frac{1}{|z| - 1} \right)$$

$$= (\rho^*)^{N(r) - p_1} \left( (\rho^*)^{p_1} |a_0| |z|^{N(r)} - |a_{p_1}| \frac{|z|^{N(r) - p_1 + 1}}{|z| - 1} \right)$$

$$= (\rho^*)^{N(r) - p_1} |z|^{N(r) - p_1 + 1} \left( |a_0| (\rho^*)^{p_1} |z|^{p_1 - 1} - |a_{p_1}| \frac{1}{|z| - 1} \right)$$

For $|z| > 1$, 

$$|F(z)| > 0 \text{ if } |a_0| (\rho^*)^{p_1} |z|^{p_1 - 1} - |a_{p_1}| \frac{1}{|z| - 1} \geq 0$$

i.e., if $|a_0| (\rho^*)^{p_1} |z|^{p_1 - 1} \geq |a_{p_1}| \frac{1}{|z| - 1}$

i.e., if $|a_0| (\rho^*)^{p_1} |z|^{p_1} - |a_0| (\rho^*)^{p_1} |z|^{p_1 - 1} - |a_{p_1}| \geq 0. \quad (22)$
Therefore on $|z| > 1, |F(z)| > 0$ if (22) holds.

Let us consider

$$f(t) = |a_0| (\rho^s t^p) - |a_0| (\rho^s t^{p-1}) - |a_p| = 0.$$  

Clearly $f(t) = 0$ has exactly one positive root and is greater than one. Let $t'_0$ be the positive root of $f(t) = 0$. Therefore $t'_0 > 1$. Obviously if $t \geq t'_0$ then $f(t) \geq 0$. So for $|F(z)| > 0, |z| \geq t'_0$. Therefore $F(z)$ does not vanish in $|z| \geq t'_0$.

Hence all the zeros of $F(z)$ lie in $|z| < t'_0$.

Let $z = z_0$ be any zero of $P(z)$. Therefore $P(z_0) = 0$. Clearly $z_0 \neq 0$ as $a_0 \neq 0$.

Now putting $z = \frac{1}{\rho^s z_0}$ in $F(z)$ we obtain that

$$F\left(\frac{1}{\rho^s z_0}\right) = (\rho^s)^{N(r)} \left(\frac{1}{\rho^s z_0}\right)^{N(r)} . P(z_0)$$

$$= \left(\frac{1}{z_0}\right)^{N(r)} . P(z_0)$$

$$= \left(\frac{1}{z_0}\right)^{N(r)} . 0$$

$$= 0.$$  

Therefore $z = \frac{1}{\rho^s z_0}$ is a zero of $F(z)$. Now

$$\left|\frac{1}{\rho^s z_0}\right| < t'_0$$

i.e., $\left|\frac{1}{z_0}\right| < \rho^s t'_0$  

i.e., $|z_0| > \frac{1}{\rho^s t'_0}$.

As $z_0$ is an arbitrary zero of $P(z)$ therefore we obtain that

all the zeros of $P(z)$ lie in $|z| > \frac{1}{\rho^s t'_0}$.  

(23)
Using (21) and (23) we get that all the zeros of $P(z)$ lie in the ring shaped region

$$\frac{1}{\rho^* t_0'} < |z| < \frac{t_0}{\rho^*}$$

where $t_0, t_0'$ are the unique positive roots of the equations $g(t) = 0$ and $f(t) = 0$ respectively whose form is given in the statement of Theorem 3. This proves the theorem.

**Corollary 2** In view of Theorem 3 we may state that all the zeros of the polynomial $P(z) = a_0 + a_{p_1}z^{p_1} + \ldots + a_{p_m}z^{p_m} + a_nz^n$ of degree $n$ with $1 \leq p_1 < p_2 < \ldots < p_m \leq n - 1, p_i$’s are integers such that

$$|a_0| \geq |a_{p_1}| \geq \ldots \geq |a_n|$$

lie in ring shaped region

$$\frac{1}{t_0'} < |z| < t_0$$

where $t_0, t_0'$ are the unique positive roots of the equations

$$g(t) \equiv |a_n| t^{n-p_m} - |a_n| t^{n-p_m-1} - |a_0| = 0$$

and

$$f(t) \equiv |a_0| t^{p_1} - |a_0| t^{p_1-1} - |a_{p_1}| = 0$$

respectively just substituting $\rho^* = 1$.

### References


Maximum Modulus and Maximum Terms-Related Growth Properties of Entire Functions Based on Relative Type and Relative Weak Type

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Research Article

Abstract: In the paper we study the comparative growth properties of composite entire function on the basis of relative order, relative type and relative weak type with respect to another entire function.

Keywords and phrases: Entire function, maximum term, maximum modulus, composition, growth, relative order, relative type, relative weak type.

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Using the inequalities \( \mu(r,f) \leq M(r,f) \leq \frac{R}{R-r} \mu(R,f) \) \{ cf. [8] \}, for \( 0 \leq r < R \) one may verify that

\[
\rho_f = \limsup_{r \to \infty} \frac{\log^{[2]} M_f(r)}{\log r} \quad \text{and} \quad \lambda_f = \liminf_{r \to \infty} \frac{\log^{[2]} M_f(r)}{\log r}.
\]

Definition 1 The order \( \rho_f \) and lower order \( \lambda_f \) of an entire function \( f \) are defined as

\[
\rho_f = \limsup_{r \to \infty} \frac{\log^{[2]} M_f(r)}{\log r} \quad \text{and} \quad \lambda_f = \liminf_{r \to \infty} \frac{\log^{[2]} M_f(r)}{\log r}.
\]

1. Introduction, Definitions and Notations

We denote by \( \mathbb{C} \) the set of all finite complex numbers. Let \( f \) be an entire function defined in the open complex plane \( \mathbb{C} \). The maximum term \( \mu(r,f) \) of \( f = \sum_{n=0}^{\infty} a_n z^n \) on \( |z| = r \) is defined by \( \mu(r,f) = \max_{|z|=r} |a_n| r^n \) and the maximum modulus \( M(r,f) \) of \( f = \sum_{n=0}^{\infty} a_n z^n \) on \( |z| = r \) is defined by \( M(r,f) = \max_{|z|=r} |f(z)| \).

In the sequel we use the following notation:

\[
\log^{[k]} x = \log (\log^{[k-1]} x) \quad \text{for} \quad k = 1, 2, 3, \ldots \quad \text{and} \quad \log^{[0]} x = x.
\]

To start our paper we just recall the following definition:

Definition 1 The order \( \rho_f \) and lower order \( \lambda_f \) of an entire function \( f \) are defined as

\[
\rho_f = \limsup_{r \to \infty} \frac{\log^{[2]} M_f(r)}{\log r} \quad \text{and} \quad \lambda_f = \liminf_{r \to \infty} \frac{\log^{[2]} M_f(r)}{\log r}.
\]

Using the inequalities \( \mu(r,f) \leq M(r,f) \leq \frac{R}{R-r} \mu(R,f) \) \{ cf. [8] \}, for \( 0 \leq r < R \) one may verify that

\[
\rho_f = \limsup_{r \to \infty} \frac{\log^{[2]} M_f(r)}{\log r} \quad \text{and} \quad \lambda_f = \liminf_{r \to \infty} \frac{\log^{[2]} M_f(r)}{\log r}.
\]

Definition 2 The type \( \sigma_f \) of an entire function \( f \) is defined as

\[
\sigma_f = \limsup_{r \to \infty} \frac{\log M_f(r)}{r^{\rho_f}}, \quad 0 < \rho_f < \infty.
\]

Datta and Jha [3] introduced the definition of weak type of a meromorphic function of finite positive lower order in the following way:

Definition 3 [3] The weak type \( \tau_f \) of an entire function \( f \) of finite positive lower order \( \lambda_f \) is defined by

\[
\tau_f = \liminf_{r \to \infty} \frac{\log M_f(r)}{r^{\lambda_f}}.
\]

If an entire function \( g \) is non-constant then \( M_g(r) \) is strictly increasing and continuous and its inverse \( M_g^{-1} : (\log(0), \infty) \to (0, \infty) \) exists and is such that

\[
\lim_{s \to \infty} M_g^{-1}(s) = \infty.
\]

Bernal [1] introduced the definition of relative order of an entire function \( f \) with respect to an entire function \( g \), denoted by \( \rho_{g}(f) \) as follows:

\[
\rho_{g}(f) = \inf \{ \mu > 0 : M_f(r) < M_g(r^\mu) \text{ for all } r > r_0(\mu) > 0 \} = \limsup_{r \to \infty} \frac{\log M_g^{-1} M_f(r)}{\log r}.
\]

The definition coincides with the classical one [9] if \( g(z) = \exp z \).
Similarly one can define the relative lower order of an entire function $f$ with respect to an entire function $g$ denoted by $\lambda_g(f)$ as follows:

$$\lambda_g(f) = \liminf_{r \to \infty} \frac{\log M_g^2(M_f(r))}{\log r}.$$  

Datta and Maji [4] gave an alternative definition of relative order and relative lower order of an entire function with respect to another entire in the following way:

**Definition 4** [4] The relative order $\rho_g(f)$ and relative lower order $\lambda_g(f)$ of an entire function $f$ with respect to another entire function $g$ are defined as follows:

$$\rho_g(f) = \limsup_{r \to \infty} \frac{\log M_g^2(M_f(r))}{\log r}$$

and

$$\lambda_g(f) = \liminf_{r \to \infty} \frac{\log M_g^2(M_f(r))}{\log r}.$$  

Recently Roy [5] introduced the notion of relative type of two entire functions in the following manner:

**Definition 5** [5] Let $f$ and $g$ be any two entire functions such that $0 < \rho_g(f) < \infty$. Then the relative type $\sigma_g(f)$ of $f$ with respect to $g$ is defined as:

$$\sigma_g(f) = \inf \{ k > 0 : M_f(r) < M_g(kr^{\rho_g(f)}) \}$$

for all sufficiently large values of $r$.

Analogously to determine the relative growth of two entire functions having same non zero finite relative lower order with respect to another entire function, one may introduce the definition of relative weak type (in the notion of Datta and Jha [3]) of an entire function $f$ with respect to another entire function $g$ of finite positive relative lower order $\lambda_g(f)$ in the following way:

**Definition 6** The relative weak type $\tau_g(f)$ of an entire function $f$ with respect to another entire function $g$ having finite positive relative lower order $\lambda_g(f)$ is defined as:

$$\tau_g(f) = \liminf_{r \to \infty} \frac{M_g^2(M_f(r))}{r^{\lambda_g(f)}}.$$  

Considering $\tau_g(f) = \exp z$ one may easily verify that Definition 5 and Definition 6 coincide with the classical Definition 2 and Definition 3 respectively.

In the paper we study some relative growth properties of maximum term and maximum modulus of composition of entire functions with respect to another entire function on the basis of relative order, relative type and relative weak type. We do not explain the standard definitions and notations in the theory of entire functions as those are available in [10].

2. Lemmas

In this section we present some lemmas which will be needed in the sequel.

**Lemma 1** [2] If $f$ and $g$ are two entire functions then for all sufficiently large values of $r$,

$$M_{fog}(r) \leq M_f(M_g(r)).$$

**Lemma 2** [7] Let $f$ and $g$ be any two entire functions. Then for every $\alpha > 1$ and $0 < r < R$,

$$\mu_{fog}(r) \leq \frac{\alpha}{a-1} \mu_f \left( \frac{ar}{r-1} \right).$$

**Lemma 3** [1] Suppose $f$ is an entire function and $\alpha > 1$, $0 < \beta < \alpha$, then for all sufficiently large $r$,

$$M_{f}(\alpha r) \geq \beta M_f(r).$$

**Lemma 4** [4] If $f$ is entire and $\alpha > 1$, $0 < \beta < \alpha$, then for all sufficiently large $r$,

$$\mu_{f}(\alpha r) \geq \beta \mu_{f}(r).$$

**Lemma 5** Let $f$ and $g$ be any two entire functions. Then for any $\alpha > 1$,

$$(i) \quad M_{h}^{-1}(M_f(r)) \leq \frac{\alpha}{(a-1)} \mu_f(\alpha r)$$

and

$$(ii) \quad \mu_{h}^{-1}(M_f(r)) \leq \alpha M_{h}^{-1} \left( \frac{\alpha}{a-1} \right).$$

Proof. Taking $R = \alpha r$ in the inequalities $\mu_{h}(r) \leq M_{h}(r) \leq \frac{\alpha}{R-r} \mu_{h}(R)$, for $0 \leq r < R$ we obtain that

$$M_{h}^{-1}(r) \leq \frac{\alpha}{(a-1)} \mu_{h}^{-1}(r)$$

and

$$\mu_{h}^{-1}(r) \leq \alpha M_{h}^{-1} \left( \frac{\alpha}{(a-1)} \right).$$

Since $M_{h}^{-1}(r)$ and $\mu_{h}^{-1}(r)$ are increasing functions of $r$, the lemma follows from the above and the inequalities $\mu_{h}(r) \leq M_{f}(r) \leq \frac{\alpha}{a-1} \mu_{f}(\alpha r)$ [cf. [8]].

2. Theorems

In this section we present the main results of the paper.

**Theorem 1** Let $f$, $g$ and $h$ be any three entire functions such that $0 < \lambda_h(f) \leq \rho_h(f) < \infty$ and $\sigma_g(f) < \infty$. Then for each $\beta > 1$

$$\limsup_{r \to \infty} \frac{\log M_g^2(M_f(r))}{\log r} \leq \frac{\mu_g(\rho_f(r))}{\lambda_g(f)}.$$  

Proof. Taking $R = \beta r$ in Lemma 2 and in view of Lemma 3 we have for all sufficiently large values of $r$ that

$$\mu_{fog}(r) \leq \left( \frac{\alpha}{(a-1)} \right) \mu_f \left( \frac{arb}{(\beta-1)} \right) \mu_g(\beta r)$$

and

$$\mu_{fog}(r) \leq \frac{\alpha}{(a-1)} \mu_f \left( \frac{arb}{(\beta-1)} \right) \mu_g(\beta r).$$

Since $\mu_{h}^{-1}(r)$ is an increasing function of $r$, it follows from above for all sufficiently large values of $r$ that

$$\mu_{h}^{-1}(M_f(r)) \leq \mu_h^{-1} \left( \frac{\alpha}{(a-1)} \right) \mu_f \left( \frac{arb}{(\beta-1)} \right) \mu_g(\beta r)$$

and

$$\mu_{h}^{-1}(M_f(r)) \leq \mu_h^{-1} \left( \frac{arb}{(\beta-1)} \right) \mu_f \left( \frac{arb}{(\beta-1)} \right) \mu_g(\beta r).$$  

(1)
Let \( \lambda_h(g) > 0 \), \( \rho_h(f) < \infty \) and \( \sigma_g < \infty \). Then for any \( \beta > 1 \),

\[
\limsup_{r \to \infty} \log \mu^{1-\beta} \mu_{\text{reg}(r)}(\mathbb{R}) \leq \frac{\sigma_g \cdot \rho_h(f)}{\lambda_h(g)},
\]

The proof is omitted.

With the help of Lemma 1 and in the line of Theorem 1 and Theorem 2 the following two theorems may be proved:

**Theorem 3** Let \( f \), \( g \), and \( h \) be any three entire functions such that \( 0 < \lambda_h(g) \leq \rho_h(f) < \infty \) and \( \sigma_g < \infty \). Then

\[
\limsup_{r \to \infty} \log \mu^{1-\beta} \mu_{\text{reg}(r)}(\mathbb{R}) \leq \frac{\sigma_g \cdot \rho_h(f)}{\lambda_h(g)}.
\]

**Theorem 4** Let \( f \), \( g \), and \( h \) be any three entire functions with \( \lambda_h(g) > 0 \), \( \rho_h(f) < \infty \) and \( \sigma_g < \infty \). Then

\[
\limsup_{r \to \infty} \log \mu^{1-\beta} \mu_{\text{reg}(r)}(\mathbb{R}) \leq \frac{\sigma_g \cdot \rho_h(f)}{\lambda_h(g)}.
\]

Using the notion of weak type, we may state the following two theorems without proof because it can be carried out in the line of Theorem 1 and Theorem 3 respectively.

**Theorem 5** Let \( f \), \( g \), and \( h \) be any three entire functions such that \( 0 < \lambda_h(g) = \rho_h(f) < \infty \) and \( \tau_f < \infty \). Then for any \( \beta > 1 \),

\[
\liminf_{r \to \infty} \log \mu^{1-\beta} \mu_{\text{reg}(r)}(\mathbb{R}) \leq \tau_f.
\]

**Theorem 6** Let \( f \), \( g \), and \( h \) be any three entire functions with \( 0 < \lambda_h(g) = \rho_h(f) < \infty \) and \( \tau_f < \infty \). Then

\[
\liminf_{r \to \infty} \log \mu^{1-\beta} \mu_{\text{reg}(r)}(\mathbb{R}) \leq \tau_f.
\]
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\[
\limsup_{r \to \infty} \frac{\log \mu_{\delta+f}(r)}{\theta_{\delta+f}(r)} \leq \frac{\alpha + (a+y-\gamma)\delta}{(a-1)\theta_{\delta+f}(r)} \cdot \sigma_f(r) + \epsilon. 
\]

**Theorem 11** Let \( f, g \) and \( h \) be any three entire functions with \((i) 0 < \rho_h(f) < \infty \), \((ii) \lambda_h(f) = \rho_g \), \((iii) \sigma_g < \infty \),(iv) \( 0 < \sigma_h(f) < \infty \). Then

\[
\lim_{r \to \infty} \frac{\log \mu_{\delta+f}(r)}{\theta_{\delta+f}(r)} \leq \frac{\rho_f(r) - \sigma_f(r)}{\theta_{\delta+f}(r)}. 
\]

**Theorem 12** Let \( f, g \) and \( h \) be any three entire functions such that \((i) 0 < \rho_h(f) < \infty \), \((ii) \lambda_h(f) = \rho_g \), \((iii) \sigma_g < \infty \),(iv) \( 0 < \rho_h(f) < \infty \). Then

\[
\lim_{r \to \infty} \frac{\log \mu_{\delta+f}(r)}{\theta_{\delta+f}(r)} \leq \frac{\rho_f(r) - \sigma_f(r)}{\theta_{\delta+f}(r)}. 
\]

**Theorem 13** Let \( f, g \) and \( h \) be any three entire functions such that \((i) \lambda_h(f) = \rho_g \), \((ii) \sigma_g < \infty \), and \((iii) 0 < \rho_h(f) < \infty \). Then

\[
\lim_{r \to \infty} \frac{\log \mu_{\delta+f}(r)}{\theta_{\delta+f}(r)} \leq \frac{\lambda_f(r)}{\theta_{\delta+f}(r)}. 
\]

**Theorem 14** Let \( f, g \) and \( h \) be any three entire functions such that \((i) 0 < \rho_h(f) < \infty \), \((ii) \lambda_h(f) = \rho_g \), \((iii) \sigma_g < \infty \), and \((iv) 0 < \rho_h(f) < \infty \). Then

\[
\lim_{r \to \infty} \frac{\log \mu_{\delta+f}(r)}{\theta_{\delta+f}(r)} \leq \frac{\rho_f(r) - \sigma_f(r)}{\theta_{\delta+f}(r)}. 
\]

The proof of Theorem 11, Theorem 12, Theorem 13 and Theorem 14 are omitted as those can be carried out in view of Lemma 1 and the line of Theorem 7, Theorem 8, Theorem 9 and Theorem 10 respectively.

**Theorem 15** Let \( f, g \) and \( h \) be any three entire functions such that \((i) 0 < \rho_h(f) < \infty \), \((ii) 0 < \sigma_h(f) < \infty \), \((iii) \rho_h(f) = \rho_g(f) \), and \((iv) \sigma_h(f) < \infty \). Then for any \( \alpha > 1 \) and \( \gamma > 0 \),

\[
\lim_{r \to \infty} \frac{\mu_{\delta+f}(r)}{\mu_{\gamma+f}(r)} \leq \frac{\alpha \theta_{\delta+f}(r)}{\theta_{\gamma+f}(r)} \cdot \sigma_f(r) \leq \frac{\rho_f(r) - \sigma_f(r)}{\theta_{\gamma+f}(r)} 
\]

\[
\text{i.e., } \mu_{\delta+f}(r) \leq \alpha \theta_{\delta+f}(r) \cdot \sigma_f(r) + \epsilon \left( \frac{\alpha - 1}{\alpha - 1} \right) \theta_{\delta+f}(r) 
\]

(7)

\[
\text{i.e., } \mu_{\delta+f}(r) \leq \sigma_f(r) + \epsilon \left( \frac{\alpha - 1}{\alpha - 1} \right) \theta_{\delta+f}(r) 
\]

(8)

Also in view of Lemma 4 and Lemma 5 we obtain for a sequence of \( r \) tending to infinity that

\[
\mu_{\delta+f}(r) \geq \frac{\mu_{\delta+f}(r)}{\theta_{\delta+f}(r)} \geq \frac{\rho_f(r) - \sigma_f(r)}{\theta_{\delta+f}(r)} 
\]

\[
\text{i.e., } \mu_{\delta+f}(r) \geq (\sigma_f(r) - \epsilon) \left( \frac{\alpha - 1}{\alpha - 1} \right) \theta_{\delta+f}(r) 
\]

Thus the theorem follows from (12) and (14).

In the line of Theorem 15 we may state the following theorem without proof:

**Theorem 16** Let \( f, g \) and \( h \) be any three entire functions with \((i) 0 < \rho_h(g) < \infty \), \((ii) 0 < \sigma_h(g) < \infty \), \((iii) \rho_h(f) = \rho_g(g) \) and \((iv) \sigma_h(f) < \infty \). Then for any \( \alpha > 1 \) and \( \gamma > 0 \),

\[
\lim_{r \to \infty} \frac{\mu_{\delta+f}(r)}{\mu_{\gamma+f}(r)} \leq \frac{\alpha \theta_{\delta+f}(r)}{\theta_{\gamma+f}(r)} \cdot \sigma_f(r) \leq \frac{\rho_f(r) - \sigma_f(r)}{\theta_{\gamma+f}(r)} 
\]

\[
\text{i.e., } \mu_{\delta+f}(r) \geq (\sigma_f(r) - \epsilon) \left( \frac{\alpha - 1}{\alpha - 1} \right) \theta_{\delta+f}(r) 
\]

Using the notion of relative weak type, we may state the following two theorems without proof because those may be carried out with the help of Lemma 3 and Lemma 5 and in the line of Theorem 15 and Theorem 16 respectively.

**Theorem 17** Let \( f, g \) and \( h \) be any three entire functions such that \((i) 0 < \lambda_h(f) < \infty \), \((ii) 0 < \rho_h(f) < \infty \), \((iii) \lambda_h(f) = \lambda_g(f) \) and \((iv) \sigma_h(f) = \sigma_g(f) < \infty \). Then for any \( \alpha > 1 \) and \( \gamma > 0 \),
Theorem 18 Let \( f, g \) and \( h \) be any three entire functions such that (i) \( 0 < \lambda_h(g) < \infty \), (ii) \( 0 < \tau_h(g) < \infty \), (iii) \( \lambda_h(fg) = \lambda_h(g) \) and (iv) \( \tau_h(fg) < \infty \). Then for any \( \alpha > 1 \) and \( \gamma > 0 \),

\[
\liminf_{r \to \infty} \frac{\mu_{\alpha}^r H_{fg}(r)}{\mu_{\alpha}^r H_0(r)} \geq \frac{(\alpha - 1)^{2\varphi(r)}}{(\alpha - 1)^{2\varphi_0(r)}} \cdot \frac{\tau_h(fg)}{\tau_h(g)}.
\]

and

\[
\frac{\tau_h(fg)}{\tau_h(g)} \leq \limsup_{r \to \infty} \frac{\mu_{\alpha}^r H_{fg}(r)}{\mu_{\alpha}^r H_0(r)}.
\]

Similarly one may state the following four theorems without proof on the basis of relative type and relative weak type of entire functions with respect to another entire functions in terms of their maximum modulus:

Theorem 19 Let \( f, g \) and \( h \) be any three entire functions with (i) \( 0 < \lambda_h^r(f) < \infty \), (ii) \( 0 < \sigma_h^r(f) < \infty \), (iii) \( \lambda_h^r(fg) = \lambda_h^r(f) \) and (iv) \( \sigma_h^r(fg) < \infty \). Then for any \( \alpha > 1 \) and \( \gamma > 0 \),

\[
\liminf_{r \to \infty} \frac{\mu_{\alpha}^r H_{fg}(r)}{\mu_{\alpha}^r H_0(r)} \geq \frac{\tau_h^r(fg)}{\tau_h^r(r)} \leq \limsup_{r \to \infty} \frac{\mu_{\alpha}^r H_{fg}(r)}{\mu_{\alpha}^r H_0(r)}.
\]

Theorem 20 Let \( f, g \) and \( h \) be any three entire functions such that (i) \( 0 < \lambda_h^r(g) < \infty \), (ii) \( 0 < \sigma_h^r(g) < \infty \), (iii) \( \lambda_h^r(fg) = \lambda_h^r(g) \) and (iv) \( \sigma_h^r(fg) < \infty \). Then for any \( \alpha > 1 \) and \( \gamma > 0 \),

\[
\liminf_{r \to \infty} \frac{\mu_{\alpha}^r H_{fg}(r)}{\mu_{\alpha}^r H_0(r)} \geq \frac{\tau_h^r(fg)}{\tau_h^r(g)} \leq \limsup_{r \to \infty} \frac{\mu_{\alpha}^r H_{fg}(r)}{\mu_{\alpha}^r H_0(r)}.
\]

Theorem 21 Let \( f, g \) and \( h \) be any three entire functions with (i) \( 0 < \lambda_h^r(f) < \infty \), (ii) \( 0 < \sigma_h^r(f) < \infty \), (iii) \( \lambda_h^r(fg) = \lambda_h^r(f) \) and (iv) \( \sigma_h^r(fg) < \infty \). Then for any \( \alpha > 1 \) and \( \gamma > 0 \),

\[
\liminf_{r \to \infty} \frac{\mu_{\alpha}^r H_{fg}(r)}{\mu_{\alpha}^r H_0(r)} \geq \frac{\tau_h^r(fg)}{\tau_h^r(g)} \leq \limsup_{r \to \infty} \frac{\mu_{\alpha}^r H_{fg}(r)}{\mu_{\alpha}^r H_0(r)}.
\]

Theorem 22 Let \( f, g \) and \( h \) be any three entire functions such that (i) \( 0 < \lambda_h^r(g) < \infty \), (ii) \( 0 < \sigma_h^r(g) < \infty \), (iii) \( \lambda_h^r(fg) = \lambda_h^r(g) \) and (iv) \( \sigma_h^r(fg) < \infty \). Then for any \( \alpha > 1 \) and \( \gamma > 0 \),

\[
\liminf_{r \to \infty} \frac{\mu_{\alpha}^r H_{fg}(r)}{\mu_{\alpha}^r H_0(r)} \geq \frac{\tau_h^r(fg)}{\tau_h^r(g)} \leq \limsup_{r \to \infty} \frac{\mu_{\alpha}^r H_{fg}(r)}{\mu_{\alpha}^r H_0(r)}.
\]

Proof. From the definition of \( \tau_h(fg) \) and in view of Lemma 4 and Lemma 5 we obtain for all sufficiently large values of \( r \) that

\[
\mu_{\alpha}^r H_{fg}(r) \geq M_{\alpha}^r M_{\alpha}^r \left( \frac{(\alpha - 1)^{2\varphi(r)}}{(\alpha - 1)^{2\varphi_0(r)}} \cdot \frac{\tau_h(fg)}{\tau_h(g)} \right)
\]

i.e., \( \mu_{\alpha}^r H_{fg}(r) \geq (\tau_h(fg) - \varepsilon) \left( \frac{(\alpha - 1)^r}{(\alpha - 1)^{2\varphi_0(r)}} \cdot \frac{\tau_h(fg)}{\tau_h(g)} \right) \)

Thus from (8) and (15) we get for all sufficiently large values of \( r \) that

\[
\mu_{\alpha}^r H_{fg}(r) \geq \frac{(\alpha - 1)^{2\varphi(r)}}{(\alpha - 1)^{2\varphi_0(r)}} \cdot \frac{\tau_h(fg)}{\tau_h(g)} \cdot \tau_h(fg) - \varepsilon \cdot \lambda_h(fg) + \varepsilon \tau_h(fg).
\]

Since \( \lambda_h(fg) \geq \lambda_h(f) \) we obtain from (16) that

\[
\liminf_{r \to \infty} \frac{\mu_{\alpha}^r H_{fg}(r)}{\mu_{\alpha}^r H_0(r)} \geq \frac{\tau_h(fg)}{\tau_h(g)} \cdot \tau_h(fg) - \varepsilon \cdot \lambda_h(fg) + \varepsilon \tau_h(fg).
\]

As \( \varepsilon \) is arbitrary, it follows from above that

\[
\liminf_{r \to \infty} \frac{\mu_{\alpha}^r H_{fg}(r)}{\mu_{\alpha}^r H_0(r)} \geq \frac{\tau_h(fg)}{\tau_h(g)} - \varepsilon \cdot \lambda_h(fg) + \varepsilon \tau_h(fg).
\]

Thus the theorem is established.

Theorem 24 Let \( f, g \) and \( h \) be any three entire functions such that (i) \( 0 < \lambda_h^r(f) < \infty \), (ii) \( \tau_h^r(f) < \infty \), (iii) \( \lambda_h(fg) = \lambda_h(f) \) and (iv) \( \sigma_h^r(fg) < \infty \). Then for any \( \alpha > 1 \) and \( \gamma > 0 \),

\[
\limsup_{r \to \infty} \frac{\mu_{\alpha}^r H_{fg}(r)}{\mu_{\alpha}^r H_0(r)} \leq \frac{(\alpha - 1)^{2\varphi(r)}}{(\alpha - 1)^{2\varphi_0(r)}} \cdot \frac{\tau_h^r(fg)}{\tau_h^r(r)}.
\]

Proof. From the definition of \( \tau_h(fg) \) and in view of Lemma 4 and Lemma 5 we obtain for all sufficiently large values of \( r \) that

\[
\mu_{\alpha}^r H_{fg}(r) \geq M_{\alpha}^r M_{\alpha}^r \left( \frac{(\alpha - 1)^{2\varphi(r)}}{(\alpha - 1)^{2\varphi_0(r)}} \cdot \frac{\tau_h(fg)}{\tau_h(g)} \right)
\]

i.e., \( \mu_{\alpha}^r H_{fg}(r) \geq (\tau_h(fg) - \varepsilon) \left( \frac{(\alpha - 1)^r}{(\alpha - 1)^{2\varphi_0(r)}} \cdot \frac{\tau_h(fg)}{\tau_h(g)} \right) \)

Thus from (7) and (17) we get for all sufficiently large values of \( r \) that

\[
\mu_{\alpha}^r H_{fg}(r) \geq \frac{(\alpha - 1)^{2\varphi(r)}}{(\alpha - 1)^{2\varphi_0(r)}} \cdot \frac{\tau_h(fg)}{\tau_h(g)} \cdot \tau_h(fg) - \varepsilon \cdot \lambda_h(fg) + \varepsilon \tau_h(fg).
\]

As \( \varepsilon \) is arbitrary, it follows from above that

\[
\limsup_{r \to \infty} \frac{\mu_{\alpha}^r H_{fg}(r)}{\mu_{\alpha}^r H_0(r)} \leq \frac{(\alpha - 1)^{2\varphi(r)}}{(\alpha - 1)^{2\varphi_0(r)}} \cdot \frac{\tau_h^r(fg)}{\tau_h^r(r)}.
\]

Thus the theorem follows from above.
Theorem 26 Let \( f, g \) and \( h \) be any three entire functions such that (i) \( 0 < \lambda_h(g) < \infty \), (ii) \( \tau_h(g) < \infty \), (iii) \( \rho_h(fog) = \lambda_h(g) \) and (iv) \( \sigma_h(fog) > 0 \). Then for any \( \alpha > 1 \) and \( y > 0 \),

\[
\limsup_{r \to \infty} \frac{ \mu_h^\alpha M_{fog}(r) }{ \mu_h^\alpha M_f(r) } \leq \frac{ (e^{\alpha y} - 1)^{2\lambda_h(g)} }{ (e^{\alpha y} - 1)^{2\lambda_h(g) + 1} } \cdot \frac{ \sigma_h(fog) }{ \tau_h(g) } .
\]

Analogously we may also state the following four theorems without proof on the basis of relative type and relative weak type of entire functions with respect to another entire functions in terms of their maximum modulus:

Theorem 27 Let \( f, g \) and \( h \) be any three entire functions such that (i) \( 0 < \rho_h(f) < \infty \), (ii) \( \sigma_h(f) < \infty \), (iii) \( \lambda_h(fog) = \rho_h(f) \), and (iv) \( \tau_h(fog) > 0 \). Then

\[
\liminf_{r \to \infty} \frac{ M_h^\alpha M_{fog}(r) }{ M_h^\alpha M_f(r) } \geq \frac{ \tau_h(fog) }{ \sigma_h(f) } .
\]

Theorem 28 Let \( f, g \) and \( h \) be any three entire functions such that (i) \( 0 < \lambda_h(f) < \infty \), (ii) \( \tau_h(f) < \infty \), (iii) \( \rho_h(fog) = \lambda_h(f) \), and (iv) \( \sigma_h(fog) > 0 \). Then

\[
\limsup_{r \to \infty} \frac{ M_h^\alpha M_{fog}(r) }{ M_h^\alpha M_f(r) } \leq \frac{ \sigma_h(fog) }{ \tau_h(f) } .
\]

Theorem 29 Let \( f, g \) and \( h \) be any three entire functions such that (i) \( 0 < \rho_h(g) < \infty \), (ii) \( \sigma_h(g) < \infty \), (iii) \( \lambda_h(fog) = \rho_h(g) \), and (iv) \( \tau_h(fog) > 0 \). Then

\[
\liminf_{r \to \infty} \frac{ M_h^\alpha M_{fog}(r) }{ M_h^\alpha M_f(r) } \geq \frac{ \tau_h(fog) }{ \sigma_h(g) } .
\]

Theorem 30 Let \( f, g \) and \( h \) be any three entire functions such that (i) \( 0 < \lambda_h(g) < \infty \), (ii) \( \tau_h(g) < \infty \), (iii) \( \rho_h(fog) = \lambda_h(g) \), and (iv) \( \sigma_h(fog) > 0 \). Then

\[
\limsup_{r \to \infty} \frac{ M_h^\alpha M_{fog}(r) }{ M_h^\alpha M_f(r) } \leq \frac{ \sigma_h(fog) }{ \tau_h(g) } .
\]
Upon the recommendation of Prof. Banerji, I am pleased to inform you that your paper titled "SOME SHARPER ESTIMATIONS OF GROWTH RELATIONSHIP OF COMPOSITE ENTIRE FUNCTIONS ON THE BASIS OF THEIR MAXIMUM TERMS," will appear in Vol 2 (2013).

Thank you for considering the PMJ for your publication.

Regards
Ayman Badawi,
Editor in chief of the PMJ

---

Dear Professor Badawi,

The paper of which my previous mail refers to, is attached herewith. Rest all details have already been mention.

Sincerely,
P. K. Banerji

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Some Sharper Estimations of Growth Relationships of Composite Entire Functions on the Basis of Their Maximum Terms

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Abstract
In this paper we discuss the growth rates of the maximum term of composition of entire functions with their corresponding left and right factors.

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1 Introduction, Definitions and Notations.

Let $f$ be an entire function defined in the open complex plane $\mathbb{C}$. The maximum term $\mu(r, f)$ of $f = \sum_{n=0}^{\infty} a_n z^n$ on $|z| = r$ is defined by $\mu(r, f) =$
max \left( |a_n| r^n \right). We do not explain the standard definitions and notations in the theory of entire function as those are available in [8]. In the sequel the following two notations are used:

\[
\log^{[k]} x = \log \left( \log^{[k-1]} x \right) \quad \text{for } k = 1, 2, 3, \ldots ;
\]
\[
\log^{[0]} x = x
\]

and

\[
\exp^{[k]} x = \exp \left( \exp^{[k-1]} x \right) \quad \text{for } k = 1, 2, 3, \ldots ;
\]
\[
\exp^{[0]} x = x.
\]

To start our paper we just recall the following definitions.

**Definition 1** The order \( \rho_f \) and lower order \( \lambda_f \) of an entire function \( f \) is defined as follows:

\[
\rho_f = \limsup_{r \to \infty} \frac{\log^{[2]} M(r, f)}{\log r} \quad \text{and} \quad \lambda_f = \liminf_{r \to \infty} \frac{\log^{[2]} M(r, f)}{\log r}.
\]

**Definition 2** [5] Let \( l \) be an integer \( \geq 2 \). The generalised order \( \rho_f^{[l]} \) and generalised lower order \( \lambda_f^{[l]} \) of an entire function \( f \) are defined as

\[
\rho_f^{[l]} = \limsup_{r \to \infty} \frac{\log^{[l]} M(r, f)}{\log r} \quad \text{and} \quad \lambda_f^{[l]} = \liminf_{r \to \infty} \frac{\log^{[l]} M(r, f)}{\log r}.
\]

When \( l = 2 \), Definition 2 coincides with Definition 1.

Juneja, Kapoor and Bajpai[3] defined the \( (p, q) \) th order and \( (p, q) \) th lower order of an entire function \( f \) respectively as follows:

\[
\rho_f (p, q) = \limsup_{r \to \infty} \frac{\log^{[p]} M(r, f)}{\log^{[q]} r} \quad \text{and} \quad \lambda_f (p, q) = \liminf_{r \to \infty} \frac{\log^{[p]} M(r, f)}{\log^{[q]} r},
\]

where \( p, q \) are positive integers with \( p > q \).

For \( p = 2 \) and \( q = 1 \) we respectively denote \( \rho_f (2, 1) \) and \( \lambda_f (2, 1) \) by \( \rho_f \) and \( \lambda_f \).
Since for $0 \leq r < R$,

$$\mu(r, f) \leq M(r, f) \leq \frac{R}{R-r} \mu(R, f) \quad \{\text{cf. [6]}\}$$

it is easy to see that

$$\rho_f = \limsup_{r \to \infty} \frac{\log^2 \mu(r, f)}{\log r}, \quad \lambda_f = \liminf_{r \to \infty} \frac{\log^2 \mu(r, f)}{\log r};$$

$$\rho_f^{[q]} = \limsup_{r \to \infty} \frac{\log^{[q]} \mu(r, f)}{\log r}, \quad \lambda_f^{[q]} = \liminf_{r \to \infty} \frac{\log^{[q]} \mu(r, f)}{\log r};$$

and

$$\rho_f(p, q) = \limsup_{r \to \infty} \frac{\log^p \mu(r, f)}{\log^{[q]} r}, \quad \lambda_f(p, q) = \liminf_{r \to \infty} \frac{\log^p \mu(r, f)}{\log^{[q]} r}.$$

**Definition 3** Let "a" be a complex number, finite or infinite. The Nevanlinna’s deficiency of "a" with respect to a meromorphic function $f$ are defined as

$$\delta(a; f) = 1 - \limsup_{r \to \infty} \frac{N(r, a; f)}{T(r, f)} = \liminf_{r \to \infty} \frac{m(r, a; f)}{T(r, f)}.$$

In this paper we wish to prove some results relating to the growth rates of maximum terms of composition of two entire functions with their corresponding left and right factors on the basis of $(p, q)$ th order ( $(p, q)$ th lower order ) where $p, q$ are positive integers with $p > q$.

## 2 Lemmas.

In this section we present some lemmas which will be needed in the sequel.

**Lemma 1** [7] Let $f$ and $g$ be any two entire functions with $g(0) = 0$. Then for all sufficiently large values of $r$,

$$\mu(r, f \circ g) \geq \frac{1}{2} \mu \left( \frac{1}{8} \mu \left( \frac{r}{4} ; g \right) - |g(0)| , f \right).$$

**Lemma 2** [1] If $f$ and $g$ are any two entire functions then for all sufficiently large values of $r$,

$$M(r, f \circ g) \leq M(M(r, g) ; f).$$
Lemma 3  [4] Let $g$ be an entire function with $\lambda_g < \infty$ and assume that $a_i(i = 1, 2, \ldots, n; n \leq \infty)$ are entire functions satisfying $T(r, a_i) = \circ \{T(r, g)\}$. If $\sum_{i=1}^{n} \delta(a_i, g) = 1$, then
\[ \lim_{r \to \infty} \frac{T(r, g)}{\log M(r, g)} = \frac{1}{\pi}. \]

Lemma 4 Let $f$ be an entire function with non zero finite generalised order $\rho_f^{[l]}$ (non zero finite generalised lower order $\lambda_f^{[l]}$). If $p - q = l - 1$, then the $(p, q)$-th order $\rho_f(p, q)$ (lower $(p, q)$-th order $\lambda_f(p, q)$) of $f$ will be equal to 1. If $p - q \neq l - 1$ then $\rho_f(p, q)$ ( $\lambda_f(p, q)$ ) is either zero or infinity.

Proof. From the definition of generalised order of an entire function $f$ we have for all sufficiently large values of $r$,
\[ \log^{[l]} \mu(r, f) \leq \left( \rho_f^{[l]} + \varepsilon \right) \log r \]  \hspace{1cm} (1)

and for a sequence of values of $r$ tending to infinity,
\[ \log^{[l]} \mu(r, f) \geq \left( \rho_f^{[l]} - \varepsilon \right) \log r. \]  \hspace{1cm} (2)

Next let $a$ and $b$ be any two positive integers.

Now from (1) we have for all sufficiently large values of $r$,
\[ i.e., \quad \frac{\log^{[l+a]} \mu(r, f)}{\log^{[l+b]} r} \leq \frac{\log^{[l+a]} r + O(1)}{\log^{[l+b]} r}. \]  \hspace{1cm} (3)

If we take $l + a = p$ and $1 + b = q$, then $p - q = (l - 1) + (a - b)$.

We discuss the following two cases:

Case I. Let $a = b$. Then from (3) we get for all sufficiently large values of $r$,
\[ i.e., \quad \limsup_{r \rightarrow \infty} \frac{\log^{[p]} \mu(r, f)}{\log^{[a]} r} \leq 1. \]  \hspace{1cm} (4)
Similarly from (2) we have for a sequence of values of $r$ tending to infinity,

\[
\frac{\log[p] \mu(r, f)}{\log[q] r} \geq 1 + \frac{O(1)}{\log^{[1+a]} r}
\]

i.e., \( \limsup_{r \to \infty} \frac{\log[p] \mu(r, f)}{\log[q] r} \geq 1. \) (5)

Now from (4) and (5) we have

\[
\rho_f(p, q) = 1 \text{ when } p - q = l - 1.
\]

**Case II.** Let \( a > b \) (i.e., \( p - q \neq l - 1 \)). Then from (3) we have for all sufficiently large values of \( r \),

\[
\limsup_{r \to \infty} \frac{\log[p] \mu(r, f)}{\log[q] r} \leq 0
\]

i.e., \( \rho_f(p, q) = 0 \) when \( p - q \neq l - 1 \).

**Case III.** Also let us choose \( a \) and \( b \) such that \( a < b \) and \( l + a > 1 + b \) (i.e., \( p - q \neq l - 1 \)). Then from (2) it can be proved for a sequence of values of \( r \) tending to infinity that

\[
\limsup_{r \to \infty} \frac{\log[p] \mu(r; f)}{\log[q] r} \geq \infty
\]

i.e., \( \rho_f(p, q) = \infty \) when \( p - q \neq l - 1 \).

Therefore combining Case II and Case III (not violating the condition \( p > q \)), it follows that \( \rho_f(p, q) \) is either zero or infinity.
Similarly we may prove the conclusion for \( \lambda_f(p, q) \).
This proves the lemma.

### 3 Theorems

In this section we present the main results of the paper.
Theorem 1  Let $f, g, h$ and $k$ be four entire functions such that $0 < \rho_f(p, q) < \infty$, $\lambda_h(a, b) > 0$, $0 < \rho_h^{[l]} < \infty$ and $\rho_g(m, n) < \rho_h^{[l]}$ where $m, n, a, b, p, q$ are all positive integers with $m > n$; $a > b$; $p > q$ and $l > 1$. Then

\[(i)\quad \limsup_{r \to \infty} \frac{\log[\mu(r, h \circ k)]}{\log[\mu(r, f \circ g)] + \log[\mu(r, g)]} = \infty \quad \text{if } b = l - 1 \text{ and } q \geq m;\]

\[(ii)\quad \limsup_{r \to \infty} \frac{\log[\mu(r, h \circ k)]}{\log[\mu(r, f \circ g)] + \log[\mu(r, g)]} = \infty \quad \text{if } b = l - 1 \text{ and } q < m;\]

\[(iii)\quad \limsup_{r \to \infty} \frac{\log[\mu(r, h \circ k)]}{\log[\mu(r, f \circ g)] + \log[\mu(r, g)]} = \infty \quad \text{if } q \geq m \text{ and } l < b + 1 < n + l;\]

\[(iv)\quad \limsup_{r \to \infty} \frac{\log[\mu(r, h \circ k)]}{\log[\mu(r, f \circ g)] + \log[\mu(r, g)]} \geq \frac{\rho_h^{[l]} \lambda_h(a, b)}{(\rho_f(p, q) + 1) \rho_g(m, n)} \quad \text{if } q \geq m, \ b = l, \text{ and } n = 1;\]

\[(v)\quad \limsup_{r \to \infty} \frac{\log[\mu(r, h \circ k)]}{\log[\mu(r, f \circ g)] + \log[\mu(r, g)]} \geq \frac{\lambda_h(a, b)}{(\rho_f(p, q) + 1) \rho_g(m, n)} \quad \text{if } q \geq m \text{ and } b - l + 1 = n > 2;\]

\[(vi)\quad \limsup_{r \to \infty} \frac{\log[\mu(r, h \circ k)]}{\log[\mu(r, f \circ g)] + \log[\mu(r, g)]} = \infty \quad \text{if } q < m \text{ and } l < b + 1 < n + l;\]

\[(vii)\quad \limsup_{r \to \infty} \frac{\log[\mu(r, h \circ k)]}{\log[\mu(r, f \circ g)] + \log[\mu(r, g)]} \geq \frac{\rho_h^{[l]} \lambda_h(a, b)}{1 + \rho_g(m, n)} \quad \text{if } q < m, \ b = l \text{ and } n = 2;\]

and

\[(viii)\quad \limsup_{r \to \infty} \frac{\log[\mu(r, h \circ k)]}{\log[\mu(r, f \circ g)] + \log[\mu(r, g)]} \geq \frac{\lambda_h(a, b)}{1 + \rho_g(m, n)} \quad \text{if } q < m \text{ and } b - l + 1 = n > 2.\]
Proof. Since for $0 \leq r < R$,

$$\mu (r, f) \leq M (r, f) \leq \frac{R}{R-r} \mu (R, f) \quad \{c.f. \ [6] \} .$$

In view of Lemma 2 and the above inequality we obtain for all sufficiently large values of $r$ that

$$\mu (r, f \circ g) \leq M (r, f \circ g) \leq M (M (r, g), f)$$

$$\log^{[p]} \mu (r, f \circ g) \leq \log^{[p]} M (M (r, g), f)$$

i.e., $\log^{[p]} \mu (r, f \circ g) \leq (\rho_f (p, q) + \varepsilon) \log^{[q]} M (r, g)$.

Now the following cases may arise:

Case I. Let $q \geq m$. Then we have from (7) for all sufficiently large values of $r$,

$$\log^{[p]} \mu (r, f \circ g) \leq (\rho_f (p, q) + \varepsilon) \log^{[m-1]} M (r, g).$$

(8)

Now from the definition of $(m, n)$ th order of $g$ we get for arbitrary positive $\varepsilon$ and for all sufficiently large values of $r$,

$$\log^{[m]} M (r, g) \leq (\rho_g (m, n) + \varepsilon) \log^{[n]} r$$

$$\text{i.e., } \log^{[m]} M (r, g) \leq (\rho_g (m, n) + \varepsilon) \log r.$$

(9)

(10)

Also for all sufficiently large values of $r$ it follows from (10) that

$$\log^{[m-1]} M (r, g) \leq r^{(\rho_g (m, n) + \varepsilon)}.$$  

(11)

So from (8) and (11) it follows for all sufficiently large values of $r$ that

$$\log^{[p]} \mu (r, f \circ g) \leq (\rho_f (p, q) + \varepsilon) r^{(\rho_g (m, n) + \varepsilon)}.$$  

(12)

Case II. Let $q < m$. Then we get from (7) for all sufficiently large values of $r$ that

$$\log^{[p]} \mu (r, f \circ g) \leq (\rho_f (p, q) + \varepsilon) \exp^{[m-q]} \log^{[m]} M (r, g).$$

(13)

Again from (10) for all sufficiently large values of $r$,

$$\exp^{[m-q]} \log^{[m]} M (r, g) \leq \exp^{[m-q]} \log r^{(\rho_g (m, n) + \varepsilon)}$$

$$\text{i.e., } \exp^{[m-q]} \log^{[m]} M (r, g) \leq \exp^{[m-q-1]} r^{(\rho_g (m, n) + \varepsilon)}.$$  

(14)
Now from (13) and (14) we obtain for all sufficiently large values of $r$ that

\[
\log^{[p]} \mu (r, f \circ g) \leq (p_f (p, q) + \varepsilon) \exp^{[m-q-1]} \mu (\rho (m,n) + \varepsilon)
\]

\[i.e., \quad \log^{[p+1]} \mu (r, f \circ g) \leq \exp^{[m-q-2]} \mu (\rho (m,n) + \varepsilon)
\]

\[i.e., \quad \log^{[p+m-q-1]} \mu (r, f \circ g) \leq \log^{[m-q-2]} \exp^{[m-q-2]} \mu (\rho (m,n) + \varepsilon)
\]

\[i.e., \quad \log^{[p+m-q-1]} \mu (r, f \circ g) \leq \mu (\rho (m,n) + \varepsilon).
\]

(15)

Since $\rho_g (m, n) < \rho_k^{[l]}$ we can choose $\varepsilon (>0)$ in such a way that

\[\rho_g (m, n) + \varepsilon < \rho_k^{[l]} - \varepsilon.
\]

(16)

By Lemma 1 we obtain for a sequence of values of $r$ tending to infinity,

\[\log^{[a]} \mu (r, h \circ k) \geq \log^{[a]} \mu \left( \frac{1}{8} \mu \left( \frac{r}{4}, k \right), h \right) + O(1)
\]

\[i.e., \quad \log^{[a]} \mu (r, h \circ k) \geq (\lambda_h (a, b) - \varepsilon) \log^{[b]} \mu \left( \frac{r}{4}, k \right) + O(1)
\]

\[i.e., \log^{[a]} \mu (r, h \circ k) \geq (\lambda_h (a, b) - \varepsilon) \log^{[b-l+1]} \log^{[l-1]} \mu \left( \frac{r}{4}, k \right) + O(1).
\]

(17)

Now the following two cases may arise:

**Case III.** Let $b = l - 1$. Then from (17) we get for a sequence of values of $r$ tending to infinity that

\[\log^{[a]} \mu (r, h \circ k) \geq (\lambda_h (a, b) - \varepsilon) \left( \frac{r}{4} \right)^{\rho_k^{[l]} - \varepsilon} + O(1).
\]

(18)

**Case IV.** Let $b - l + 1 = d > 0$. Then from (17) it follows for a sequence of values of $r$ tending to infinity that

\[\log^{[a]} \mu (r, h \circ k) \geq (\lambda_h (a, b) - \varepsilon) \log^{[d]} \left( \frac{r}{4} \right)^{\rho_k^{[l]} - \varepsilon}.
\]

(19)

Now from the definition of $(m, n)$ th order of $g$ we have for arbitrary positive $\varepsilon$ and for all sufficiently large values of $r$,

\[\log^{[m]} \mu (r, g) \leq (\rho (m, n) + \varepsilon) \log^{[m]} r.
\]

(20)
Let \( q \geq m \). Then we have from (7) and (9) for all sufficiently large values of \( r \),
\[
\log^{[p]} \mu (r, f \circ g) \leq (\rho_f (p, q) + \varepsilon) (\rho_g (m, n) + \varepsilon) \log^{[n]} r.
\] (21)

Now if \( b = l - 1 \) and \( q \geq m \), we get from (12), (18), (20) and in view of (16) for a sequence of values of \( r \) tending to infinity,
\[
\frac{\log^{[a]} \mu (r, h \circ k)}{\log^{[p]} \mu (r, f \circ g) + \log^{[m]} \mu (r, g)} \geq \frac{(\lambda_h (a, b) - \varepsilon) (\varepsilon)}{\rho_f (p, q) + \varepsilon) (\rho_g (m, n) + \varepsilon) \log^{[n]} r + O(1)}
\]
\[= \frac{\log^{[a]} \mu (r, h \circ k)}{\log^{[p]} \mu (r, f \circ g) + \log^{[m]} \mu (r, g)} = \infty,
\]
which proves the first part of the theorem.

Again we obtain from (15), (16), (18), and (20), for a sequence of values of \( r \) tending to infinity when \( b = l - 1 \) and \( q < m \),
\[
\frac{\log^{[a]} \mu (r, h \circ k)}{\log^{[p+m-q-1]} \mu (r, f \circ g) + \log^{[m]} \mu (r, g)} \geq \frac{(\lambda_h (a, b) - \varepsilon) (\varepsilon)}{\rho_f (p, q) + \varepsilon) (\rho_g (m, n) + \varepsilon) \log^{[n]} r + O(1)}
\]
\[= \frac{\log^{[a]} \mu (r, h \circ k)}{\log^{[p+m-q-1]} \mu (r, f \circ g) + \log^{[m]} \mu (r, g)} = \infty.
\]
This proves the second part of the theorem.

When \( b > l - 1 \) and \( q \geq m \), from (19), (20), and (21) we get for a sequence of values of \( r \) tending to infinity,
\[
\frac{\log^{[a]} \mu (r, h \circ k)}{\log^{[p]} \mu (r, f \circ g) + \log^{[m]} \mu (r, g)} \geq \frac{(\lambda_h (a, b) - \varepsilon) (\varepsilon)}{\rho_f (p, q) + \varepsilon) (\rho_g (m, n) + \varepsilon) \log^{[n]} r + O(1)}
\]
\[= \frac{\log^{[a]} \mu (r, h \circ k)}{\log^{[p]} \mu (r, f \circ g) + \log^{[m]} \mu (r, g)} = \infty \text{ if } l < b + 1 < n + l;
\]
again
\[
\limsup_{r \to \infty} \frac{\log[a] \mu(r, h \circ k)}{\log[p] \mu(r, f \circ g) + \log[m] \mu(r, g)} \geq \frac{\rho_k \lambda_h(a, b)}{(\rho_f(p, q) + 1) \rho_g(m, n)} \text{ if } b = l, n = 1;
\]
and also
\[
\limsup_{r \to \infty} \frac{\log[a] \mu(r, h \circ k)}{\log[p] \mu(r, f \circ g) + \log[m] \mu(r, g)} \geq \frac{\lambda_h(a, b)}{(\rho_f(p, q) + 1) \rho_g(m, n)} \text{ if } b - l + 1 = n > 2.
\]

This respectively proves the third, fourth and fifth part of the theorem.

Again when \( b > l - 1 \) and \( q < m \), combining (15), (19) and (20) we obtain for a sequence of values of \( r \) tending to infinity,
\[
\limsup_{r \to \infty} \frac{\log[a] \mu(r, h \circ k)}{\log[p] \mu(r, f \circ g) + \log[m] \mu(r, g)} = \infty \text{ if } l < b + 1 < n + l;
\]
i.e.,
\[
\limsup_{r \to \infty} \frac{\log[a] \mu(r, h \circ k)}{\log[p] \mu(r, f \circ g) + \log[m] \mu(r, g)} = \infty \text{ if } l < b + 1 < n + l;
\]
also
\[
\limsup_{r \to \infty} \frac{\log[a] \mu(r, h \circ k)}{\log[p] \mu(r, f \circ g) + \log[m] \mu(r, g)} \geq \frac{\rho_k \lambda_h(a, b)}{1 + \rho_g(m, n)} \text{ if } b = l, n = 1
\]
and again
\[
\limsup_{r \to \infty} \frac{\log[a] \mu(r, h \circ k)}{\log[p] \mu(r, f \circ g) + \log[m] \mu(r, g)} \geq \frac{\lambda_h(a, b)}{1 + \rho_g(m, n)} \text{ if } b - l + 1 = n > 2.
\]
from which the sixth, seventh and eighth part of the theorem follows respectively.
Remark 1 The condition \( \rho_g(m, n) < \rho_k \) and \( \rho_f(p, q) < \infty \) in Theorem 1 are essential as we see in the following examples.

Example 1 Let
\[
f = g = h = k = \exp z.
\]
Also let
\[
p = m = a = 2 \text{ and } q = n = b = 1.
\]
Then
\[
\rho_f = 1, \quad \rho_g = 1 = \rho_k \text{ and } \lambda_h = 1.
\]
Now in view of the inequality \( \mu(r, f) \leq M(r, f) \leq \frac{R}{R-r} \mu(R, f) \) {cf. [6]} and \( T(r, g) \leq \log^+ M(r, g) \) we get that
\[
\log \mu(r, h \circ k) \leq \log M(r, h \circ k)
\]
\[
\leq 3T(2r, h \circ k) \sim \frac{3 \exp(2r)}{(4\pi^3 r)^{\frac{3}{2}}}
\]
\[
i.e., \log^{[2]} \mu(r, h \circ k) \leq 2r - \frac{1}{2} \log r + O(1)
\]
and
\[
\log \mu(r, f \circ g) \geq \log M(\frac{r}{2}, f \circ g) + O(1)
\]
\[
\geq T(\frac{r}{2}, f \circ g) + O(1) \sim \frac{\exp \left( \frac{r}{2} \right)}{(2\pi^3 r)^{\frac{3}{2}}}
\]
\[
i.e., \log^{[2]} \mu(r, f \circ g) \geq \frac{r}{2} - \frac{1}{2} \log r + O(1).
\]
So
\[
\limsup_{r \to \infty} \frac{\log^{[2]} \mu(r, h \circ k)}{\log^{[2]} \mu(r, f \circ g) + \log^{[2]} \mu(r, g)} \leq \limsup_{r \to \infty} \frac{2r - \frac{1}{2} \log r + O(1)}{\frac{r}{2} - \frac{1}{2} \log r + O(1) + \log \frac{r}{2}}
\]
\[
i.e., \limsup_{r \to \infty} \frac{\log^{[2]} \mu(r, h \circ k)}{\log^{[2]} \mu(r, f \circ g) + \log^{[2]} \mu(r, g)} \leq \limsup_{r \to \infty} \frac{2r - \frac{1}{2} \log r + O(1)}{\frac{r}{2} + \frac{1}{2} \log r + O(1)} = 1,
\]
which is contrary to Theorem 1.
Example 2 Let
\[
f = \exp^{[2]} z, \quad g = h = \exp z \quad \text{and} \quad k = \exp (z^2)
\]
and
\[
p = m = a = 2 \quad \text{and} \quad q = n = b = 1.
\]
Then
\[
\rho_f = \infty, \quad \rho_g = 1 < 2 = \rho_k \quad \text{and} \quad \lambda_h = 1.
\]
Now
\[
\log \mu(r, h \circ k) \leq \log M(r, h \circ k) = \log \exp^{[2]} (r^2)
\]
i.e.,
\[
\log \mu(r, h \circ k) \leq \exp (r^2),
\]
and
\[
\log \mu(r, f \circ g) \geq \log M\left(\frac{r}{2}, f \circ g\right)
\]
i.e.,
\[
\log \mu(r, f \circ g) \geq \log \exp^{[3]} \left(\frac{r}{2}\right) = \exp^{[2]} \left(\frac{r}{2}\right)
\]
and
\[
\log^{[2]} \mu(r, g) \geq \log^{[2]} M\left(\frac{r}{2}, g\right) = 1.
\]
Therefore
\[
\frac{\log^{[2]} \mu(r, h \circ k)}{\log^{[2]} \mu(r, f \circ g) + \log^{[2]} \mu(r, g)} \leq \frac{\log \exp (r^2)}{\log \exp^{[2]} \left(\frac{r}{2}\right) + O(1) + \log^{[2]} \exp r}
\]
i.e.,
\[
\log^{[2]} \mu(r, h \circ k) \leq \frac{\log \exp^{[2]} \left(\frac{r}{2}\right) + O(1) + \log^{[2]} \exp r}{r^2}
\]
and
\[
\log^{[2]} \mu(r, f \circ g) + \log^{[2]} \mu(r, g) \leq \exp \left(\frac{r}{2}\right) + \log r + O(1)
\]
i.e.,
\[
\limsup_{r \to \infty} \frac{\log^{[2]} \mu(r, h \circ k)}{\log^{[2]} \mu(r, f \circ g) + \log^{[2]} \mu(r, g)} = 0,
\]
which is contrary to Theorem 1.

Remark 2 The condition \( \rho_g (m, n) < \lambda_k \) in Theorem 1 is necessary which is true in general only if \( \rho_f (p, q) > 0 \) otherwise the condition \( \rho_g (m, n) < \rho_k \) will be violated. The following example ensures this comment.

Example 3 Let
\[
f = h = k = \exp z \quad \text{and} \quad g = \exp (z^3)
\]
Also let
\[ p = 3, \quad m = a = 2 \text{ and } q = n = b = 1. \]

Then
\[ \bar{\rho}_f = \rho_f(3,1) = 0 < \infty, \quad \rho_g = 3 > 1 = \rho_k \text{ and } \lambda_h = 1. \]

Now
\[
\log \mu(r, h \circ k) \geq \log M\left(\frac{r}{2}, h \circ k\right) + O(1)
\begin{align*}
&\geq T\left(\frac{r}{2}, h \circ k\right) + O(1) \\
&\sim \frac{\exp\left(\frac{r}{2}\right)}{\left(2\pi^{3}\frac{r}{2}\right)^{\frac{1}{2}}}
\end{align*}
\]
i.e., \( \log^{[2]} \mu(r, h \circ k) \geq \frac{r}{2} - \frac{1}{2} \log r + O(1). \)

\[
\log \mu(r, f \circ g) \leq \log M(r, f \circ g) = \exp r^{3}
\]
i.e., \( \log^{[2]} \mu(r, f \circ g) \leq 3 \log r. \)

Therefore
\[
\frac{\log^{[2]} \mu(r, h \circ k)}{\log^{[2]} \mu(r, f \circ g) + \log^{[2]} \mu(r, g)} \geq \frac{\frac{r}{2} - \frac{1}{2} \log r + O(1)}{6 \log r}
\]
i.e., \( \lim_{r \to \infty} \frac{\log^{[2]} \mu(r, h \circ k)}{\log^{[2]} \mu(r, f \circ g) + \log^{[2]} \mu(r, g)} = \infty. \)

**Theorem 2** Let \( f \) and \( g \) be any two entire functions such that \( \lambda_f(p,q) \) and \( \lambda_g \) are both finite and \( p,q \) are any two positive integers with \( p > q \). Also suppose that there exist entire functions \( a_i(i = 1, 2, ..., n; n \leq \infty) \) satisfying

(A) \( T(r, a_i) = o\{T(r, g)\} \) as \( r \to \infty \) and

(B) \( \sum_{i=1}^{n} \delta(a_i, g) = 1. \) Then for any \( \beta > \alpha > 1, \)

\[
\liminf_{r \to \infty} \frac{\log^{[p]} \mu(r, f \circ g)}{\log^{[p]} \mu(r, \beta r, g)} \leq \left(\frac{\alpha + 1}{\alpha - 1}\right) \frac{\lambda_f(p,q)}{\pi}. \]

**Proof.** In view of (6) we have from Lemma 2, for all sufficiently large values of \( r, \)

\[
\log^{[p]} \mu(r, f \circ g) \leq \log^{[p]} M(M(r, g), f). \]  \( \text{(22)} \)
Since $\varepsilon(>0)$ is arbitrary and $T(r, g) \leq \log^+ M(r, g) \leq \left(\frac{a+1}{a-1}\right) T(\alpha r, g)$, cf. [2] from (22) and in view of (6) we get for a sequence of values of $r$ tending to infinity that

\[
\log^{[p]} \mu(r, f \circ g) \leq (\lambda_f(p, q) + \varepsilon) \log^{[q]} M(r, g)
\]

i.e.,

\[
\log^{[p]} \mu(r, f \circ g) \leq (\lambda_f(p, q) + \varepsilon) \log M(r, g)
\]

i.e.,

\[
\log^{[p]} \mu(r, f \circ g) \leq \left(\frac{a + 1}{a - 1}\right) (\lambda_f(p, q) + \varepsilon) T(\alpha r, g)
\]

i.e.,

\[
\frac{\log^{[p]} \mu(r, f \circ g)}{\log \mu(\beta r, g)} \leq \left(\frac{a + 1}{a - 1}\right) \frac{\lambda_f(p, q) + \varepsilon) T(\alpha r, g)}{\log M(\alpha r, g)}.
\]

(23)

Since $\varepsilon(>0)$ is arbitrary, by Lemma 3 it follows from (23) that

\[
\liminf_{r \to \infty} \frac{\log^{[p]} \mu(r, f \circ g)}{\log \mu(\beta r, g)} \leq \left(\frac{a + 1}{a - 1}\right) \frac{\lambda_f(p, q)}{\pi}.
\]

This proves the theorem.

**Theorem 3** Let $f$ and $g$ be two entire functions such that $\rho_f(p, q)$ and $\lambda_g$ are both finite and $p, q$ are any two positive integers with $p > q$. Also suppose that there exist entire functions $a_i(i = 1, 2, \ldots, n; n \leq \infty)$ satisfying

(A) $T(r, a_i) = o\{T(r, g)\}$ as $r \to \infty$ and

(B) $\sum_{i=1}^{n} \delta(a_i, g) = 1$. Then for any $\beta > \alpha > 1$,

\[
\limsup_{r \to \infty} \frac{\log^{[p]} \mu(r, f \circ g)}{\log \mu(\beta r, g)} \leq \left(\frac{a + 1}{a - 1}\right) \frac{\rho_f(p, q)}{\pi}.
\]

The proof of Theorem 3 is omitted as it can be carried out in the line of Theorem 2.

**Theorem 4** Let $f$ be an entire function such that $\rho_f(p, q) < \infty$ where $p, q$ are positive integers with $p > q > 1$. Also let $g$ be entire. If $\lambda_{fg}(p, q) = \infty$ then for every positive number $\beta$,

\[
\lim_{r \to \infty} \frac{\log^{[p]} \mu(r, f \circ g)}{\log^{[p]} \mu(\beta r, f)} = \infty.
\]
Proof. Let us suppose that the conclusion of the theorem does not hold. Then we can find a constant \( \alpha > 0 \) such that for a sequence of values of \( r \) tending to infinity

\[
\log^{[p]} \mu(r, f \circ g) \leq \alpha \log^{[p]} \mu(r^{\beta}, f). \tag{24}
\]

Again for \( q > 1 \) from the definition of \( \rho_f(p, q) \) it follows that for all sufficiently large values of \( r \)

\[
\log^{[p]} \mu(r^{\beta}, f) \leq (\rho_f(p, q) + \varepsilon) \log^{[q]} (r^{\beta})
\]

i.e.,

\[
\log^{[p]} \mu(r^{\beta}, f) \leq (\rho_f(p, q) + \varepsilon) \log^{[q]} r + O(1). \tag{25}
\]

Thus from (24) and (25) we have for a sequence of values of \( r \) tending to infinity that

\[
\log^{[p]} \mu(r, f \circ g) \leq \alpha (\rho_f(p, q) + \varepsilon) \frac{\log^{[q]} r + O(1)}{\log^{[q]} r}
\]

i.e.,

\[
\liminf_{r \to \infty} \frac{\log^{[p]} \mu(r, f \circ g)}{\log^{[q]} r} = \lambda_{f \circ g}(p, q) < \infty.
\]

This is a contradiction.

This proves the theorem.

Remark 3 Theorem 4 is also valid with “limit superior” instead of “limit” if \( \lambda_{f \circ g}(p, q) = \infty \) is replaced by \( \rho_{f \circ g}(p, q) = \infty \) and the other conditions remaining the same.

Corollary 1 Under the assumptions of Remark 3,

\[
\limsup_{r \to \infty} \frac{\log^{[p-1]} \mu(r, f \circ g)}{\log^{[p-1]} \mu(r^{\beta}, f)} = \infty.
\]

Proof. From Remark 3 we obtain for all sufficiently large values of \( r \) and for \( K > 1 \),

\[
\log^{[p]} \mu(r, f \circ g) > K \log^{[p]} \mu(r^{\beta}, f)
\]

i.e.,

\[
\log^{[p-1]} \mu(r, f \circ g) > \left\{ \log^{[p-1]} \mu(r^{\beta}, f) \right\}^K,
\]

from which the corollary follows.
Corollary 2  Under the same conditions of Theorem 4 if \( q = 1 \)

\[
\lim_{r \to \infty} \frac{\log^{[p]} \mu(r, f \circ g)}{\log^{[p]} \mu(r^\beta, f)} = \infty.
\]

Corollary 3  Under the same conditions of Remark 3 if \( q = 1 \)

\[
\limsup_{r \to \infty} \frac{\log^{[p]} \mu(r, f \circ g)}{\log^{[p]} \mu(r^\beta, f)} = \infty.
\]

Remark 4  The condition \( \lambda_{fg}(p, 1) = \infty \) in Corollary 2 is necessary as we see in the following example.

Example 4  Let \( f = \exp z, g = z \) and \( p = 2, q = 1, \beta = 1 \).
Then \( \rho_f(p, 1) = \lambda_{fg}(p, 1) = 1 \).

Now

\[
\log \mu(r, f \circ g) \leq \log M(r, f \circ g) = \log M(r, \exp z) = r
\]

and

\[
\log \mu(r, f \circ g) \geq \log M\left(\frac{r}{2}, f \circ g\right) = \log M\left(\frac{r}{2}, \exp z\right) = \frac{r}{2}.
\]

Then

\[
\lim_{r \to \infty} \frac{\log^{[p]} \mu(r, f \circ g)}{\log^{[p]} \mu(r^\beta, f)} \\
\leq \lim_{r \to \infty} \frac{\log^{[2]} \mu(r, f \circ g)}{\log^{[2]} \mu(r, f)} \\
\leq \lim_{r \to \infty} \frac{\log r}{\log \frac{r}{2}} \\
\leq 1 \neq \infty, \text{ which is contrary to Corollary 2.}
\]

Remark 5  Considering \( f = \exp z, g = z \) and \( p = 2, q = 1, \beta = 1 \) one can easily verify that the condition \( \lambda_{fg}(p, 1) = \infty \) in Corollary 3 is essential.

Theorem 5  Let \( f \) and \( g \) be any two entire functions such that \( \rho_g(m, n) = 1 < \lambda_f(p, q) \leq \rho_f(p, q) < \infty \) where \( p, q, m, n \) are positive integers with \( p > q \) and \( m - n = 1 \). Then for any \( R > r \)

\[
(i) \quad \lim_{r \to \infty} \frac{\left\{ \log^{[p]} \mu(\exp^{[1]} r, f \circ g) \right\}^2}{\log^{[p-1]} \mu(\exp^{[1]} r, f) \log^{[q]} \mu(\exp^{[1]} R, g)}
\]
= 0 if \( q \geq m \)

and

\[
(ii) \lim_{r \to \infty} \frac{\log^{[p+m-q-1]} \mu(\exp^{[n-1]} r, f \circ g) \log^{[p]} \mu(\exp^{[n-1]} r, f \circ g)}{\log^{[p-1]} \mu(\exp^{[q-1]} r, f) \log^{[q]} \mu(\exp^{[n-1]} R, g)} = 0 \text{ if } q < m.
\]

**Proof.** From the definition of \((p, q)\) th lower order of \(f\) we have for arbitrary positive \(\varepsilon\) and for all sufficiently large values of \(r\),

\[
\log^{[p]} \mu(\exp^{[q-1]} r, f) \geq (\lambda_f(p, q) - \varepsilon) \log^{[q]} \exp^{[q-1]} r
\]

i.e., \(\log^{[p]} \mu(\exp^{[q-1]} r, f) \geq (\lambda_f(p, q) - \varepsilon) \log r\)

i.e., \(\log^{[p-1]} \mu(\exp^{[q-1]} r, f) \geq r^{(\lambda_f(p, q) - \varepsilon)}\). \hspace{1cm} (26)

Again in view of (6) we obtain from Lemma 2, for all sufficiently large values of \(r\) that

\[
\log^{[p]} \mu(\exp^{[n-1]} r, f \circ g) \leq \log^{[p]} M(\exp^{[n-1]} r, g, f)
\]

i.e., \(\log^{[p]} \mu(\exp^{[n-1]} r, f \circ g)

\leq (\rho_f(p, q + \varepsilon) \log^{[q]} M(\exp^{[n-1]} r, g). \hspace{1cm} (27)

Now the following two cases may arise:

**Case I.** Let \( q \geq m \). Then from (27) we get for all sufficiently large values of \(r\) that

\[
\log^{[p]} \mu(\exp^{[n-1]} r, f \circ g) \leq (\rho_f(p, q + \varepsilon) \log^{[m]} M(\exp^{[n-1]} r, g). \hspace{1cm} (28)
\]

Now for all sufficiently large values of \(r\),

\[
\log^{[m]} M(\exp^{[n-1]} r, g) \leq (\rho_g(m, n) + \varepsilon) \log r
\]

i.e., \(\log^{[m-1]} M(\exp^{[n-1]} r, g) \leq r^{\rho_g(m, n) + \varepsilon}. \hspace{1cm} (29)

From (28) and (29) it follows for all sufficiently large values of \(r\),

\[
\log^{[p]} \mu(\exp^{[n-1]} r, f \circ g) \leq (\rho_f(p, q + \varepsilon) r^{\rho_g(m, n) + \varepsilon}. \hspace{1cm} (30)
\]
Case II. Let $q < m$. Then from (27) we have for all sufficiently large values of $r$ that
\[
\log^p \mu(\exp^{[n-1]} r, f \circ g) \leq (\rho_f (p, q) + \varepsilon) \exp^{[m-q]} \log^m M(\exp^{[n-1]} r, g).
\tag{31}
\]

Now for all sufficiently large values of $r$,
\[
\log^m M(\exp^{[n-1]} r, g) \leq (\rho_g (m, n) + \varepsilon) \log^m \log r
\]

\[\text{i.e.,} \quad \log^m M(\exp^{[n-1]} r, g) \leq \exp^{[m-q]} \log r (\rho_g (m, n) + \varepsilon)
\]

\[\text{i.e.} \quad \exp^{[m-q]} \log^m M(\exp^{[n-1]} r, g) \leq \exp^{[m-q]} r (\rho_g (m, n) + \varepsilon).
\tag{32}
\]

Now from (31) and (32) we get for all sufficiently large values of $r$ that
\[
\log^p \mu(\exp^{[n-1]} r, f \circ g) \leq (\rho_f (p, q) + \varepsilon) \exp^{[m-q]} \log^m (\rho_g (m, n) + \varepsilon)
\]

\[\text{i.e.,} \quad \log^{[p+1]} \mu(\exp^{[n-1]} r, f \circ g) \leq \exp^{[m-q-2]} r (\rho_g (m, n) + \varepsilon)
\]

\[\text{i.e.,} \quad \log^{[p+m-q-1]} \mu(\exp^{[n-1]} r, f \circ g) \leq \exp^{[m-q]} r (\rho_g (m, n) + \varepsilon).
\tag{33}
\]

As $\rho_g (m, n) < \lambda_f (p, q)$ we can choose $\varepsilon (> 0)$ in such a way that
\[
\rho_g (m, n) < \lambda_f (p, q) - \varepsilon.
\tag{34}
\]

Now combining (30) of Case I and (26) we have for all sufficiently large values of $r$,
\[
\frac{\log^p \mu(\exp^{[n-1]} r, f \circ g)}{\log^{[p-1]} \mu(\exp^{[q-1]} r, f)} \leq \frac{(\rho_f (p, q + \varepsilon) r (\rho_g (m, n) + \varepsilon)}{r (\lambda_f (p, q) - \varepsilon)}.
\]

In view of (34) we get from above that
\[
\limsup_{r \to \infty} \frac{\log^p \mu(\exp^{[n-1]} r, f \circ g)}{\log^{[p-1]} \mu(\exp^{[q-1]} r, f)} = 0
\]

\[\text{i.e.,} \quad \lim_{r \to \infty} \frac{\log^p \mu(\exp^{[n-1]} r, f \circ g)}{\log^{[p-1]} \mu(\exp^{[q-1]} r, f)} = 0.
\tag{35}
\]
Again combining (33) of Case II and (26) it follows for all sufficiently large values of $r$ that
\[
\frac{\log^{[p+m-q-1]} \mu \left( \exp^{[n-1]} r, f \circ g \right)}{\log^{[p-1]} \mu \left( \exp^{[q-1]} r, f \right)} \leq \frac{r^{(\rho_f(m,n)+\varepsilon)}}{r^{(\lambda_f(p,q)-\varepsilon)}}.
\]

Now in view of (34) we obtain from above that
\[
\limsup_{r \to \infty} \frac{\log^{[p+m-q-1]} \mu \left( \exp^{[n-1]} r, f \circ g \right)}{\log^{[p-1]} \mu \left( \exp^{[q-1]} r, f \right)} = 0
\]
i.e.,
\[
\lim_{r \to \infty} \frac{\log^{[p+m-q-1]} \mu \left( \exp^{[n-1]} r, f \circ g \right)}{\log^{[p-1]} \mu \left( \exp^{[q-1]} r, f \right)} = 0. \tag{36}
\]

Since $M(r, g) \leq \frac{R}{R-r} \mu(R, f)$, from (27) we have for all sufficiently large values of $r$,
\[
\log^{[p]} \mu \left( \exp^{[n-1]} r, f \circ g \right) \leq \left( \rho_f(p, q + \varepsilon) \right) \log^{[q]} \mu \left( \exp^{[n-1]} R, g \right)
\]
i.e.,
\[
\frac{\log^{[p]} \mu \left( \exp^{[n-1]} r, f \circ g \right)}{\log^{[q]} \mu \left( \exp^{[n-1]} R, g \right)} \leq \left( \rho_f(p, q + \varepsilon) \right).
\]

Since $\varepsilon(> 0)$ is arbitrary, we get from above that
\[
\limsup_{r \to \infty} \frac{\log^{[p]} \mu \left( \exp^{[n-1]} r, f \circ g \right)}{\log^{[q]} \mu \left( \exp^{[n-1]} R, g \right)} \leq \rho_f(p, q). \tag{37}
\]

From (35) and (37) we obtain for all sufficiently large values of $r$ that
\[
\limsup_{r \to \infty} \frac{\left\{ \log^{[p]} \mu \left( \exp^{[n-1]} r, f \circ g \right) \right\}^2}{\log^{[p]} \log^{[q]} \mu \left( \exp^{[n-1]} r, f \circ g \right)} = \lim_{r \to \infty} \frac{\log^{[p]} \mu \left( \exp^{[n-1]} r, f \circ g \right)}{\log^{[q]} \mu \left( \exp^{[n-1]} R, g \right)} \limsup_{r \to \infty} \frac{\log^{[p]} \mu \left( \exp^{[n-1]} r, f \circ g \right)}{\log^{[q]} \mu \left( \exp^{[n-1]} R, g \right)} \leq 0, \rho_f(p, q) = 0.
\]

This proves the first part of the theorem.

Again from (36) and (37) we get for all sufficiently large values of $r$ that
\[
\limsup_{r \to \infty} \frac{\log^{[p+m-q-1]} \mu \left( \exp^{[n-1]} r, f \circ g \right) \log^{[p]} \mu \left( \exp^{[n-1]} r, f \circ g \right)}{\log^{[p-1]} \mu \left( \exp^{[q-1]} r, f \right) \log^{[q]} \mu \left( \exp^{[n-1]} R, g \right)} = \lim_{r \to \infty} \frac{\log^{[p+m-q-1]} \mu \left( \exp^{[n-1]} r, f \circ g \right)}{\log^{[p-1]} \mu \left( \exp^{[q-1]} r, f \right)} \limsup_{r \to \infty} \frac{\log^{[p]} \mu \left( \exp^{[n-1]} r, f \circ g \right)}{\log^{[q]} \mu \left( \exp^{[n-1]} R, g \right)} \leq 0, \rho_f(p, q) = 0
\]
Thus the second part of the theorem is established.

**Theorem 6** \ Let \( f \) and \( g \) be any two entire functions such that \( \rho_f(p, q) < \infty \)
and \( \rho_{f \circ g}(a, b) < \infty \) where \( p; q; a, b \) are all positive integers with \( p > q \) and \( a > b \). Also let \( \lambda_g < \infty \). Then for any two positive integers \( m, n \) with \( m - n = 1, m > 2 \) and any \( R > r \)

\[
\limsup_{r \to \infty} \frac{\log^p \mu(\exp^{[n-1]} r, f \circ g) \log^q \mu(\exp^{[n-1]} r, f) \log^a \mu(\exp^{[n-1]} R, g)}{\log^p \mu(\exp^{[n-1]} r, f) \log^q \mu(\exp^{[n-1]} r, f \circ g) \log^a \mu(\exp^{[n-1]} R, g)} = 0.
\]

**Proof.** For all sufficiently large values of \( r \) we have

\[
\log^a \mu(\exp^{[b-1]} r, f \circ g) \leq (\rho_{f \circ g}(a, b) + \varepsilon) \log^b \exp^{[b-1]} r
\]
i.e., \( \log^a \mu(\exp^{[b-1]} r, f \circ g) \leq (\rho_{f \circ g}(a, b) + \varepsilon) \log r. \) \( (38) \)

Again for all sufficiently large values of \( r \) it follows that

\[
\log^m \mu(\exp^{[n-1]} r, g) \geq (\lambda_g(m, n) - \varepsilon) \log^m \exp^{[n-1]} r
\]
i.e., \( \log^m \mu(\exp^{[n-1]} r, g) \geq (\lambda_g(m, n) - \varepsilon) \log r. \) \( (39) \)

Now combining (38) and (39) we have for all sufficiently large values of \( r \) that

\[
\frac{\log^a \mu(\exp^{[b-1]} r, f \circ g)}{\log^m \mu(\exp^{[n-1]} r, g)} \leq \frac{\rho_{f \circ g}(a, b) + \varepsilon}{\lambda_g(m, n) - \varepsilon}.
\]

As \( \varepsilon (> 0) \) is arbitrary we get from above that

\[
\limsup_{r \to \infty} \frac{\log^a \mu(\exp^{[b-1]} r, f \circ g)}{\log^m \mu(\exp^{[n-1]} r, g)} \leq \frac{\rho_{f \circ g}(a, b)}{\lambda_g(m, n)}.
\] \( (40) \)

Now from (37) and (40) we obtain that

\[
\limsup_{r \to \infty} \frac{\log^p \mu(\exp^{[n-1]} r, f \circ g) \log^q \mu(\exp^{[n-1]} r, f \circ g)}{\log^p \mu(\exp^{[n-1]} r, f) \log^q \mu(\exp^{[n-1]} r, f) \log^a \mu(\exp^{[n-1]} R, g) \log^m \mu(\exp^{[n-1]} R, g)} = 0.
\]
\[ \limsup_{r \to \infty} \frac{\log^{[a]} \mu(\exp^{[b-1]} r, f \circ g)}{\log^{[m]} \mu(\exp^{[a-1]} r, g)} \cdot \limsup_{r \to \infty} \frac{\log^{[p]} \mu(\exp^{[a-1]} r, f \circ g)}{\log^{[q]} \mu(\exp^{[a-1]} R, g)} \]
\[ \leq \frac{\rho_{f \circ g}(a, b) \rho_f(p, q)}{\lambda_g(m, n)}. \]

Thus in view of Lemma 4, the theorem follows from above.

**Corollary 4** Under the same conditions of Theorem 6 when \( m = 2 \),
\[ \limsup_{r \to \infty} \frac{\log^{[p]} \mu(r, f \circ g) \log^{[a]} \mu(\exp^{[b-1]} r, f \circ g)}{\log^{[q]} \mu(R, g) \log^{[p]} \mu(r, g)} \]
\[ \leq \frac{\rho_{f \circ g}(a, b) \rho_f(p, q)}{\lambda_g}. \]

**Theorem 7** Let \( f \) and \( g \) be any two entire functions such that \( \lambda_f(p, q) \) and \( \lambda_g \) are both finite and \( p, q \) are any two positive integers with \( p > q \). Then for any \( R > r \),
\[ (i) \quad \liminf_{r \to \infty} \frac{\log^{[p+1]} \mu(r, f \circ g)}{\log^{[q+1]} \mu(R, g)} \leq 1 \]
and
\[ (ii) \quad \liminf_{r \to \infty} \frac{\log^{[p]} \mu(r, f \circ g)}{\log^{[q]} \mu(R, g)} \leq \lambda_f(p, q). \]

**Proof.** In view of (6) we get from (22) for a sequence of values of \( r \) tending to infinity that
\[ \log^{[p]} \mu(r, f \circ g) \leq (\lambda_f(p, q) + \varepsilon) \log^{[q]} M(r, g) \]
\( i.e., \)
\[ \log^{[p]} \mu(r, f \circ g) \leq (\lambda_f(p, q) + \varepsilon) \log^{[q]} \mu(R, g) \] (41)
\( i.e., \)
\[ \log^{[p+1]} \mu(r, f \circ g) \leq \log^{[q+1]} \mu(R, g) + O(1) \]
\[ i.e., \]
\[ \frac{\log^{[p+1]} \mu(r, f \circ g)}{\log^{[q+1]} \mu(R, g)} \leq \frac{\log^{[q+1]} \mu(R, g) + O(1)}{\log^{[q+1]} \mu(R, g)}. \] (42)

Also from (41) it follows for a sequence of values of \( r \) tending to infinity that
\[ \frac{\log^{[p]} \mu(r, f \circ g)}{\log^{[q]} \mu(R, g)} \leq \frac{\lambda_f(p, q) + \varepsilon}{\log^{[q]} \mu(R, g)}. \] (43)
Since $\varepsilon (> 0)$ is arbitrary, it follows from (42) that
\[
\liminf_{r \to \infty} \frac{\log^{[p+1]} \mu (r, f \circ g)}{\log^{[q+1]} \mu (R, g)} \leq 1.
\]
This proves the first part of the theorem.

As $\varepsilon (> 0)$ is arbitrary we obtain from (43) that
\[
\liminf_{r \to \infty} \frac{\log^{[p]} \mu (r, f \circ g)}{\log^{[q]} \mu (R, g)} \leq \lambda_f (p, q).
\]
Thus the second part of the theorem follows.

In the line of Theorem 7 the following theorem may be deduced.

**Theorem 8** Let $f$ and $g$ be any two entire functions such that $\rho_f (p, q)$ and $\lambda_g$ are both finite and $p, q$ are any two positive integers with $p > q$. Then for any $R > r$,

1. \[ \limsup_{r \to \infty} \frac{\log^{[p]} \mu (r, f \circ g)}{\log^{[q]} \mu (R, g)} \leq \rho_f (p, q) \]
2. \[ \limsup_{r \to \infty} \frac{\log^{[p+1]} \mu (r, f \circ g)}{\log^{[q+1]} \mu (R, g)} \leq 1. \]

**References**


From: Editorial office (Tamkang J. Math.) <editor@staff.tku.edu.tw>
Subject: [Tamkang-J-Math] Editor Decision on Submission ID #1198
To: "Sanjib Kumar Datta" <sanjib.kr_datta@yahoo.co.in>
Date: Friday, 2 November, 2012, 6:32 AM

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We have reached a decision regarding your submission to Tamkang Journal of Mathematics, "Few Relations on the Growth Rates of Composite Entire Functions using their \((p,q)\)th Order".

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Some Relations on the Growth Rates of Composite Entire Functions using their (p,q) th Order

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Abstract

In this paper we discuss the growth rates of the maximum term of composition of entire functions with their corresponding left and right factors.

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1 Introduction, Definitions and Notations.

Let $f$ be an entire function defined in the open complex plane $\mathbb{C}$. The maximum term $\mu(r, f)$ of $f = \sum_{n=0}^{\infty} a_n z^n$ on $|z| = r$ is defined by $\mu(r, f) = \max_{n \geq 0} (|a_n| r^n)$. We do not explain the standard definitions and notations in the theory of entire function as those are available in [8] In the sequel the following two notations are used.

$$\log[k] x = \log (\log[k-1] x) \quad \text{for } k = 1, 2, 3, \cdots;$$
$$\log[0] x = x$$

and

$$\exp[k] x = \exp (\exp[k-1] x) \quad \text{for } k = 1, 2, 3, \cdots;$$
$$\exp[0] x = x.$$

To start our paper we just recall the following definitions.

**Definition 1** The order $\rho_f$ and lower order $\lambda_f$ of an entire function $f$ is defined as follows:

$$\rho_f = \limsup_{r \to \infty} \frac{\log[2] M(r, f)}{\log r} \quad \text{and} \quad \lambda_f = \liminf_{r \to \infty} \frac{\log[2] M(r, f)}{\log r}.$$

**Definition 2** [4] Let $l$ be an integer $\geq 2$. The generalised order $\rho_f[l]$ and generalised lower order $\lambda_f[l]$ of an entire function $f$ are defined as

$$\rho_f[l] = \limsup_{r \to \infty} \frac{\log[l] M(r, f)}{\log r} \quad \text{and} \quad \lambda_f[l] = \liminf_{r \to \infty} \frac{\log[l] M(r, f)}{\log r}.$$

When $l = 2$, Definition 2 coincides with Definition 1.

Juneja, Kapoor and Bajpai[2] defined the $(p, q)$ th order and $(p, q)$ th lower order of an entire function $f$ respectively as follows:

$$\rho_f(p, q) = \limsup_{r \to \infty} \frac{\log[p] M(r, f)}{\log[q] r} \quad \text{and} \quad \lambda_f(p, q) = \liminf_{r \to \infty} \frac{\log[p] M(r, f)}{\log[q] r},$$
where $p, q$ are positive integers with $p > q$.

For $p = 2$ and $q = 1$ we respectively denote $\rho_f (2, 1)$ and $\lambda_f (2, 1)$ by $\rho_f$ and $\lambda_f$.

Since for $0 \leq r < R$,

$$\mu (r, f) \leq M (r, f) \leq \frac{R}{R - r} \mu (R, f) \quad \{ \text{cf. [7]} \} \quad (1)$$

it is easy to see that

$$\rho_f = \limsup_{r \to \infty} \frac{\log^2 \mu (r, f)}{\log r}, \quad \lambda_f = \liminf_{r \to \infty} \frac{\log^2 \mu (r, f)}{\log r};$$

$$\rho_f^{[q]} = \limsup_{r \to \infty} \frac{\log^q \mu (r, f)}{\log r}, \quad \lambda_f^{[q]} = \liminf_{r \to \infty} \frac{\log^q \mu (r, f)}{\log r};$$

and

$$\rho_f (p, q) = \limsup_{r \to \infty} \frac{\log^p \mu (r, f)}{\log^q r}, \quad \lambda_f (p, q) = \liminf_{r \to \infty} \frac{\log^p \mu (r, f)}{\log^q r}.$$

In this paper we wish to prove some results relating to the growth rates of maximum terms of composition of two entire functions with their corresponding left and right factors on the basis of $(p, q)$ th order and $(p, q)$ th lower order where $p, q$ are positive integers with $p > q$.

## 2 Lemmas.

In this section we present some lemmas which will be needed in the sequel.

**Lemma 1** [6] Let $f$ and $g$ be two entire functions with $g(0) = 0$. Then for all sufficiently large values of $r$,

$$\mu (r, f \circ g) \geq \frac{1}{2} \mu \left( \frac{1}{8} \mu \left( \frac{r}{4}, g \right), f \right).$$

**Lemma 2** [1] If $f$ and $g$ are two entire functions then for all sufficiently large values of $r$,

$$M (r, f \circ g) \leq M (M (r, g), f).$$
Lemma 3 Let $f$ be an entire function with non zero finite generalised order $\rho_f^{[q]}$ (non zero finite generalised lower order $\lambda_f^{[q]}$). If $p - q = l - 1$, then the $(p,q)$-th order $\rho_f(p,q)$ (lower $(p,q)$-th order $\lambda_f(p,q)$) of $f$ will be equal to 1. If $p - q \neq l - 1$ then $\rho_f(p,q)$ ( $\lambda_f(p,q)$ ) is either zero or infinity.

Proof. From the definition of generalised order of an entire function $f$ we have for all sufficiently large values of $r$,

$$\log^{[q]} \mu(r,f) \leq (\rho_f^{[q]} + \varepsilon) \log r \quad (2)$$

and for a sequence of values of $r$ tending to infinity,

$$\log^{[q]} \mu(r,f) \geq (\rho_f^{[q]} - \varepsilon) \log r. \quad (3)$$

Next we let $a$ and $b$ any two positive integers. Now from (2) we have for all sufficiently large values of $r$,

$$\log^{[q+a]} \mu(r,f) \leq \log^{[q+a]} r + O(1)$$

i.e.,

$$\frac{\log^{[q+a]} \mu(r,f)}{\log^{[q+b]} r} \leq \frac{\log^{[q+a]} r + O(1)}{\log^{[q+b]} r}. \quad (4)$$

If we take $l + a = p$ and $1 + b = q$, then $p - q = (l - 1) + (a - b)$. We discuss the following three cases:

Case I. Let $a = b$. Then from (4) we get for all sufficiently large values of $r$,

$$\frac{\log^{[p]} \mu(r,f)}{\log^{[q]} r} \leq 1 + \frac{O(1)}{\log^{[q+b]} r}$$

i.e.,

$$\limsup_{r \to \infty} \frac{\log^{[p]} \mu(r,f)}{\log^{[q]} r} \leq 1. \quad (5)$$

Similarly from (3) we have for a sequence of values of $r$ tending to infinity,

$$\frac{\log^{[p]} \mu(r,f)}{\log^{[q]} r} \geq 1 + \frac{O(1)}{\log^{[q+b]} r}$$

i.e.,

$$\limsup_{r \to \infty} \frac{\log^{[p]} \mu(r,f)}{\log^{[q]} r} \geq 1. \quad (6)$$
Now from (5) and (6) we have
\[ \rho_f(p, q) = 1 \text{ when } p - q = l - 1. \]

**Case II.** Let \( a > b \) (i.e., \( p - q \neq l - 1 \)). Then from (4) we have for all sufficiently large values of \( r \),
\[
\limsup_{r \to \infty} \frac{\log^{[p]} \mu(r, f)}{\log^{[q]} r} \leq 0
\]
i.e., \( \rho_f(p, q) = 0 \text{ when } p - q \neq l - 1. \)

**Case III.** Also let us choose \( a \) and \( b \) such that \( a < b \) and \( l + a > 1 + b \)
(i.e., \( p - q \neq l - 1 \)). Then from (3) it can be proved for a sequence of values of \( r \) tending to infinity that
\[
\limsup_{r \to \infty} \frac{\log^{[p]} \mu(r, f)}{\log^{[q]} r} \geq \infty
\]
i.e., \( \rho_f(p, q) = \infty \text{ when } p - q \neq l - 1. \)

Therefore combining Case II and Case III (not violating the condition \( p > q \)), it follows that \( \rho_f(p, q) \) is either zero or infinity.
Similarly we may prove the conclusion for \( \lambda_f(p, q) \).
This proves the lemma.

### 3 Main Results.

In this section we present the main results of the paper.

**Theorem 1** Let \( f \) and \( g \) be any two entire functions such that \( \rho_g(m, n) < \lambda_f(p, q) \leq \rho_f(p, q) < \infty \) where \( p, q, m, n \) are positive integers with \( p > q, m > n \). Then
\[
(i) \lim_{r \to \infty} \frac{\log^{[p]} \mu(\exp^{[n-1]} r, f \circ g)}{\log^{[p-1]} \mu(\exp^{[q-1]} r, f)} = 0 \text{ if } q \geq m
\]
and
\[
(ii) \lim_{r \to \infty} \frac{\log^{[p+m-q-1]} \mu(\exp^{[n-1]} r, f \circ g)}{\log^{[p-1]} \mu(\exp^{[q-1]} r, f)} = 0 \text{ if } q < m.
\]
Proof. Since $\rho_g(m,n) < \lambda_f(p,q)$ we can choose $\varepsilon (>0)$ is such a way that

$$\rho_g(m,n) + \varepsilon < \lambda_f(p,q) - \varepsilon. \tag{7}$$

Now in view of the inequality (1), we have from Lemma 2 for all sufficiently large values of $r$,

$$\mu(r, f \circ g) \leq M(r, f \circ g) \leq M(M(r, g), f)$$

i.e.,

$$\log[^{[p]}\mu \left( \exp[^{[n-1]}]r, f \circ g \right) \leq \log[^{[p]}]M \left( \exp[^{[n-1]}]r, g \right).$$

i.e.,

$$\log[^{[p]}\mu \left( \exp[^{[n-1]}]r, f \circ g \right) \leq (\rho_f(p, q) + \varepsilon) \log[^{[q]}]M \left( \exp[^{[n-1]}]r, g \right). \tag{8}$$

Now the following two cases may arise.

Case I. Let $q \geq m$.

Then we have from (8) for all sufficiently large values of $r$,

$$\log[^{[p]}\mu \left( \exp[^{[n-1]}]r, f \circ g \right) \leq (\rho_f(p, q) + \varepsilon) \log[^{[m-1]}]M \left( \exp[^{[n-1]}]r, g \right). \tag{9}$$

Again for all sufficiently large values of $r$,

$$\log[^{[m]}]M \left( \exp[^{[n-1]}]r, g \right) \leq (\rho_g(m, n) + \varepsilon) \log[^{[m]}]\exp[^{[n-1]}]r$$

i.e.,

$$\log[^{[m]}]M \left( \exp[^{[n-1]}]r, g \right) \leq (\rho_g(m, n) + \varepsilon) \log r$$

i.e.,

$$\log[^{[m]}]M \left( \exp[^{[n-1]}]r, g \right) \leq \log r(\rho_g(m, n) + \varepsilon)$$

i.e.,

$$\log[^{[m-1]}]M \left( \exp[^{[n-1]}]r, g \right) \leq r(\rho_g(m, n) + \varepsilon). \tag{10}$$

Now from (9) and (10) we have for all sufficiently large values of $r$,

$$\log[^{[p]}\mu \left( \exp[^{[n-1]}]r, f \circ g \right) \leq (\rho_f(p, q) + \varepsilon) r(\rho_g(m, n) + \varepsilon). \tag{11}$$

Case II. Let $q < m$.

Then for all sufficiently large values of $r$ we get from (8) that

$$\log[^{[p]}\mu \left( \exp[^{[n-1]}]r, f \circ g \right) \leq (\rho_f(p, q) + \varepsilon) \log[^{[m]}]\exp[^{[n-1]}]r \log[^{[m]}]M \left( \exp[^{[n-1]}]r, g \right). \tag{12}$$
Again for all sufficiently large values of $r$,

\[
\log^m M (\exp^{[n-1]} r, g) \leq (\rho_g(m, n) + \varepsilon) \log^n \exp^{[n-1]} r
\]

i.e., \[
\log^m M (\exp^{[n-1]} r, g) \leq (\rho_g(m, n) + \varepsilon) \log r
\]

i.e., \[
\exp^{[m-q]} \log^m M (\exp^{[n-1]} r, g) \leq \exp^{[m-q]} \log r^{\rho_g(m, n) + \varepsilon}
\]

i.e., \[
\exp^{[m-q]} \log^m M (\exp^{[n-1]} r, g) \leq \exp^{[m-q] - 1} r^{\rho_g(m, n) + \varepsilon}.
\]

(13)

Now from (12) and (13) we have for all sufficiently large values of $r$,

\[
\log^p \mu (\exp^{[n-1]} r, f \circ g) \leq (\rho_f(p, q) + \varepsilon) \exp^{[m-q-1]} r^{\rho_g(m, n) + \varepsilon}
\]

i.e., \[
\log^{[p+1]} (\exp^{[n-1]} r, f \circ g) \leq \exp^{[m-q-2]} r^{\rho_g(m, n) + \varepsilon}
\]

i.e., \[
\log^{[p+m-q-1]} \mu (\exp^{[n-1]} r, f \circ g) \leq \log^{[m-q-2]} \exp^{[m-q-2]} r^{\rho_g(m, n) + \varepsilon}
\]

i.e., \[
\log^{[p+m-q-1]} \mu (\exp^{[n-1]} r, f \circ g) \leq r^{\rho_g(m, n) + \varepsilon}.
\]

(14)

Again for all sufficiently large values of $r$, we obtain that

\[
\log^p \mu (\exp^{[q-1]} r, f) \geq (\lambda_f(p, q) - \varepsilon) \log^q \exp^{[q-1]} r
\]

i.e., \[
\log^p \mu (\exp^{[q-1]} r, f) \geq (\lambda_f(p, q) - \varepsilon) \log r
\]

i.e., \[
\log^p \mu (\exp^{[q-1]} r, f) \geq \log^{(\lambda_f(p, q) - \varepsilon)} r
\]

i.e., \[
\log^{[p-1]} \mu (\exp^{[q-1]} r, f) \geq r^{(\lambda_f(p, q) - \varepsilon)}.
\]

(15)

Now combining (11) of Case I and (15) we get for all sufficiently large values of $r$ that

\[
\frac{\log^p \mu (\exp^{[n-1]} r, f \circ g)}{\log^{[p-1]} \mu (\exp^{[q-1]} r, f)} \leq \frac{(\rho_f(p, q) + \varepsilon) r^{(\rho_g(m, n) + \varepsilon)}}{r^{(\lambda_f(p, q) - \varepsilon)}}.
\]

(16)

Now in view of (7) it follows from (16) that

\[
\lim_{r \to \infty} \sup \frac{\log^p \mu (\exp^{[n-1]} r, f \circ g)}{\log^{[p-1]} \mu (\exp^{[q-1]} r, f)} = 0
\]

i.e., \[
\lim_{r \to \infty} \frac{\log^p \mu (\exp^{[n-1]} r, f \circ g)}{\log^{[p-1]} \mu (\exp^{[q-1]} r, f)} = 0.
\]
This proves the first part of the theorem. Again combining (14) of Case II and (15) we obtain for all sufficiently large values of $r$ that
\[
\frac{\log^{[p+m-q-1]} \mu \left( \exp^{[n-1]} r, f \circ g \right)}{\log^{[p-1]} \mu \left( \exp^{[q-1]} r, f \right)} \leq \frac{r^{\rho_g(m,n)+\epsilon} + O(1)}{r^{(\lambda_f(p,q)-\epsilon)}}. \tag{17}
\]
Now in view of (7) it follows from (17) that
\[
\limsup_{r \to \infty} \frac{\log^{[p+m-q-1]} \mu \left( \exp^{[n-1]} r, f \circ g \right)}{\log^{[p-1]} \mu \left( \exp^{[q-1]} r, f \right)} = 0
\]
i.e.,
\[
\lim_{r \to \infty} \frac{\log^{[p+m-q-1]} \mu \left( \exp^{[n-1]} r, f \circ g \right)}{\log^{[p-1]} \mu \left( \exp^{[q-1]} r, f \right)} = 0.
\]
This establishes the second part of the theorem.

**Remark 1** The condition $\rho_g (m, n) < \lambda_f (p, q)$ in Theorem 1 is essential as we see in the following example.

**Example 1** Let $f = g = \exp z$ and $p = m = 2$, $q = n = 1$.

Then $\rho_g (m, n) = \lambda_f (p, q) = \rho_f (p, q) = 1$.

Now
\[
\log \mu (r, f \circ g) \geq \log M \left( \frac{r}{2}, f \circ g \right) + O(1) \geq T \left( \frac{r}{2}, f \circ g \right) + O(1)
\]
\[
= T \left( \frac{r}{2}, \exp^{[2]} z \right) + O(1) \sim \exp \left( \frac{r}{2} \right) \frac{\exp \left( \frac{r}{2} \right)}{(2\pi \frac{r}{2})^{\frac{1}{2}}} + O(1) \quad (r \to \infty)
\]
i.e.,
\[
\log^{[2]} \mu (r, f \circ g) \geq \frac{r}{2} - \frac{1}{2} \log r + O(1).
\]
and $\log \mu (r, f) \leq \log M (r, f) = \log M (r, \exp z) = r$.

Then
\[
\lim_{r \to \infty} \frac{\log^{[p+m-q-1]} \mu \left( \exp^{[n-1]} r, f \circ g \right)}{\log^{[p-1]} \mu \left( \exp^{[q-1]} r, f \right)} = \lim_{r \to \infty} \frac{\log^{[2]} \mu (r, f \circ g)}{\log \mu (r, f)}
\]
\[
\geq \lim_{r \to \infty} \frac{\frac{r}{2} - \frac{1}{2} \log r + O(1)}{r} = \frac{1}{2} \neq 0, \text{ which is contrary to Theorem 1.}
Theorem 2 Let \( f \) and \( g \) be two entire functions such that \( \lambda_g(m, n) < \lambda_f(p, q) \leq \rho_f(p, q) < \infty \) where \( p, q, m, n \) are positive integers with \( p > q, m > n \). Then

\[
(i) \liminf_{r \to \infty} \frac{\log^{[p]} \mu \left( \exp^{[n-1]} r, f \circ g \right)}{\log^{[p-1]} \mu(\exp^{[q-1]} r, f)} = 0 \text{ if } q \geq m
\]

and

\[
(ii) \liminf_{r \to \infty} \frac{\log^{[p+m-q-1]} \mu \left( \exp^{[n-1]} r, f \circ g \right)}{\log^{[p-1]} \mu(\exp^{[q-1]} r, f)} = 0 \text{ if } q < m.
\]

Proof. For a sequence of values of \( r \) tending to infinity that

\[
\log^{[m]} M \left( \exp^{[n-1]} r, g \right) \leq (\lambda_g(m, n) + \varepsilon) \log^{[n]} \exp^{[n-1]} r
\]

i.e., \( \log^{[m]} M \left( \exp^{[n-1]} r, g \right) \leq (\lambda_g(m, n) + \varepsilon) \log r \)

i.e., \( \log^{[m]} M \left( \exp^{[n-1]} r, g \right) \leq \log r^{\lambda_g(m,n)+\varepsilon} \)

i.e., \( \log^{[m]} M \left( \exp^{[n-1]} r, g \right) \leq \log r^{\lambda_g(m,n)+\varepsilon}. \) (18)

Now from (9) and (18) we have for a sequence of values of \( r \) tending to infinity that

\[
\log^{[p]} \mu \left( \exp^{[n-1]} r, f \circ g \right) \leq (\rho_f(p, q) + \varepsilon) r^{\lambda_g(m,n)+\varepsilon}.
\]

Combining (15) and (19) we get for a sequence of values of \( r \) tending to infinity that

\[
\frac{\log^{[p]} \mu \left( \exp^{[n-1]} r, f \circ g \right)}{\log^{[p-1]} \mu(\exp^{[q-1]} r, f)} \leq \frac{(\rho_f(p, q) + \varepsilon) r^{\lambda_g(m,n)+\varepsilon}}{r^{(\lambda_f(p,q)-\varepsilon)}}.
\]

Now in view of (7) it follows from (20) that

\[
\liminf_{r \to \infty} \frac{\log^{[p]} \mu \left( \exp^{[n-1]} r, f \circ g \right)}{\log^{[p-1]} \mu(\exp^{[q-1]} r, f)} = 0.
\]

This proves the first part of the theorem. Again for a sequence of values of \( r \) tending to infinity that

\[
\log^{[m]} M \left( \exp^{[n-1]} r, g \right) \leq (\lambda_g(m, n) + \varepsilon) \log^{[n]} \exp^{[n-1]} r
\]

i.e., \( \log^{[m]} M \left( \exp^{[n-1]} r, g \right) \leq (\lambda_g(m, n) + \varepsilon) \log r \)

i.e., \( \log^{[m]} M \left( \exp^{[n-1]} r, g \right) \leq \log r^{(\lambda_g(m,n)+\varepsilon)} \)

i.e., \( \exp^{[m-q]} \log^{[m]} M \left( \exp^{[n-1]} r, g \right) \leq \exp^{[m-q]} \log r^{(\lambda_g(m,n)+\varepsilon)} \)

i.e., \( \exp^{[m-q]} \log^{[m]} M \left( \exp^{[n-1]} r, g \right) \leq \exp^{[m-q]} \log r^{(\lambda_g(m,n)+\varepsilon)}. \) (21)
Now from (12) and (21) we have for a sequence of values of $r$ tending to infinity that
\[
\log^{[p]} \mu (\exp^{[n-1]} r, f \circ g) \leq (\rho_f (p, q) + \varepsilon) \exp^{[m-q-1]} r (\lambda_p (m,n) + \varepsilon)
\]
i.e., \[ \log^{[p+1]} \mu (\exp^{[n-1]} r, f \circ g) \leq \exp^{[m-q-2]} r (\lambda_p (m,n) + \varepsilon)
\]
i.e., \[ \log^{[p+m-q-1]} \mu (\exp^{[n-1]} r, f \circ g) \leq \log^{[m-q-2]} \exp^{[m-q-2]} r (\lambda_p (m,n) + \varepsilon)
\]
i.e., \[ \log^{[p+m-q-1]} \mu (\exp^{[n-1]} r, f \circ g) \leq r (\lambda_p (m,n) + \varepsilon).
\] (22)
Combining (15) and (22) we obtain for a sequence of values of $r$ tending to infinity that
\[
\frac{\log^{[p+m-q-1]} \mu (\exp^{[n-1]} r, f \circ g)}{\log^{[p-1]} \mu (\exp^{[q-1]} r, f)} \leq \frac{r \lambda_p (m,n) + \varepsilon}{r^{\lambda_f (p,q) - \varepsilon}}.
\] (23)
Now in view of (7) it follows from (23) that
\[
\liminf_{r \to \infty} \frac{\log^{[p+m-q-1]} \mu (\exp^{[n-1]} r, f \circ g)}{\log^{[p-1]} \mu (\exp^{[q-1]} r, f)} = 0.
\]
This establishes the second part of the theorem.

**Remark 2** The condition $\lambda_p (m,n) < \lambda_f (p,q)$ in Theorem 2 is necessary which is evident from the following example.

**Example 2** Let $f = g = \exp z$ and $p = m = 2$, $q = n = 1$.

Then $\lambda_p (m,n) = \lambda_f (p,q) = \rho_f (p,q) = 1$.

Now $\log \mu (r, f \circ g) \geq \log M (\frac{r}{2}, f \circ g) + O(1) \geq T (\frac{r}{2}, f \circ g) + O(1)$
\[
= T (\frac{r}{2}, \exp [2] z) + O(1) \sim \frac{\exp (\frac{r}{2})}{(2\pi)^{\frac{3}{2}}} + O(1) \quad (r \to \infty)
\]
and $\log \mu (r, f) \leq \log M (r, f) = \log M (r, \exp z) = r$. 

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Therefore

\[
\liminf_{r \to \infty} \frac{\log^{p+m-q-1} \mu (\exp^{n-1} r, f \circ g)}{\log^{p-1} \mu (\exp^{q-1} r, f)}
\]

= \liminf_{r \to \infty} \frac{\log^2 \mu (r, f \circ g)}{\log \mu (r, f)}

\geq \liminf_{r \to \infty} \frac{r^2 - \frac{1}{2} \log r + O(1)}{r}

= \frac{1}{2} \neq 0, \text{ which is contrary to Theorem 2.}

**Theorem 3** Let \( f \) and \( g \) be any two entire functions such that

(A) \( \rho_g < \infty \), (B) \( \lambda_f^{[l]} > 0 \) and (C) \( \lambda_{f,g}(a,b) > 0 \) where \( l, a, b \) are all positive integers with \( l \geq 2 \), and \( a > b \). Also let \( A < \rho_g \). Then for any two positive integers \( m, n \) such that \( m - n = 1 \) and \( m > 2 \)

\[
\limsup_{r \to \infty} \frac{\log^m \mu (\exp^{n-1} r, f \circ g) \log^l \mu (r, f \circ g)}{\log^m \mu (\exp^n (r^A), g) \log^l \mu (\exp^{n-1} r, g)} = \infty.
\]

**Proof.** Let us choose \( 0 < \varepsilon < \min \{ \lambda_f^{[l]}, \lambda_{f,g}(a,b), \rho_g \} \).

Now from Lemma 1 we get for a sequence of values of \( r \) tending to infinity that

\[
\log^m \mu (r, f \circ g) \geq \log^m \mu \left( \frac{1}{8} \mu \left( \frac{r}{4}, g \right) - |g(0)|, f \right) + O(1)
\]

i.e., \( \log^m \mu (r, f \circ g) \geq (\lambda_f^{[l]} - \varepsilon) \log \mu \left( \frac{r}{4}, g \right) + O(1) \)

i.e., \( \log^m \mu (r, f \circ g) \geq (\lambda_f^{[l]} - \varepsilon) \left( \frac{r}{4} \right)^{\rho_g - \varepsilon} \). \hspace{1cm} (24)

Since \( \rho_g(m, n) = 1 \), in view of Lemma 3 it follows for all sufficiently large values of \( r \) that

\[
\log^m \mu (\exp^n (r^A), g) \leq (1 + \varepsilon) \log^n \exp^n (r^A)
\]

i.e., \( \log^m \mu (\exp^n (r^A), g) \leq (1 + \varepsilon) (r^A) \). \hspace{1cm} (25)

Now from (24) and (25) we get for a sequence of values of \( r \) tending to infinity that

\[
\frac{\log^l \mu (r, f \circ g)}{\log^m \mu (\exp^n (r^A), g)} \geq \frac{(\lambda_f^{[l]} - \varepsilon) \left( \frac{r}{4} \right)^{\rho_g - \varepsilon}}{(1 + \varepsilon) (r^A)}. \hspace{1cm} (26)
\]
Again for all sufficiently large values of $r$,

$$\log^a \mu(\exp^{b-1} r, f \circ g) \geq (\lambda_{f \circ g}(a, b) - \varepsilon) \log r$$

and

$$\log^m \mu(\exp^{n-1} r, g) \leq (1 + \varepsilon) \log r.$$ 

Therefore from the above two inequalities we get for sufficiently all large values of $r$ that

$$\frac{\log^a \mu(\exp^{b-1} r, f \circ g)}{\log^m \mu(\exp^{n-1} r, g)} \geq \frac{\lambda_{f \circ g}(a, b) - \varepsilon}{1 + \varepsilon}.$$ 

Since $\varepsilon (> 0)$ is arbitrary it follows from above that

$$\liminf_{r \to \infty} \frac{\log^a \mu(\exp^{b-1} r, f \circ g)}{\log^m \mu(\exp^{n-1} r, g)} \geq \lambda_{f \circ g}(a, b). \quad (27)$$

As $A < \rho_g$ we can choose $\varepsilon$ in such a way that

$$A < \rho_g - \varepsilon. \quad (28)$$

Now from (26) and (28) it follows that

$$\limsup_{r \to \infty} \frac{\log^l \mu(r, f \circ g)}{\log^m \mu(\exp^{n-1} (r-A), g)} = \infty. \quad (29)$$

Thus the theorem follows from (27) and (29).

**Corollary 1** Under the same conditions of Theorem 3 when $m = 2$

$$\limsup_{r \to \infty} \frac{\log^a \mu(\exp^{b-1} r, f \circ g) \log^l \mu(r, f \circ g)}{\left\{ \log^2 \mu(\exp^{n-1} (r-A), g) \right\} \left\{ \log^2 \mu(r, g) \right\}} = \infty.$$ 

**Remark 3** The condition $\lambda^l_f > 0$ in Corollary 1 is essential as we see in the following example.

**Example 3** Let

$$f = z \text{ and } g = \exp z.$$
Also let 
\[ l = a = 2 \text{ and } b = 1. \]

Then 
\[ \lambda_f = 0 \text{ and } \rho_g = \lambda_{fog} = 1. \]

Now 
\[ \log^2 \mu(r, f \circ g) \leq \log^2 M(r, f \circ g) = \log^2 M(r, g) = \log r \]
and 
\[ \log^2 \mu(r, g) \geq \log^2 M\left(\frac{T}{2}, g\right) = \log r + O(1). \]

Hence 
\[ \limsup_{r \to \infty} \frac{\left\{ \log^2 \mu(r, f \circ g) \right\}^2}{\log^2 \mu(\exp(r^A), g) \log^2 \mu(r, g)} \leq \limsup_{r \to \infty} \frac{\{\log r\}^2}{\frac{rA}{2} \{\log r + O(1)\}} = 0, \text{ which is contrary to Corollary 1.} \]

**Theorem 4** Let \( f \) be an entire function such that \( 0 < \lambda_f(p, q) \leq \rho_f(p, q) < \infty \). Also let \( g \) be an entire function with \( \rho_g^{[l]} > 0 \). Then

(i) \[ \limsup_{r \to \infty} \frac{\log^p \mu(r, f \circ g)}{\log^p \mu(\exp^{[l-1]}(r^\mu), f)} = \infty \text{ if } q = l - 1 \]

(ii) \[ \limsup_{r \to \infty} \frac{\log^p \mu(r, f \circ g)}{\log^p \mu(\exp^{[l-1]}(r^\mu), f)} \geq \frac{\rho_g^{[l]} \lambda_f(p, q)}{\mu \rho_f(p, q)} \text{ if } q = l \]

and

(iii) \[ \limsup_{r \to \infty} \frac{\log^p \mu(r, f \circ g)}{\log^p \mu(\exp^{[l-1]}(r^\mu), f)} \geq \frac{\lambda_f(p, q)}{\rho_f(p, q)} \text{ if } q > l \]

where \( \mu < \rho_g^{[l]} \) and \( p, q; l \) are positive integers with \( p > q \).
Proof. By Lemma 1 we obtain for a sequence of values of \( r \) tending to infinity,

\[
\log^{|p|} \mu(r, f \circ g) \geq \log^{|p|} \mu \left( \frac{1}{8} \mu \left( \frac{r}{4}, g \right), f \right) + O(1)
\]
i.e., \( \log^{|p|} \mu(r, f \circ g) \geq (\lambda_f (p, q) - \varepsilon) \log^{|q|} \mu \left( \frac{r}{4}, g \right) + O(1) \)
i.e., \( \log^{|p|} \mu(r, f \circ g) \geq (\lambda_f (p, q) - \varepsilon) \log^{q-l+1} \log^{l-1} \mu \left( \frac{r}{4}, g \right) + O(1) \)
i.e., \( \log^{|p|} \mu(r, f \circ g) \geq (\lambda_f (p, q) - \varepsilon) \log^{q-l+1} \left( \frac{r}{4} \right) \rho_g^{l-1} + O(1). \) (30)

Again from the definition of \( \rho_f (p, q) \) it follows for all sufficiently large values of \( r \) that

\[
\log^{|p|} \mu(\exp^{[l-1]} (r^\mu), f) \leq (\rho_f (p, q) + \varepsilon) \log^{[q]} \exp^{[l-1]} (r^\mu)
\]
i.e., \( \log^{|p|} \mu(\exp^{[l-1]} (r^\mu), f) \leq (\rho_f (p, q) + \varepsilon) \log^{q-l+1} (r^\mu). \) (31)

Thus from (30) and (31) we have for a sequence of values of \( r \) tending to infinity that

\[
\frac{\log^{|p|} \mu(r, f \circ g)}{\log^{|p|} \mu(\exp^{[l-1]} (r^\mu), f)} \geq \frac{(\lambda_f (p, q) - \varepsilon) \log^{q-l+1} \left( \frac{r}{4} \right) \rho_g^{l-1} + O(1)}{(\rho_f (p, q) + \varepsilon) \log^{q-l+1} (r^\mu)}. \) (32)

Since \( \mu < \rho_g^{l-1} \), the theorem follows from (32).

Remark 4 The condition \( \mu < \rho_g \) in Theorem 4 is essential as we see in the following example.

Example 4 Let \( f = g = \exp z \) and \( p = m = 2, q = n = 1, l = 2 \). Also let \( \mu = 1 \).

Then \( \lambda_f (p, q) = \rho_f (p, q) = 1 \) and \( \rho_g^{[l]} = \rho_g = 1 \).

Now

\[
\log \mu (r, f \circ g) \leq \log M (r, f \circ g) \leq 3T(2r, f \circ g) \sim \frac{3 \exp (2r)}{(4\pi^3 r)^{\frac{1}{2}}} \quad (r \to \infty)
\]

and

\[
\mu(\exp r, f) \leq M(\exp r, f) = \exp^{[2]} r.
\]
Therefore

\[
\limsup_{r \to \infty} \frac{\log^p \mu(r, f \circ g)}{\log^p \mu(\exp^{[p]}(r^n), f)} = \limsup_{r \to \infty} \frac{\log^2 \mu(r, f \circ g)}{\log^2 \mu(\exp r, f)} \leq 2r - \frac{1}{2} \log r + O(1)
\]

\[
= 2 \neq \infty, \text{which is contrary to Theorem 4.}
\]

**Theorem 5** Let \( f \) and \( g \) be two entire functions such that \( 0 < \lambda_f(p, q) \leq \rho_f(p, q) < \infty \) and \( \rho_g(m, n) < \infty \) where \( p, q, m, n \) are positive integers with \( p > q, m > n \). Then

\[
(i) \limsup_{r \to \infty} \frac{\log^{p+1} \mu(\exp^{[p]}(r^n), f \circ g)}{\log^p \mu(\exp^{[p]}(r^n), f)} \leq \frac{\rho_g(m, n)}{\lambda_f(p, q)} \quad \text{if} \ q \geq m
\]

and

\[
(ii) \limsup_{r \to \infty} \frac{\log^{p+m-q} \mu(\exp^{[p]}(r^n), f \circ g)}{\log^p \mu(\exp^{[p]}(r^n), f)} \leq \frac{\rho_g(m, n)}{\lambda_f(p, q)} \quad \text{if} \ q < m.
\]

**Proof.** We have for all sufficiently large values of \( r \)

\[
\log^p \mu(\exp^{[q]}(r^n), f) \geq (\lambda_f(p, q) - \varepsilon) \log^q \exp^{[q]}(r)
\]

* i.e., \( \log^p \mu(\exp^{[q]}(r^n), f) \geq (\lambda_f(p, q) - \varepsilon) \log r. \) (33)

**Case I.** If \( q \geq m \), then from (11) and (33) we get for all sufficiently large values of \( r \) that

\[
\limsup_{r \to \infty} \frac{\log^{p+1} \mu(\exp^{[p]}(r^n), f \circ g)}{\log^p \mu(\exp^{[p]}(r^n), f)} \leq \frac{\rho_g(m, n) + \varepsilon}{\lambda_f(p, q) - \varepsilon} \log r + O(1)
\]

* i.e., \( \limsup_{r \to \infty} \frac{\log^{p+1} \mu(\exp^{[p]}(r^n), f \circ g)}{\log^p \mu(\exp^{[p]}(r^n), f)} \leq \frac{\rho_g(m, n) + \varepsilon}{\lambda_f(p, q) - \varepsilon}. \)

Since \( \varepsilon > 0 \) is arbitrary, it follows from above that

\[
\limsup_{r \to \infty} \frac{\log^{p+1} \mu(\exp^{[p]}(r^n), f \circ g)}{\log^p \mu(\exp^{[p]}(r^n), f)} \leq \frac{\rho_g(m, n)}{\lambda_f(p, q)}.
\]
This proves the first part of the theorem.

**Case II.** If \( q < m \) then from (14) and (33) we obtain for all sufficiently large values of \( r \) that

\[
\frac{\log^{[p+m-q]} \mu \left( \exp^{[n-1]} r, f \circ g \right)}{\log^{[p]} \mu(\exp^{[q-1]} r, f)} \leq \frac{(\rho_g(m,n) + \varepsilon) \log r + O(1)}{(\lambda_f(p,q) - \varepsilon) \log r}
\]

i.e.,

\[
\limsup_{r \to \infty} \frac{\log^{[p+m-q]} \mu \left( \exp^{[n-1]} r, f \circ g \right)}{\log^{[p]} \mu(\exp^{[q-1]} r, f)} \leq \frac{\rho_g(m,n) + \varepsilon}{\lambda_f(p,q) - \varepsilon}.
\]

As \( \varepsilon (> 0) \) is arbitrary, it follows from above that

\[
\limsup_{r \to \infty} \frac{\log^{[p+m-q]} \mu \left( \exp^{[n-1]} r, f \circ g \right)}{\log^{[p]} \mu(\exp^{[q-1]} r, f)} \leq \frac{\rho_g(m,n)}{\lambda_f(p,q)}.
\]

Thus the second part of the theorem is established.

**Remark 5** The condition \( \rho_g(m,n) < \infty \) in Theorem 5 is necessary which is evident from the following example.

**Example 5** Let \( f = \exp z, \; g = \exp^{[2]} z \) and \( p = m = 2, \; q = n = 1 \).

Then \( \lambda_f(p,q) = \rho_f(p,q) = 1 \) and \( \rho_g(m,n) = \infty \).

Now

\[
\log^{[3]} \mu(r, f \circ g) \geq \log^{[3]} M\left(\frac{r}{2}, f \circ g\right) + O(1)
\]

i.e.,

\[
\log^{[3]} \mu(r, f \circ g) \geq \log^{[3]} M\left(\frac{r}{2}, f \circ g\right) + O(1)
\]

i.e.,

\[
\log^{[3]} \mu(r, f \circ g) \geq \log^{[3]} \exp^{[3]} \left(\frac{r}{2}\right) + O(1)
\]

i.e.,

\[
\log^{[3]} \mu(r, f \circ g) \geq \left(\frac{r}{2}\right) + O(1)
\]

and

\[
\log^{[2]} \mu(r, f) \leq \log^{[2]} M(r, f) = \log r.
\]

Therefore

\[
\limsup_{r \to \infty} \frac{\log^{[p+m-q]} \mu \left( \exp^{[n-1]} r, f \circ g \right)}{\log^{[p]} \mu(\exp^{[q-1]} r, f)} = \limsup_{r \to \infty} \frac{\log^{[2]} T(r, f \circ g)}{\log^{[2]} \mu(r, f)}
\]

i.e.,

\[
\limsup_{r \to \infty} \frac{\log^{[p+m-q]} \mu \left( \exp^{[n-1]} r, f \circ g \right)}{\log^{[p]} \mu(\exp^{[q-1]} r, f)} \geq \limsup_{r \to \infty} \frac{\frac{r}{2} + O(1)}{\log r}
\]

i.e.,

\[
\limsup_{r \to \infty} \frac{\log^{[p+m-q]} \mu \left( \exp^{[n-1]} r, f \circ g \right)}{\log^{[p]} \mu(\exp^{[q-1]} r, f)} = \infty,
\]

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which is contrary to Theorem 5.

References


GROWTH ANALYSIS OF DIFFERENTIAL MONOMIALS AND DIFFERENTIAL POLYNOMIALS GENERATED BY ENTIRE OR MEROMORPHIC FUNCTIONS

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ABSTRACT

In this paper we investigate the comparative growth of composite entire or meromorphic functions and differential monomials, differential polynomials generated by one of the factors which improves some earlier results.

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1 INTRODUCTION, DEFINITIONS AND NOTATIONS

For any two transcendental entire functions \( f \) and \( g \) defined in the open complex plane \( \mathbb{C} \), Clunie [3] proved that

\[
\lim_{r \to \infty} \frac{T(r, fg)}{T(r, g)} = \infty \quad \text{and} \quad \lim_{r \to \infty} \frac{T(r, fg)}{T(r, f)} = \infty.
\]

Singh [14] proved some comparative growth properties of \( \log T(r, fg) \) and \( T(r, f) \). He also raised the problem of investigating the comparative growth of \( \log T(r, fg) \) and \( T(r, g) \) which he was unable to solve. However, some results on the comparative growth of \( \log T(r, fg) \) and \( T(r, g) \) are proved in [8].

Let \( f \) be a non-constant meromorphic function defined in the open complex plane \( \mathbb{C} \). Also let \( n_{ij}, n_{ij}, \ldots, n_{ij} \) (\( k \geq 1 \)) be non-negative integers such that for each \( j \), \( \sum_{i=0}^{k} n_{ij} \geq 1 \). We call \( M_j[f] = A_j \left( f \right)^{n_{ij}} \left( f^{(1)} \right)^{n_{ij}} \ldots \left( f^{(k)} \right)^{n_{ij}} \)

where \( T(r, A_j) = S(r, f) \) to be a differential monomial generated by \( f \). The numbers \( \gamma_{M_j} = \sum_{i=0}^{k} n_{ij} \) are called respectively the degree and weight of \( M_j[f] \) \([5],[13]\). The expression

\[
P[f] = \sum_{j=1}^{\infty} M_j[f]
\]

called a differential polynomial generated by \( f \) The numbers \( \gamma_P = \max_{1 \leq j \leq s} \gamma_{M_j} \) and

\[
\Gamma_P = \max_{1 \leq j \leq s} \Gamma_{M_j}
\]

called respectively the degree and weight of \( P[f] \) \([5],[13]\). Also we call the numbers

\[
\gamma_P = \min_{1 \leq j \leq s} \gamma_{M_j}
\]

describe \( f \) (the order of the highest derivative of \( f \)) the lower degree and the order of \( P[f] \) respectively. If \( \gamma_P = \gamma_P \), \( P[f] \) is called a homogeneous differential polynomial. In the paper we further investigate the question of Singh [14] mentioned earlier and prove some new results relating to the comparative growths of composite entire or meromorphic functions and differential monomials, differential polynomials generated by one of
the factors. We do not explain the standard notations and definitions of the theory of entire and meromorphic functions because those are available in [17] and [6]. Throughout the paper we consider only the non-constant differential polynomials and we denote by $P_e[f]$ a differential polynomial not containing $f$, i.e., for which $n_{ej} = 0$ for $j = 1, 2, \ldots s$. We consider only those $P[f], P_e[f]$ singularities of whose individual terms do not cancel each other.

We also denote by $M[f]$ a differential monomial generated by a transcendental meromorphic function $f$.

The following definitions are well known.

**Definition 1.** The order $\rho_f$ and lower order $\lambda_f$ of a meromorphic function $f$ are defined as

$$
\rho_f = \limsup_{r \to \infty} \frac{\log T(r, f)}{\log r} \quad \text{and} \quad \lambda_f = \liminf_{r \to \infty} \frac{\log T(r, f)}{\log r},
$$

If $f$ is entire, one can easily verify that

$$
\rho_f = \limsup_{r \to \infty} \frac{\log^{(2)} M(r, f)}{\log r} \quad \text{and} \quad \lambda_f = \liminf_{r \to \infty} \frac{\log^{(2)} M(r, f)}{\log r},
$$

where $\log^k x = \log (\log^{k-1} x)$ for $k = 1, 2, 3, \ldots$ and $\log^0 x = x$.

If $\rho_f < \infty$ then $f$ is of finite order. Also $\rho_f = 0$ means that $f$ is of order zero. In this connection Datta and Biswas [4] gave the following definition:

**Definition 2.** [4] Let $f$ be a meromorphic function of order zero. Then the quantities $\rho_f^*$ and $\lambda_f^*$ of $f$ are defined by

$$
\rho_f^* = \limsup_{r \to \infty} \frac{T(r, f)}{\log r} \quad \text{and} \quad \lambda_f^* = \liminf_{r \to \infty} \frac{T(r, f)}{\log r},
$$

If $f$ is an entire function then clearly

$$
\rho_f^* = \limsup_{r \to \infty} \frac{M(r, f)}{\log r} \quad \text{and} \quad \lambda_f^* = \liminf_{r \to \infty} \frac{M(r, f)}{\log r}.
$$

**Definition 3.** Let $a$ be a complex number, finite or infinite. The Nevanlinna’s deficiency and the Valiron deficiency of $a$ with respect to a meromorphic function $f$ are defined as

$$
\delta(a; f) = 1 - \limsup_{r \to \infty} \frac{N(r, a; f)}{T(r, f)} = \liminf_{r \to \infty} \frac{N(r, a; f)}{T(r, f)}
$$

and

$$
\Delta(a; f) = 1 - \liminf_{r \to \infty} \frac{N(r, a; f)}{T(r, f)} = \limsup_{r \to \infty} \frac{N(r, a; f)}{T(r, f)}.
$$

**Definition 4.** The quantity $\Theta(a; f)$ of a meromorphic function $f$ is defined as follows

$$
\Theta(a; f) = 1 - \limsup_{r \to \infty} \frac{N(r, a; f)}{T(r, f)}.
$$

**Definition 5.** [16] For $a \in \mathbb{C} \cup \{\infty\}$, we denote by $n(r, a; f \mid 1)$, the number of simple zeros of $f - a$ in $|z| \leq r$. $N(r, a; f \mid 1)$ is defined in terms of $n(r, a; f \mid 1)$ in the usual way. We put

$$
\delta_1(a; f) = 1 - \limsup_{r \to \infty} \frac{N(r, a; f \mid 1)}{T(r, f)},
$$

the deficiency of $a$ corresponding to the simple $a$-points of $f$, i.e., simple zeros of $f - a$.

Yang [15] proved that there exists at most a denumerable number of complex numbers $a \in \mathbb{C} \cup \{\infty\}$ for which $\delta_1(a; f) > 0$ and

$$
\sum_{a \in \mathbb{C} \cup \{\infty\}} \delta_1(a; f) \leq 4.
$$

**Definition 6.** [9] For $a \in \mathbb{C} \cup \{\infty\}$, let $\eta_0(r, a; f)$ denote the number of zeros of $f - a$ in $|z| \leq r$, where a zero of multiplicity $< p$ is counted according to its multiplicity and a zero of multiplicity $\geq p$ is counted exactly $p$ times; and $N_0(r, a; f)$ is defined in terms of $\eta_0(r, a; f)$ in the usual way. We define

$$
\delta_0(a; f) = 1 - \limsup_{r \to \infty} \frac{\eta_0(r, a; f)}{T(r, f)}.
$$
Definition 7. [2] \( P_f \) is said to be admissible if
(i) \( P_f \) is homogeneous, or
(ii) \( P_f \) is non-homogeneous and \( m(r, f) = S(r, f) \).

Definition 8. A function \( \rho_f(r) \) is called a proximate order of \( f \) relative to \( T(r, f) \) if
(i) \( \rho_f(r) \) is non-negative and continuous for \( r \geq r_0 \), say.
(ii) \( \rho_f(r) \) is differentiable for \( r \geq r_0 \) except possibly at isolated points at which \( \rho_f'(r - 0) \) and \( \rho_f'(r + 0) \) exist.
(iii) \( \lim_{r \to \infty} \rho_f(r) = \rho_f < \infty \)
(iv) \( \lim_{r \to \infty} \rho_f'(r) \log r = 0 \) and
(v) \( \limsup_{r \to \infty} \frac{T(r, f)}{r^{\rho_f(r)}} = 1 \).

Definition 9. A function \( \lambda_f(r) \) is called a lower proximate order of \( f \) relative to \( T(r, f) \) if
(i) \( \lambda_f(r) \) is non-negative and continuous for \( r \geq r_0 \), say,
(ii) \( \lambda_f(r) \) is differentiable for \( r \geq r_0 \) except possibly at isolated points at which \( \lambda_f'(r - 0) \) and \( \lambda_f'(r + 0) \) exists
(iii) \( \lim_{r \to \infty} \lambda_f(r) = \lambda_f < \infty \),
(iv) \( \lim_{r \to \infty} r \lambda_f'(r) \log r = 0 \) and
(v) \( \liminf_{r \to \infty} \frac{T(r, f)}{r^{\lambda_f(r)}} = 1 \).

2. LEMMAS

In this section we present some lemmas which will be needed in the sequel.

Lemma 1. [1] If \( f \) is meromorphic and \( g \) is entire then for all sufficiently large values of \( r \),
\[
T(r, f, g) \leq \left\{ 1 + o(1) \right\} \frac{T(r, g)}{\log M(r, g)} T(M(r, g), f).
\]

Lemma 2. [8] If \( f \) be an entire function then for \( \delta(> 0) \) the function \( r^{\rho_f + \delta - \rho_f} \) is ultimately an increasing function of \( r \).

Lemma 3. [11] Let \( f \) be an entire function. Then for \( \delta(> 0) \) the function \( r^{\lambda_f + \delta - \lambda_f(r)} \) is ultimately an increasing function of \( r \).

Lemma 4. [2] Let \( P_0[f] \) be admissible. If \( f \) is of finite order or of non-zero lower order and
\[
\sum_{\alpha \neq 0} \Theta(a; f) = 2 \quad \text{then} \quad \lim_{r \to \infty} \frac{T(r, P_0[f])}{T(r, f)} = \Gamma P_0[f].
\]

Lemma 5. [2] Let \( f \) be either of finite order or of non-zero lower order such that
\[
\Theta(\infty; f) = \sum_{\alpha \neq 0} \delta_p(a; f) = 1 \quad \text{or} \quad \delta(\infty; f) = \sum_{\alpha \neq 0} \delta(a; f) = 1.
\]
Then for homogeneous \( P_0[f] \),
\[
\lim_{r \to \infty} \frac{T(r, P_0[f])}{T(r, f)} = \gamma P_0[f].
\]

Lemma 6. Let \( f \) be a meromorphic function of finite order or of non-zero lower order. If \( \sum_{\alpha \neq 0} \Theta(a; f) = 2 \), then the order (lower order) of homogeneous \( P_0[f] \) is same as that of \( f \) if \( f \) is of positive finite order.

Proof. By Lemma 5, \( \lim_{r \to \infty} \frac{T(r, P_0[f])}{T(r, f)} \) exists and is equal to 1.
\[
\rho_{P_0[f]} = \limsup_{r \to \infty} \frac{\log T(r, P_0[f])}{\log r} = \limsup_{r \to \infty} \frac{\log T(r, f)}{\log r} \cdot \lim_{r \to \infty} \frac{T(r, P_0[f])}{T(r, f)} = \rho_f, 1 = \rho_f.
\]
In a similar manner, \( \lambda_{0}(f) = \lambda_{f} \).

This proves the lemma.

**Lemma 7.** Let \( f \) be a meromorphic function of finite order or of non-zero lower order such that \( \sum_{\alpha \neq \infty} \delta_{\alpha} = 1 \). Then the order (lower order) of homogeneous \( P(\alpha) \) and \( \alpha \) are same when \( f \) is of finite positive order.

We omit the proof of the lemma because it can be carried out in the line of Lemma 6 and with the help of Lemma 9.

In a similar manner we can state the following lemma without proof.

**Lemma 8.** Let \( f \) be a meromorphic function of finite order or of non-zero lower order such that \( \delta(\infty; f) \sum_{\alpha \neq \infty} \delta_{\alpha} = 1 \). Then for every homogeneous \( P(f) \), the order (lower order) of \( P(f) \) is same as that of \( f \) when \( f \) is of finite positive order.

**Lemma 9.** [10] Let \( f \) be a transcendental meromorphic function of finite order or of non-zero lower order and \( \sum_{\alpha \in \mathbb{C} \cup \{ \infty \}} \delta_{\alpha}(a; f) = 4 \). Then
\[
\lim_{r \to \infty} \frac{T(r,M(f))}{T(r,f)} = \Gamma_{M} - (\Gamma_{M} - \gamma_{M}) \Theta(\infty; f),
\]
where
\[
\Theta(\infty; f) = 1 - \limsup_{r \to \infty} \frac{T(r,f)}{T(r,M(f))},
\]

**Lemma 10.** If \( f \) be a transcendental meromorphic function of finite order or of non-zero lower order and \( \sum_{\alpha \in \mathbb{C} \cup \{ \infty \}} \delta_{\alpha}(a; f) = 4 \), then the order and lower order of \( M(f) \) are same as those of \( f \).

We omit the proof of the lemma because it can be carried out in the line of Lemma 6 and with the help of Lemma 9.

**Lemma 11.** [7] Let \( g \) be an entire function with \( \lambda_{g} < \infty \) and assume that \( a_{i} (i = 1, 2, ..., n; n \leq \infty) \) are entire functions satisfying \( T(r,a_{i}) = o(T(r,g)) \). If \( \sum_{i=1}^{n} \delta(a_{i};g) = 1 \), then
\[
\lim_{r \to \infty} \frac{T(r,g)}{\log M(r,g)} = \frac{1}{\pi}.
\]

3. THEOREMS

In this section we present the main results of the paper.

**Theorem 1.** Let \( f \) be a meromorphic function of order zero and \( g \) be entire such that \( p_{g} \) is finite. Also let \( \sum_{\alpha \neq \infty} \Theta(a; g) = 2 \). Then for any \( \alpha > 1 \)
\[
\liminf_{r \to \infty} \frac{T(r,f(g))}{T(r,g)} \leq \left(1 + o(1)\right) \frac{p_{f}^{\alpha} \alpha^{\alpha}}{\log p_{g}(\alpha)}.
\]

**Proof.** If \( p_{f}^{\alpha} = \infty \), then the result is obvious. So we suppose that \( p_{f}^{\alpha} < \infty \). Since \( T(r,g) \leq \log^{+} M(r,g) \), we obtain by Lemma 1 for \( \varepsilon > 0 \) and for all large values of \( r \),
\[
T(r,f(g)) \leq \left(1 + o(1)\right) \left(p_{f}^{\alpha} + \varepsilon\right) \log M(r,g)
\]
Since \( \varepsilon > 0 \) is arbitrary, it follows from above that
\[
\liminf_{r \to \infty} \frac{T(r,g)}{T(r,f(g))} \leq \left(1 + o(1)\right) p_{f}^{\alpha} \liminf_{r \to \infty} \frac{\log M(r,g)}{T(r,g)}.
\]

(1)

Since \( \limsup_{r \to \infty} T(r,g) = 1 \), for given \( \varepsilon (0 < \varepsilon < 1) \) we get for all sufficiently large values of \( r \)
\[
T(r,g) < \left(1 + \varepsilon\right) r p_{g}(\varepsilon)
\]
and for a sequence of values of \( r \) tending to infinity
\[
T(r,g) > (1 - \varepsilon) r p_{g}(\varepsilon).
\]

(3)
Thus from (1) and (5) it follows that
\[ \liminf_{r \to \infty} \frac{\log M(r,g)}{T(r,g)} \leq \left( \frac{a+1}{a-1} \right) \cdot \alpha^g \cdot a^{\rho_g} \cdot \frac{\rho_g^{\rho_g}}{T_{\mathcal{P}\{g\}}(\rho_g)} \]

because by Lemma 2, \( r^\rho_g + \delta - \rho_g(r) \) is ultimately an increasing function of \( r \). Since \( \varepsilon(>0) \) and \( \delta(>0) \) are arbitrary, we obtain that
\[ \liminf_{r \to \infty} \frac{\log M(r,g)}{T(r,g)} \leq \left( \frac{a+1}{a-1} \right) \cdot a^{\rho_g} \cdot \frac{\rho_g^{\rho_g}}{T_{\mathcal{P}\{g\}}(\rho_g)} \]  

(4) Now in view of (4) and Lemma 4 we get that
\[ \liminf_{r \to \infty} \frac{\log M(r,g)}{T(r,g)} \leq \liminf_{r \to \infty} \frac{\log M(r,g)}{T(r,g)} \cdot \lim_{r \to \infty} \frac{T(r,g)}{T(r,P_{\mathcal{M}\{g\}}(\rho_g))} = \left( \frac{a+1}{a-1} \right) \cdot a^{\rho_g} \cdot \frac{\rho_g^{\rho_g}}{T_{\mathcal{P}\{g\}}(\rho_g)} \]  

(5) Thus from (1) and (5) it follows that
\[ \liminf_{r \to \infty} \frac{T(r,f;g)}{T(r,P_{\mathcal{M}\{g\}}(\rho_g))} \leq (1 + o(1)) \cdot \left( \frac{a+1}{a-1} \right) \cdot a^{\rho_g} \cdot \frac{\rho_g^{\rho_g}}{T_{\mathcal{P}\{g\}}(\rho_g)} \]

This proves the theorem.

**Remark 1.** If we take \( \Theta(\infty;g) = \sum_{a \neq \infty} \delta_a(a; g) = 1 \) or \( \delta(\infty; g) = \sum_{a \neq \infty} \delta(a; g) = 1 \) instead of \( \sum_{a \neq \infty} \sum_{a \neq \infty} \delta_a(a; g) = 2 \) in Theorem 1 and the other conditions remain the same then one can easily prove that
\[ \liminf_{r \to \infty} \frac{T(r,f;g)}{T(r,P_{\mathcal{M}\{g\}}(\rho_g))} \leq (1 + o(1)) \cdot \left( \frac{a+1}{a-1} \right) \cdot a^{\rho_g} \cdot \frac{\rho_g^{\rho_g}}{T_{\mathcal{P}\{g\}}(\rho_g)} \]

In the line of Theorem 1 and with the help of Lemma 9 we may state the following theorem without proof.

**Theorem 2.** Let \( f \) be a meromorphic function of order zero and \( g \) be entire such that \( \rho_g \) is finite. Also let \( \sum_{a \in \mathbb{C} \cup \{\infty\}} \delta_a(a; g) = 4 \). Then for any \( \alpha > 1 \)
\[ \liminf_{r \to \infty} \frac{T(r,f;g)}{T(r,P_{\mathcal{M}\{g\}}(\rho_g))} \leq (1 + o(1)) \cdot \left( \frac{a+1}{a-1} \right) \cdot a^{\rho_g} \cdot \frac{\rho_g^{\rho_g}}{T_{\mathcal{P}\{g\}}(\rho_g)} \]

In the line of Theorem 1 one can easily prove the following theorem using the definition of lower proximate order.

**Theorem 3.** Let \( f \) be a meromorphic function of order zero and \( g \) be entire with \( \lambda_g < \infty \). Also let \( \Theta(\infty; g) = \sum_{a \neq \infty} \delta_a(a; g) = 1 \) or \( \delta(\infty; g) = \sum_{a \neq \infty} \delta(a; g) = 1 \). Then for any \( \alpha > 1 \)
\[ \liminf_{r \to \infty} \frac{T(r,f;g)}{T(r,P_{\mathcal{M}\{g\}}(\rho_g))} \leq (1 + o(1)) \cdot \left( \frac{a+1}{a-1} \right) \cdot a^{\rho_g} \cdot \frac{\rho_g^{\rho_g}}{T_{\mathcal{P}\{g\}}(\rho_g)} \]

**Remark 2.** If we take \( \sum_{a \neq \infty} \delta_a(a; g) = 2 \) instead of \( \alpha > 1 \) in Theorem 3 and the other conditions remain the same then one can easily prove that
\[ \liminf_{r \to \infty} \frac{T(r,f;g)}{T(r,P_{\mathcal{M}\{g\}}(\rho_g))} \leq (1 + o(1)) \cdot \left( \frac{a+1}{a-1} \right) \cdot a^{\rho_g} \cdot \frac{\rho_g^{\rho_g}}{T_{\mathcal{P}\{g\}}(\rho_g)} \]

**Theorem 4.** Let \( f \) be a meromorphic function of order zero and \( g \) be entire with \( \lambda_g < \infty \). Also let \( \sum_{a \in \mathbb{C} \cup \{\infty\}} \delta_a(a; g) = 4 \). Then for any \( \alpha > 1 \)
\[ \liminf_{r \to \infty} \frac{T(r,f;g)}{T(r,P_{\mathcal{M}\{g\}}(\rho_g))} \leq (1 + o(1)) \cdot \left( \frac{a+1}{a-1} \right) \cdot a^{\rho_g} \cdot \frac{\rho_g^{\rho_g}}{T_{\mathcal{P}\{g\}}(\rho_g)} \]

The proof of the theorem can be established in the line of Theorem 3 and with the help of Lemma 9 and therefore it is omitted.
Theorem 5. Let \( f \) and \( g \) be two non constant entire functions such that \( f \) is of lower order zero and \( \lambda_f^* \) and \( \lambda_g^* \) are finite. Also let \( \Theta(\infty; g) = \sum_{a \neq \infty} \delta_{a}(g) = 1 \) or \( \delta(\infty; g) = \sum_{a \neq \infty} \delta(a; g) = 1 \). Then

\[
\limsup_{r \to \infty} \frac{T(r, f g)}{T(r, P_f)} \geq (1 + o(1)) \cdot \frac{1}{3} \frac{\lambda_f^*}{4^s}.
\]

Proof. If \( \lambda_f^* = 0 \) then the result is obvious. So we suppose that \( \lambda_f^* > 0 \).

For all sufficiently large values of \( r \) we know that

\[
T(r, f g) \geq \frac{1}{2} \log M \left( \frac{1}{n} M \left( \frac{r}{n}, g \right) + o(1) \right) \quad \text{cf. [12]}
\]

For \( \varepsilon (0 < \varepsilon < \min \{ \lambda_f^+, I \}) \) we get for all sufficiently large values of \( r \),

\[
T(r, f g) \geq \frac{1}{2} \left( \lambda_f^+ - \varepsilon \right) \log \left( \frac{1}{n} M \left( \frac{r}{n}, g \right) + o(1) \right)
\]

i.e.,

\[
T(r, f g) \geq \frac{1}{2} \left( \lambda_f^+ - \varepsilon \right) \log \left( \frac{1}{n} M \left( \frac{r}{n}, g \right) \right)
\]

i.e.,

\[
T(r, f g) \geq \frac{1}{2} \left( \lambda_f^+ - \varepsilon \right) \log M \left( \frac{r}{n}, g \right) + \frac{1}{3} (\lambda_f^+ - \varepsilon) \log \frac{1}{n}
\]

i.e.,

\[
T(r, f g) \geq \frac{1}{3} (\lambda_f^+ - \varepsilon) T \left( \frac{r}{n}, g \right) + O(1).
\]

Since \( \liminf_{r \to \infty} \frac{T(r, g)}{T(r, g \log g)} = 1 \), for given \( \varepsilon(> 0) \) we get for all sufficiently large values of \( r \)

\[
T(r, g) > (1 - \varepsilon) r^{1/g(r)}
\]

and for a sequence of values of \( r \) tending to infinity

\[
T(r, g) < (1 + \varepsilon) r^{1/g(r)}.
\]

(6)

From (6) and (7) we get for \( \delta(> 0) \) and for all sufficiently large values of \( r \)

\[
T(r, f g) \geq \frac{1}{3} (\lambda_f^+ - \varepsilon) (1 - \varepsilon) (1 + o(1)) \frac{T(r, g) - r^{1/g + \delta}}{r^{1/g + \delta} - r^{1/g}}.
\]

Since \( r^{1/g + \delta - \lambda_g(r)} \) is ultimately an increasing function of \( r \) it follows for all sufficiently large values of \( r \) that

\[
T(r, f g) \geq \frac{1}{3} (\lambda_f^+ - \varepsilon) (1 - \varepsilon) (1 + o(1)) \frac{T(r, g)}{4^s + \pi^2}.
\]

(9)

So by (8) and (9) we get for a sequence of values of \( r \) tending to infinity

\[
T(r, f g) \geq \frac{1}{3} (\lambda_f^+ - \varepsilon) \left( \frac{1 - \varepsilon}{1 + \varepsilon} \right) (1 + o(1)) \frac{T(r, g)}{4^s + \pi^2}
\]

i.e.,

\[
T(r, f g) \geq \frac{1}{3} (\lambda_f^+ - \varepsilon) \left( \frac{1 - \varepsilon}{1 + \varepsilon} \right) (1 + o(1)) \frac{T(r, f g)}{4^s + \pi^2}
\]

Since \( \varepsilon(> 0) \) and \( \delta(> 0) \) are arbitrary, in view of Lemma 5 it follows from above that

\[
\limsup_{r \to \infty} \frac{T(r, f g)}{T(r, P_f)} \geq (1 + o(1)) \cdot \frac{1}{3} \frac{\lambda_f^*}{4^s}.
\]

Thus the theorem is proved.

Remark 3. If we take \( \sum_{a \neq \infty} \Theta(a; g) = 2 \) instead of \( \Theta(\infty; g) = \sum_{a \neq \infty} \delta_{a}(g) = 1 \) or \( \delta(\infty; g) = \sum_{a \neq \infty} \delta_{a}; g \equiv 1 \) in Theorem 5 and the other conditions remain the same then one can easily prove that

\[
\limsup_{r \to \infty} \frac{T(r, f g)}{T(r, P_f)} \geq (1 + o(1)) \cdot \frac{1}{3} \frac{\lambda_f^*}{4^s}.
\]

In the line of Theorem 5 and with the help of Lemma 9 we may state the following theorem without proof.
Theorem 6. Let $f$ and $g$ be two non constant entire functions such that $f$ is of lower order zero and $\lambda_1^*$ and $\lambda_2^*$ are finite. Also let $\sum_{a \in \mathbb{C} \cup \{\infty\}} \delta_t(a; g) = 4$. Then

$$\limsup_{r \to \infty} \frac{T(r, f \circ g)}{T(r, M_{1, 2}(g))} \geq (1 + o(1)) \cdot \frac{1}{3^{1/2} \Gamma_{1, 2}(r, g)} \cdot \lambda_1^*. $$

Theorem 7. Let $f$ and $g$ be two non constant entire functions such that $\rho_1^*$ and $\lambda_2^*$ are finite. Also suppose that there exist entire functions $a_i (i = 1, 2, \ldots, n; n \leq \infty)$ satisfying

(i) $T(r, a_i) = o(T(r, g))$ as $r \to \infty$ for $i = 1, 2, \ldots, n$.

(ii) $\sum_{a \neq \infty} \delta_t(a; g) = 1$ and

(iii) $\Theta(a; g) = 2$.

Then

$$\frac{\pi \lambda_2^*}{3^{t/2} \Gamma_{1, 2}(g)} \leq \limsup_{r \to \infty} \frac{T(r, f \circ g)}{T(r, M_{1, 2}(g))} \leq \frac{\pi \rho_1^*}{\Gamma_{1, 2}(g)}. $$

Proof. For any two entire functions $f$ and $g$, the following two inequalities are well known:

$$T(r, f) \leq \log^+ M(r, f) \leq 3 \cdot T(2r, f). \quad \{ \text{cf. [6]} \} \tag{10}$$

and

$$\log M(r, f \circ g) \leq \log M(M(r, g), f). \quad \{ \text{cf. [3]} \} \tag{11}$$

For $\varepsilon > 0$ we get from (10) and (11) for all sufficiently large values of $r$,

$$T(r, f \circ g) \leq \log M(M(r, g), f)$$

i.e., $T(r, f \circ g) \leq (\rho_1^* + \varepsilon) \log M(r, g)$

i.e.,

$$\frac{T(r, f \circ g)}{T(r, M_{1, 2}(g))} \leq \frac{(\rho_1^* + \varepsilon) \log M(r, g)}{T(r, M_{1, 2}(g))}. $$

Hence we get from above that

$$\limsup_{r \to \infty} \frac{T(r, f \circ g)}{T(r, M_{1, 2}(g))} \leq (\rho_1^* + \varepsilon) \limsup_{r \to \infty} \frac{\log M(r, g)}{T(r, M_{1, 2}(g))}$$

i.e.,

$$\limsup_{r \to \infty} \frac{T(r, f \circ g)}{T(r, M_{1, 2}(g))} \leq (\rho_1^* + \varepsilon) \limsup_{r \to \infty} \frac{\log M(r, g)}{T(r, g)} \cdot \limsup_{r \to \infty} \frac{T(r, g)}{T(r, M_{1, 2}(g))}. $$

Since $\varepsilon > 0$ is arbitrary, it follows from Lemma 11 and Lemma 4 that

$$\limsup_{r \to \infty} \frac{T(r, f \circ g)}{T(r, M_{1, 2}(g))} \leq \frac{\pi \rho_1^*}{\Gamma_{1, 2}(g)}. \tag{12}$$

Now suppose that $0 < \varepsilon < \min \{ \lambda_1^*, 1 \}$ we get from (6) for all sufficiently large values of $r$ that

$$\frac{T(r, f \circ g)}{T(r, M_{1, 2}(g))} \geq \frac{1}{2} (\lambda_1^* - \varepsilon) \frac{\log M(r, g)}{T(r, g)} \cdot \frac{T(r, g)}{T(r, M_{1, 2}(g))} + O(1). \tag{13}$$

From (7) and (8) and in the line of Lemma 3 we get for a sequence of values of $r$ tending to infinity and for $\delta > 0$

$$\frac{T(r, g)}{T(r, g)} \geq \frac{1}{r^{1-\delta}} \cdot \left( \begin{array}{c} \frac{1}{\log r} \\ \frac{1}{\log r} \end{array} \right).$$

$$\geq \frac{1}{r^{1-\delta}} \cdot \left( \begin{array}{c} \frac{1}{\log r} \\ \frac{1}{\log r} \end{array} \right) \cdot \frac{1}{r^{1-\delta}} \cdot \frac{1}{r^{1-\delta}} \cdot \frac{1}{r^{1-\delta}}.$$

Since $\varepsilon > 0$ and $\delta > 0$ are arbitrary we get from (13), Lemma 4, Lemma 11 and above that

$$\limsup_{r \to \infty} \frac{T(r, f \circ g)}{T(r, M_{1, 2}(g))} \geq \frac{\pi \lambda_2^*}{3^{t/2} \Gamma_{1, 2}(g)}. \tag{14}$$

Thus the theorem follows from (12) and (14).
Sanjib Kumar Datta*, Tanmay Biswas† and Manab Biswas‡ / Growth Analysis of Differential Monomials and Differential Polynomials Generated by Entire or Meromorphic Functions/ IJMA- 3(9), Sept.-2012.

Remark 4. If we take “$\Theta(\infty; g) = \sum_{a \neq \infty} \delta_p(a; g) = 1$ or $\delta(\infty; g) = \sum_{a \neq \infty} \delta(a; g) = 1$” instead of “$\sum_{a \neq \infty} \Theta(a; g) = 2$” in Theorem 7 and the other conditions remain the same then one can easily prove that

$$\limsup_{r \to \infty} \frac{\pi \lambda^*}{3A^2 - \Gamma_M(\Gamma_M + \gamma_M) \Theta(\infty; g)} \leq \limsup_{r \to \infty} \frac{T(r; f, g)}{\log T(r; P_0[f])} \leq \frac{\pi \rho^*_g}{\log T(r; P_0[f])}. $$

In the line of Theorem 7 and with the help of Lemma 9 we may state the following theorem without proof.

Theorem 8. Let $f$ and $g$ be two non constant entire functions such that $\rho^*_g$ and $\lambda$ are finite. Also suppose that there exist entire functions $a_i$ ($i = 1, 2, \ldots, n; n \leq \infty$) satisfying

(i) $T(r, a_i) = o(T(r, g))$ as $r \to \infty$ for $i = 1, 2, \ldots, n$,

(ii) $\sum_{a \neq \infty} \delta(a; g) = 1$ and

(iii) $\sum_{a \in \mathbb{U} \cup \{\infty\}} \delta_1(a; g) = 4$.

Then

$$\limsup_{r \to \infty} \frac{\pi \lambda^*}{3A^2 - \Gamma_M(\Gamma_M + \gamma_M) \Theta(\infty; g)} \leq \limsup_{r \to \infty} \frac{T(r; f, g)}{T(r, \mathcal{M}[g])} \leq \frac{\pi \rho^*_g}{\Gamma_M(\Gamma_M + \gamma_M) \Theta(\infty; g)}.$$  

Theorem 9. Let $f$ and $g$ be any two entire functions such that $0 < \lambda_f \leq \rho_f < \infty$ and $\rho^*_g > 0$. Also let $\sum_{a \neq \infty} \Theta(a; f) = 2$. Then

$$\limsup_{r \to \infty} \frac{\log T(r; f, g)}{\log T(r, P_0[f])} \geq \frac{\lambda_f \rho^*_g}{A \rho_f},$$

where $A$ is any positive real number.

Proof. We know that for $r > 0$ [12]

$$T(r, f, g) \geq \frac{1}{3} \log M \left\{ \frac{1}{8} M \left( \frac{\Gamma}{4}, g \right) + o(1), f \right\}. $$

Let us suppose that $0 < \epsilon < \min \{ \lambda_f, \rho^*_g \}$.

Now from (15) we have for a sequence of values of $r$ tending to infinity that

$$\log T(r, f, g) \geq (\lambda_f - \epsilon) \log M \left( \frac{\Gamma}{4}, g \right) + O(1).$$

i.e.,$\log T(r, f, g) \geq (\lambda_f - \epsilon)(\rho^*_g - \epsilon) \log r + O(1).$ (16)

Again from the definition of $\rho_0[f]$ we have for arbitrary positive $\epsilon$ and for all sufficiently large values of $r$,

$$\log T(r, P_0[f]) \leq A (\rho_0[f] + \epsilon) \log r$$

i.e., $\log T(r, P_0[f]) \leq A (\rho_f + \epsilon) \log r. $ (17)

So combining (16) and (17) we get for a sequence of values of $r$ tending to infinity that

$$\limsup_{r \to \infty} \frac{\log T(r; f, g)}{\log T(r, P_0[f])} \geq \frac{(\lambda_f - \epsilon)(\rho^*_g - \epsilon) \log r + O(1)}{A \rho_f \log r}.$$ 

i.e., $\limsup_{r \to \infty} \frac{\log T(r; f, g)}{\log T(r, P_0[f])} \geq \frac{\lambda_f \rho^*_g}{A \rho_f}. $

This completes the proof.

Remark 5. Under the same conditions of Theorem 9, if $f$ is of regular growth then

$$\limsup_{r \to \infty} \frac{\log T(r; f, g)}{\log T(r, P_0[f])} \geq \frac{\rho^*_g}{A}. $
Remark 6. In Theorem 9 if we take \( \lambda_g^* > 0 \) instead of \( \rho_g^* > 0 \) and the other conditions remain the same then it can be shown that

\[
\liminf_{r \to \infty} \frac{\log T(r, f[g])}{\log T(r^\alpha, P_0[f])} = \frac{\lambda_f^* \lambda_g^*}{\alpha \rho_f}.
\]

In addition if \( f \) is of regular growth then

\[
\lim_{r \to \infty} \frac{T(r, f[g])}{T(r^\alpha, P_0[f])} = \frac{\lambda_f^*}{\alpha}.
\]

Remark 7. Also if we consider \( 0 < \lambda_g < \infty \) or \( 0 < \rho_g < \infty \) instead of \( 0 < \lambda_g \leq \rho_g < \infty \) in Theorem 9 and the other conditions remain the same, then one can easily verify that

\[
\limsup_{r \to \infty} \frac{\log T(r, f[g])}{\log T(r^\alpha, P_0[f])} = \frac{\lambda_f^*}{\alpha}.
\]

Remark 8. The conclusions of Theorem 9, Remark 5, Remark 6 and Remark 7 can also be drawn under the hypothesis \( \Theta(\infty; f) = \sum_{\alpha \not= \infty} \delta_\infty(a, f) = 1 \) or \( \delta(\infty; f) = \sum_{\alpha \not= \infty} \delta(a, f) = 1 \) instead of \( \sum_{\alpha \not= \infty} \Theta(a, f) = 2 \).

Theorem 10. Let \( f \) and \( g \) be two entire functions such that \( 0 < \lambda_f < \infty \) and \( \rho_g^* > 0 \). Also let \( \sum \delta_1(a, f) = 4 \). Then

\[
\limsup_{r \to \infty} \frac{\log T(r, f[g])}{\log T(r^\alpha, M[f])} \geq \frac{\lambda_f \rho_g^*}{\alpha \rho_f}.
\]

where \( A \) is any positive real number.

The proof is omitted because it can be carried out in the line of Theorem 10 and with the help of Lemma 10.

Remark 9. Under the same conditions of Theorem 10 if \( f \) is of regular growth then

\[
\limsup_{r \to \infty} \frac{\log T(r, f[g])}{\log T(r^\alpha, M[f])} = \frac{\rho_g^*}{\alpha}.
\]

Remark 10. In Theorem 10 if we take \( \lambda_g^* > 0 \) instead of \( \rho_g^* > 0 \) and the other conditions remain the same then it can be shown that

\[
\lim_{r \to \infty} \frac{\log T(r, f[g])}{\log T(r^\alpha, M[f])} = \frac{\lambda_f^* \lambda_g^*}{\alpha \rho_f}.
\]

In addition if \( f \) is of regular growth then

\[
\lim_{r \to \infty} \frac{T(r, f[g])}{T(r^\alpha, M[f])} = \frac{\lambda_f^*}{\alpha}.
\]

Remark 11. Further if we consider \( 0 < \lambda_g < \infty \) or \( 0 < \rho_g < \infty \) instead of \( 0 < \lambda_g \leq \rho_g < \infty \) in Theorem 10 and the other conditions remain the same, then one can easily verify that

\[
\limsup_{r \to \infty} \frac{\log T(r, f[g])}{\log T(r^\alpha, P_0[f])} \geq \frac{\lambda_f^*}{\alpha}.
\]

Theorem 11. Let \( f \) be a transcendental meromorphic function with the maximum deficiency sum and \( g \) be an entire function such that \( 0 < \lambda_f \leq \rho_f < \infty \) and \( \rho_g^* < \infty \). Also let \( \delta(\infty; f) = \sum \delta(a, f) = 1 \). Then

\[
\limsup_{r \to \infty} \frac{\log T(r, f[g])}{\log T(r^\alpha, P_0[f])} \leq \frac{\rho_f \rho_g^*}{\alpha \lambda_f}.
\]

for any positive real number \( A \).

Proof. In view of Lemma 1 and the inequality \( T(r, g) \leq \log^+ M(r, g) \) we get for all sufficiently large values of \( r \),

\[
T(r, f[g]) \leq (1 + o(1)) T(M(r, g), f)
\]

i.e.,

\[
\log T(r, f[g]) \leq (\rho_f + \epsilon) \log M(r, g) + O(1)
\]

i.e.,

\[
\log T(r, f[g]) \leq (\rho_f + \epsilon) (\rho_g^* + \epsilon) + O(1).
\]

(18)
From the definition of \( \lambda_{p[f]} \), we have for arbitrary positive \( \varepsilon \) and for all sufficiently large values of \( r, \)

\[
\log T(r^A, P_0[f]) \geq A (\lambda_{p[f]} - \varepsilon) \log r
\]

de i.e., \( \log T(r^A, P_0[f]) \geq A (\lambda_f - \varepsilon) \log r \).

Now combining (18) and (19) we get for all sufficiently large values of \( r, \)

\[
\frac{\log T(r, f_0g)}{\log T(r^A, P_0[f])} \leq \frac{\rho_f + \varepsilon(\rho_f^* + \varepsilon) + O(1)}{A (\lambda_f - \varepsilon) \log r}
\]

de i.e., \( \limsup_{r \to \infty} \frac{\log T(r, f_0g)}{\log T(r^A, P_0[f])} \leq \frac{\rho_f + \rho_f^*}{A \lambda_f} \).

This completes the proof.

**Remark 12.** Under the same conditions of Theorem 11 if \( f \) is of regular growth then

\[
\limsup_{r \to \infty} \frac{\log T(r, f_0g)}{\log T(r^A, P_0[f])} \leq \frac{\rho_f^*}{A}.
\]

**Remark 13.** In Theorem 11 if we take \( \lambda_f^* < \infty \) instead of \( \rho_f^* < \infty \) and the other conditions remain the same then can be shown that

\[
\liminf_{r \to \infty} \frac{\log T(r, f_0g)}{\log T(r^A, f_0g)} \leq \frac{\rho_f \lambda_f^*}{A \rho_f}.
\]

In addition if \( f \) is of regular growth then

\[
\liminf_{r \to \infty} \frac{\log T(r, f_0g)}{\log T(r^A, P_0[f])} \leq \frac{\rho_f \lambda_f^*}{A \rho_f}.
\]

**Remark 14.** If we take \( 0 < \rho_f < \infty \) or \( 0 < \lambda_f < \infty \) instead of \( 0 < \lambda_f \leq \rho_f < \infty \) in Theorem 11 and the other conditions remain the same, one can easily verify that

\[
\liminf_{r \to \infty} \frac{\log T(r, f_0g)}{\log T(r^A, P_0[f])} \leq \frac{\rho_f^*}{A}.
\]

**Remark 15.** The conclusions of Theorem 11, Remark 12, Remark 13 and Remark 14 can also be deduced if we replace \( \delta(x; f) = \sum_{\alpha \neq \infty} \delta(\alpha; f) = 1 \) by \( \Theta(x; f) = \sum_{\alpha \neq \infty} \delta_0(\alpha; f) = 1 \) or \( \Theta(\alpha; f) = 2 \) respectively.

In the line of Theorem 11 and with the help of Lemma 10 we may state the following theorem without proof.

**Theorem 12.** Let \( f \) be a transcendental meromorphic function with the maximum deficiency sum and \( g \) be an entire function such that \( 0 < \lambda_f \leq \rho_f < \infty \) and \( \rho_f^* < \infty \). Also let \( \sum_{\alpha \in \mathbb{C} \cup \{\infty\}} \delta_1(\alpha; f) = 4 \). Then

\[
\limsup_{r \to \infty} \frac{\log T(r, f_0g)}{\log T(r^A, \mathbb{M}[f])} \leq \frac{\rho_f + \rho_f^*}{A \lambda_f},
\]

where \( A \) is any positive real number.

**Remark 16.** Under the same conditions of Theorem 12 if \( f \) is of regular growth then

\[
\limsup_{r \to \infty} \frac{\log T(r, f_0g)}{\log T(r^A, \mathbb{M}[f])} \leq \frac{\rho_f^*}{A}.
\]

**Remark 17.** In Theorem 12 if we take \( \lambda_f^* < \infty \) instead of \( \rho_f^* < \infty \) and the other conditions remain the same then it can be shown that

\[
\liminf_{r \to \infty} \frac{\log T(r, f_0g)}{\log T(r^A, \mathbb{M}[f])} \leq \frac{\rho_f \lambda_f^*}{A \rho_f}.
\]

In addition if \( f \) is of regular growth then

\[
\liminf_{r \to \infty} \frac{\log T(r, f_0g)}{\log T(r^A, \mathbb{M}[f])} \leq \frac{\lambda_f^*}{A}.
\]

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Remark 18. If we take \( 0 < \rho_f < \infty \) or \( 0 < \lambda_f < \infty \) instead of \( 0 < \lambda_f \leq \rho_f < \infty \) in Theorem 12 and the other conditions remain the same, then one can easily verify that

\[
\liminf_{r \to \infty} \log T(r, f/g) = \frac{\rho_f^\lambda}{\lambda}.
\]

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Further Growth Estimations of Differential Monomials and Differential Polynomials in the Light of Zero Order and Weak Type

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Abstract

In this paper we investigate the comparative growth of composite entire or meromorphic functions and differential monomials, differential polynomials generated by one of the factors which improves some earlier results.

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1 Introduction, Definitions and Notations.

For any two transcendental entire functions $f$ and $g$ defined in the open complex plane $C$, Clunie [4] proved that

$$\lim_{r \to \infty} \frac{T(r, f g)}{T(r, f)} = \infty \quad \text{and} \quad \lim_{r \to \infty} \frac{T(r, f g)}{T(r, g)} = \infty.$$  

Singh [15] proved some comparative growth properties of $\log T(r, f g)$ and $T(r, f)$. He also raised the problem of investigating the comparative growth of $\log T(r, f g)$ and $T(r, g)$ which he was unable to solve. However, some results on the comparative growth of $\log T(r, f g)$ and $T(r, g)$ are proved in [11].

Let $f$ be a non-constant meromorphic function defined in the open complex plane $C$.

Also let $n_{ij}, n_{ij}, \ldots, n_{kj}$ ($k \geq 1$) be non-negative integers such that for each $j$, $\sum_{i=0}^{k} n_{ij} \geq 1$.

We call $M_{j}[f] = A_{j}(f)^{n_{ij}} (f^{(1)})^{n_{ij}} \ldots (f^{(k)})^{n_{ij}}$ where $T(r, A_{j}) = S(r, f)$ to be a differential monomial generated by $f$. The numbers $\gamma_{M_{j}} = \sum_{i=0}^{k} n_{ij}$ and $\Gamma_{M_{j}} = \sum_{i=0}^{k} (i+1)n_{ij}$ are called
respectively the degree and weight of \( M_j[f] \) \((\{8\}, [14]\)) . The expression \( P[f] = \sum_{j=1}^{s} M_j[f] \) is called a differential polynomial generated by \( f \). The numbers \( \gamma_p = \max_{1 \leq j \leq s} \gamma_{M_j} \) and \( \Gamma_p = \max_{1 \leq j \leq s} \Gamma_{M_j} \) are called respectively the degree and weight of \( P[f] \) \((\{8\}, [14]\)) . Also we call the numbers \( \gamma_p = \min_{1 \leq j \leq s} \gamma_{M_j} \) and \( k \) (the order of the highest derivative of \( f \)) the lower degree and the order of \( P[f] \) respectively . If \( \gamma_p = \gamma_p, P[f] \) is called a homogeneous differential polynomial . In the paper we further investigate the question of Singh [15] mentioned earlier and prove some new results relating to the comparative growths of composite entire or meromorphic functions and differential monomials, differential polynomials generated by one of the factors . We do not explain the standard notations and definitions of the theory of entire and meromorphic functions because those are available in [18] and [9] . Throughout the paper we consider only the non-constant differential polynomials and we denote by \( P_0[f] \) a differential polynomial not containing \( f \) i.e., for which \( n_{oj} = 0 \) for \( j = 1, 2, ..., s \). We consider only those \( P[f] , P_0[f] \) singularities of whose individual terms do not cancel each other . We also denote by \( M[f] \) a differential monomial generated by a transcendental meromorphic function \( f \).

The following definitions are well known .

**Definition 1** The order \( \rho_f \) and lower order \( \lambda_f \) of a meromorphic function \( f \) are defined as

\[
\rho_f = \limsup_{r \to \infty} \frac{\log T(r,f)}{\log r} \quad \text{and} \quad \lambda_f = \liminf_{r \to \infty} \frac{\log T(r,f)}{\log r} .
\]

If \( f \) is entire, one can easily verify that

\[
\rho_f = \limsup_{r \to \infty} \frac{\log^{[2]} M(r,f)}{\log r} \quad \text{and} \quad \lambda_f = \liminf_{r \to \infty} \frac{\log^{[2]} M(r,f)}{\log r} ,
\]

where \( \log^{[k]} x = \log (\log^{[k-1]} x) \) for \( k = 1, 2, 3, ..., \) and \( \log^{[0]} x = x \).

If \( \rho_f < \infty \) then \( f \) is of finite order . Also \( \rho_f = 0 \) means that \( f \) is of order zero . In this connection Datta and Biswas [6] gave the following definition .

**Definition 2** [6] Let \( f \) be a meromorphic function of order zero . Then the quantities \( \rho_f^{**} \) and \( \lambda_f^{**} \) of \( f \) are defined by :

\[
\rho_f^{**} = \limsup_{r \to \infty} \frac{T(r,f)}{\log^2 r} \quad \text{and} \quad \lambda_f^{**} = \liminf_{r \to \infty} \frac{T(r,f)}{\log^2 r} .
\]

If \( f \) is an entire function then clearly

\[
\rho_f^{**} = \limsup_{r \to \infty} \frac{\log M(r,f)}{\log r} \quad \text{and} \quad \lambda_f^{**} = \liminf_{r \to \infty} \frac{\log M(r,f)}{\log r} .
\]

**Definition 3** The type \( \sigma_f \) and lower type \( \bar{\sigma}_f \) of a meromorphic function \( f \) are defined as

\[
\sigma_f = \limsup_{r \to \infty} \frac{T(r,f)}{r^{\rho_f}} \quad \text{and} \quad \bar{\sigma}_f = \liminf_{r \to \infty} \frac{T(r,f)}{r^{\rho_f}}, \quad 0 < \rho_f < \infty .
\]
When \( f \) is entire, it can be easily verified that
\[
\sigma_f = \limsup_{r \to \infty} \frac{\log M(r, f)}{r^\rho_f} \quad \text{and} \quad \overline{\sigma}_f = \liminf_{r \to \infty} \frac{\log M(r, f)}{r^\rho_f} , \quad 0 < \rho_f < \infty.
\]

Datta and Jha [5] gave the definition of weak type of a meromorphic function of finite positive lower order in the following way:

**Definition 4** [5] The weak type \( \tau_f \) of a meromorphic function \( f \) of finite positive lower order \( \lambda_f \) is defined by
\[
\tau_f = \liminf_{r \to \infty} \frac{T(r, f)}{r^\lambda_f}.
\]

For entire \( f \),
\[
\tau_f = \liminf_{r \to \infty} \frac{\log M(r, f)}{r^\lambda_f} , \quad 0 < \lambda_f < \infty.
\]

Similarly one can define the growth indicator \( \overline{\tau}_f \) of a meromorphic function \( f \) of finite positive lower order \( \lambda_f \) as
\[
\overline{\tau}_f = \limsup_{r \to \infty} \frac{T(r, f)}{r^\lambda_f}.
\]

When \( f \) is entire, it can be easily verified that
\[
\overline{\tau}_f = \limsup_{r \to \infty} \frac{\log M(r, f)}{r^\lambda_f} , \quad 0 < \lambda_f < \infty.
\]

**Definition 5** Let "\( a \)" be a complex number, finite or infinite. The Nevanlinna's deficiency and the Valiron deficiency of "\( a \)" with respect to a meromorphic function \( f \) are defined as
\[
\delta(a; f) = 1 - \limsup_{r \to \infty} \frac{N(r, a; f)}{T(r, f)} = \liminf_{r \to \infty} \frac{m(r, a; f)}{T(r, f)}
\]
and
\[
\Delta(a; f) = 1 - \liminf_{r \to \infty} \frac{N(r, a; f)}{T(r, f)} = \limsup_{r \to \infty} \frac{m(r, a; f)}{T(r, f)}.
\]

**Definition 6** The quantity \( \Theta(a; f) \) of a meromorphic function \( f \) is defined as follows
\[
\Theta(a; f) = 1 - \limsup_{r \to \infty} \frac{\overline{N}(r, a; f)}{T(r, f)}.
\]

**Definition 7** [17] For \( a \in \mathbb{C} \cup \{\infty\} \), we denote by \( n(r, a; f |1\) \), the number of simple zeros of \( f - a \) in \(|z| \leq r \). \( N(r, a; f |1\) \) is defined in terms of \( n(r, a; f |1\) \) in the usual way. We put
\[
\delta_1(a; f) = 1 - \limsup_{r \to \infty} \frac{N(r, a; f |1\)}{T(r, f)} ,
\]
the deficiency of 'a' corresponding to the simple a-points of \( f \) i.e., simple zeros of \( f - a \).
Yang [16] proved that there exists at most a denumerable number of complex numbers $a \in \mathbb{C} \cup \{\infty\}$ for which $\delta_1(a; f) > 0$ and $\sum_{a \in \mathbb{C} \cup \{\infty\}} \delta_1(a; f) \leq 4$.

**Definition 8** [12] For $a \in \mathbb{C} \cup \{\infty\}$, let $n_p(r, a; f)$ denotes the number of zeros of $f - a$ in $|z| \leq r$, where a zero of multiplicity $< p$ is counted according to its multiplicity and a zero of multiplicity $\geq p$ is counted exactly $p$ times; and $N_p(r, a; f)$ is defined in terms of $n_p(r, a; f)$ in the usual way. We define

$$\delta_p(a; f) = 1 - \limsup_{r \to \infty} \frac{N_p(r, a; f)}{T(r, f)}.$$

**Definition 9** [3] $P[f]$ is said to be admissible if

(i) $P[f]$ is homogeneous, or

(ii) $P[f]$ is non homogeneous and $m(r, f) = S(r, f)$.

## 2 Lemmas.

In this section we present some lemmas which will be needed in the sequel.

**Lemma 1** [1] If $f$ is meromorphic and $g$ is entire then for all sufficiently large values of $r$,

$$T(r, f o g) \leq \{1 + o(1)\} \frac{T(r, g)}{\log M(r, g)} T(M(r, g), f).$$

**Lemma 2** [2] Let $f$ be meromorphic and $g$ be entire and suppose that $0 < \mu < \rho_g \leq \infty$. Then for a sequence of values of $r$ tending to infinity,

$$T(r, f o g) \geq T(\exp(r^\mu), f).$$

**Lemma 3** [10] Let $f$ be meromorphic and $g$ be entire such that $0 < \mu < \rho_g \leq \infty$ and $\lambda_f > 0$. Then for a sequence of values of $r$ tending to infinity,

$$T(r, f o g) > T(\exp(r^\mu), g).$$

**Lemma 4** [7] Let $f$ be a meromorphic function and $g$ be an entire function such that $\lambda_g < \mu < \infty$ and $0 < \lambda_f \leq \rho_f < \infty$. Then for a sequence of values of $r$ tending to infinity,

$$T(r, f o g) < T(\exp(r^\mu), f).$$

**Lemma 5** [7] Let $f$ be a meromorphic function of finite order and $g$ be an entire function with $0 < \lambda_g < \mu < \infty$. Then for a sequence of values of $r$ tending to infinity,

$$T(r, f o g) < T(\exp(r^\mu), g).$$

**Lemma 6** [3] Let $P_0[f]$ be admissible. If $f$ is of finite order or of non-zero lower order and

$$\sum_{a \neq \infty} \Theta(a; f) = 2 \text{ then}$$

$$\lim_{r \to \infty} \frac{T(r, P_0[f])}{T(r, f)} = \Gamma_0[f].$$

**Lemma 7** [3] Let $f$ be either of finite order or of non-zero lower order such that $\Theta(\infty; f) =$
\[
\sum_{a \neq \infty} \delta_p(a; f) = 1 \text{ or } \delta(\infty; f) = \sum_{a \neq \infty} \delta(a; f) = 1. \text{ Then for homogeneous } P_0[f],
\]
\[
\lim_{r \to \infty} \frac{T(r, P_0[f])}{T(r, f)} = \gamma_{P_0[f]}.
\]

**Lemma 8** Let \( f \) be a meromorphic function of finite order or of non-zero lower order. If \( \sum_a \Theta(a; f) = 2 \), then the order (lower order) of homogeneous \( P_0[f] \) is same as that of \( f \). Also \( \sigma_{P_0[f]}, \overline{\sigma}_{P_0[f]}, \tau_{P_0[f]} \) and \( \overline{\tau}_{P_0[f]} \) are \( I_{P_0[f]} \) times that of \( f \) if \( f \) is of positive finite order.

**Proof.** By Lemma 6, \( \lim_{r \to \infty} \frac{\log T(r, P_0[f])}{\log T(r, f)} \) exists and is equal to 1.

\[
\rho_{P_0[f]} = \limsup_{r \to \infty} \frac{\log T(r, P_0[f])}{\log r} = \lim_{r \to \infty} \frac{\log T(r, f)}{\log r} \cdot \lim_{r \to \infty} \frac{\log T(r, P_0[f])}{\log T(r, f)} = \rho_f, 1 = \rho_f.
\]

In a similar manner, \( \lambda_{P_0[f]} = \lambda_f \).

Again by Lemma 6,

\[
\sigma_{P_0[f]} = \limsup_{r \to \infty} \frac{T(r, P_0[f])}{r^{\rho_{P_0[f]}}} = \lim_{r \to \infty} \frac{T(r, f)}{r^{\rho_f}} \cdot \limsup_{r \to \infty} \frac{T(r, P_0[f])}{r^{\rho_f}} = \Gamma_{P_0[f]} \cdot \sigma_f.
\]

Similarly \( \overline{\sigma}_{P_0[f]} = \Gamma_{P_0} \cdot \overline{\sigma}_f \).

Also

\[
\tau_{P_0[f]} = \liminf_{r \to \infty} \frac{T(r, P_0[f])}{r^{\lambda_{P_0[f]}}} = \lim_{r \to \infty} \frac{T(r, f)}{r^{\lambda_f}} \cdot \liminf_{r \to \infty} \frac{T(r, f)}{r^{\lambda_f}} = \Gamma_{P_0} \cdot \tau_f.
\]

Analogously \( \overline{\tau}_{P_0[f]} = \Gamma_{P_0[f]} \cdot \overline{\tau}_f \).

This proves the lemma.

**Lemma 9** Let \( f \) be a meromorphic function of finite order or of non-zero lower order such that \( \Theta(\infty; f) = \sum_a \delta_p(a; f) = 1 \). Then the order (lower order) of homogeneous \( P_0[f] \) and \( f \) are same. Also \( \sigma_{P_0[f]}, \overline{\sigma}_{P_0[f]}, \tau_{P_0[f]} \) and \( \overline{\tau}_{P_0[f]} \) are \( \gamma_{P_0[f]} \) times that of \( f \) when \( f \) is of finite positive order.
We omit the proof of the lemma because it can be carried out in the line of Lemma 8 and with the help of Lemma 7.

In a similar manner we can state the following lemma without proof.

**Lemma 10** Let \( f \) be a meromorphic function of finite order or of non-zero lower order such that \( \delta(\infty; f) = \sum_{a \neq \infty} \delta(a; f) = 1 \). Then for every homogeneous \( P_0[f] \) the order (lower order) of \( P_0[f] \) is same as that of \( f \). Also the \( \sigma_{P_0[f]}, \bar{\sigma}_{P_0[f]}, \tau_{P_0[f]} \) and \( \bar{\tau}_{P_0[f]} \) are \( \gamma_{P_0[f]} \) times that of \( f \) when \( f \) is of finite positive order.

**Lemma 11** [13] Let \( f \) be a transcendental meromorphic function of finite order or of non-zero lower order and \( \sum_{a \in C \cup \{ \infty \}} \delta_1(a; f) \leq 4 \), then

\[
\lim_{r \to \infty} \frac{T(r,M[f])}{T(r,f)} = \Gamma_M - (\Gamma_M - \gamma_M) \Theta(\infty; f),
\]

where

\[
\Theta(\infty; f) = 1 - \limsup_{r \to \infty} \frac{N(r,f)}{T(r,f)}.
\]

**Lemma 12** If \( f \) be a transcendental meromorphic function of finite order or of non-zero lower order and \( \sum_{a \in C \cup \{ \infty \}} \delta_1(a; f) \leq 4 \), then the order and lower order of \( M[f] \) are same as those of \( f \). Also \( \sigma_{M[f]}, \bar{\sigma}_{M[f]}, \tau_{M[f]} \) and \( \bar{\tau}_{M[f]} \) are \( \{ \Gamma_M - (\Gamma_M - \gamma_M) \Theta(\infty; f) \} \) times that of \( f \) when \( f \) is of finite positive order.

We omit the proof of the lemma because it can be carried out in the line of Lemma 8 and with the help of Lemma 11.

### 3 Theorems.

In this section we present the main results of the paper.

**Theorem 1** Let \( f \) be a meromorphic function and \( g \) be an entire function such that (i) \( 0 < \lambda_f \leq \rho_f < \infty \), (ii) \( \lambda_f = \lambda_g \), (iii) \( \tau_f > 0 \), (iv) \( \tau_g < \infty \) and (v) \( \lambda_f < \rho_g \leq \infty \). Also let \( \sum_{a \neq \infty} \Theta(a; f) = 2 \). Then

\[
\max \{ \lambda_f, \lambda_g \} \leq \limsup_{r \to \infty} \frac{\log T(r, f\circ g)}{T(r, P_0[f])} \leq \rho_f \frac{\bar{\tau}_g}{\Gamma_{P_0[f]} \bar{\tau}_f}.
\]

**Proof.** Let us suppose that \( 0 < \varepsilon < \min \{ \lambda_f, \Gamma_{P_0[f]} \tau_f \} \).

Since \( \lambda_f < \rho_g \), in view of Lemma 2 we obtain for a sequence of values of \( r \) tending to infinity that

\[
\log T(r, f\circ g) \geq \log T(\exp (r^{\lambda_f}), f)
\]

i.e., \( \log T(r, f\circ g) \geq (\lambda_f - \varepsilon) \log \exp (r^{\lambda_f}) \)

i.e., \( \log T(r, f\circ g) \geq (\lambda_f - \varepsilon) r^{\lambda_f} \).

\[
\text{(1)}
\]
Again by Lemma 8, we have for all sufficiently large values of \( r \),
\[
T(r, P_0[f]) \leq (\tau_{P_0[f]} + \varepsilon) r^{\lambda_{P_0[f]}}
\]
\( i.e., T(r, P_0[f]) \leq (\tau_{P_0[f]} + \varepsilon) r^{\lambda_f} \).

(2)

Therefore from (1) and (2) it follows for a sequence of values of \( r \) tending to infinity that
\[
\log \frac{T(r, f \circ g)}{T(r, P_0[f])} \geq \frac{(\lambda_f - \varepsilon) r^{\lambda_f}}{(\tau_{P_0[f]} + \varepsilon) r^{\lambda_f}}
\]
\( i.e., \limsup_{r \to \infty} \log \frac{T(r, f \circ g)}{T(r, P_0[f])} \geq \frac{\lambda_f}{\tau_{P_0[f]} + \varepsilon}. \)

(3)

Similarly in view of Lemma 3 we get that
\[
\limsup_{r \to \infty} \log \frac{T(r, f \circ g)}{T(r, P_0[f])} \geq \frac{\lambda_g}{\tau_{P_0[f]} + \varepsilon}. \)

(4)

Again we have from Lemma 1 for all sufficiently large values of \( r \),
\[
T(r, f \circ g) \leq (1 + o(1)) T(M(r, g), f)
\]
\( i.e., \log T(r, f \circ g) \leq (\rho_f + \varepsilon) \log M(r, g) + o(1)\)

\( i.e., \liminf_{r \to \infty} \log \frac{T(r, f \circ g)}{T(r, P_0[f])} \leq (\rho_f + \varepsilon) \liminf_{r \to \infty} \log \frac{M(r, g)}{T(r, P_0[f])}. \)

(5)

Also for all sufficiently large values of \( r \)
\[
\log M(r, g) \leq (\tau_g + \varepsilon) r^{\lambda_g}.
\]

(6)

Again in view of Lemma 8 we obtain for a sequence of values of \( r \) tending to infinity that
\[
T(r, P_0[f]) \geq (\tau_{P_0[f]} - \varepsilon) r^{\lambda_{P_0[f]}}
\]
\( i.e., T(r, P_0[f]) \geq (\tau_{P_0[f]} - \varepsilon) r^{\lambda_f}. \)

(7)

Since \( \lambda_f = \lambda_g \) we get from (6) and (7) for a sequence of values of \( r \) tending to infinity that
\[
\liminf_{r \to \infty} \frac{\log M(r, g)}{T(r, P_0[f])} \leq \frac{\tau_g}{\Gamma_{P_0} \tau_f}.
\]

(8)

Since \( \varepsilon(>0) \) is arbitrary, from (6) and (8) we obtain that
\[
\liminf_{r \to \infty} \frac{\log T(r, f \circ g)}{T(r, P_0[f])} \leq \rho_f \frac{\tau_g}{\Gamma_{P_0} \tau_f}.
\]

(9)
Thus the theorem follows from (3), (4) and (9).

**Remark 1** If we take \( \Theta(\infty; f) = \sum \delta_p(a; f) = 1 \) or \( \delta(\infty; f) = \sum \delta(a; f) = 1 \)” instead of “ \( \sum \Theta(a; f) = 2 \)” in Theorem 1 and the other conditions remain the same then one can easily prove that

\[
\max \left\{ \frac{\lambda_f, \lambda_g}{\gamma_{P_0[f]} \cdot \bar{\tau}_f} \right\} \leq \limsup_{r \to \infty} \frac{\log T(r, f; g)}{T(r, P_0[f])} \leq \frac{\bar{\tau}_g}{\gamma_{P_0[f]} \cdot \bar{\tau}_f}.
\]

In the line of Theorem 1 and with the help of Lemma 12 we may state the following theorem without proof.

**Theorem 2** Let \( f \) be a transcendental meromorphic function and \( g \) be an entire function such that (i) \( 0 < \lambda_f \leq \rho_f < \infty \), (ii) \( \lambda_f = \lambda_g \), (iii) \( \tau_f > 0 \), (iv) \( \overline{\tau}_g < \infty \) and (v) \( \lambda_f < \rho_g \). Also let \( \sum \delta_1(a; f) \leq 4 \). Then

\[
\max \left\{ \frac{\lambda_f, \lambda_g}{\Gamma_M - (\Gamma_M - \gamma_M) \Theta(\infty; f) \cdot \bar{\tau}_f} \right\} \leq \limsup_{r \to \infty} \frac{\log T(r, f; g)}{T(r, M[f])} \leq \frac{\bar{\tau}_g}{\Gamma_M - (\Gamma_M - \gamma_M) \Theta(\infty; f) \cdot \tau_f}.
\]

In the line of Theorem 1 we may also state the following theorem without proof.

**Theorem 3** Let \( f \) be a meromorphic function and \( g \) be an entire function with (i) \( 0 < \lambda_f \leq \rho_f < \infty \), (ii) \( 0 < \lambda_g < \rho_g < \infty \), (iii) \( 0 < \overline{\tau}_g \leq \tau_g < \infty \), and (iv) \( 0 < \overline{\tau}_g \leq \tau_g < \infty \). Also let \( \Theta(\infty; g) = \sum a \neq \infty \delta_p(a; g) = 1 \) or \( \delta(\infty; g) = \sum a \neq \infty \delta(a; g) = 1 \). Then

\[
\max \left\{ \frac{\lambda_f, \lambda_g}{\gamma_{P_0[g]} \cdot \bar{\tau}_g} \right\} \leq \limsup_{r \to \infty} \frac{\log T(r, f; g)}{T(r, P_0[g])} \leq \frac{\rho_f}{\gamma_{P_0[g]}} \min \left\{ \frac{\sigma_g}{\sigma_g}, \frac{\tau_g}{\tau_g} \right\}.
\]

**Remark 2** In addition to the conditions of Theorem 3 if \( f \) be a meromorphic function with \( 0 < \lambda_f^{**} \leq \rho_f^{**} < \infty \) then by Definition 2 and similar process of Theorem 1 one can verify that

\[
\max \left\{ \frac{\lambda_f^{**}, \lambda_g}{\gamma_{P_0[g]} \cdot \bar{\tau}_g} \right\} \leq \limsup_{r \to \infty} \frac{T(r, f; g)}{T(r, P_0[g])} \leq \frac{\{1 + o(1)\} \rho_f^{**}}{\gamma_{P_0[g]}} \min \left\{ \frac{\sigma_g}{\sigma_g}, \frac{\tau_g}{\tau_g} \right\}.
\]

**Remark 3** Under the same condition of Theorem 3, if we take “ \( \sum a \neq \infty \Theta(a; f) = 2 \)” instead of “ \( \Theta(\infty; g) = \sum a \neq \infty \delta_p(a; g) = 1 \) or \( \delta(\infty; g) = \sum a \neq \infty \delta(a; g) = 1 \)”, then the following result holds:

\[
\frac{1}{\Gamma_{P_0[g]} \cdot \bar{\tau}_g} \max(\lambda_f, \lambda_g) \leq \limsup_{r \to \infty} \frac{\log T(r, f; g)}{T(r, P_0[g])} \leq \frac{\rho_f}{\gamma_{P_0[g]}} \min \left\{ \frac{\sigma_g}{\sigma_g}, \frac{\tau_g}{\tau_g} \right\}.
\]
Remark 4. In Remark 2 if we take \( 0 < \lambda_f^* \leq \rho_f^* < \infty \) instead of \( 0 < \lambda_f \leq \rho_f < \infty \) and the other conditions remain the same then it can be shown that

\[
\frac{\lambda_f^{**}}{\Gamma_{P_0[g]}(\bar{r}_g)} \leq \limsup_{r \to \infty} \frac{T(r,f \circ g)}{T(r,M[g])} \leq \frac{1 + o(1)}{\rho_f^*} \min \left\{ \frac{\sigma_g}{\sigma_g + \bar{r}_g}, \frac{\bar{r}_g}{\sigma_g + \bar{r}_g} \right\}.
\]

Theorem 4. Let \( f \) be a meromorphic function and \( g \) be a transcendental entire function such that (i) \( 0 < \lambda_f \leq \rho_f < \infty \), (ii) \( 0 < \lambda_g \leq \rho_g < \infty \), (iii) \( 0 < \sigma_g \leq \sigma_g < \infty \), (iv) \( 0 < \bar{r}_g \leq \bar{r}_g < \infty \). Also let \( \sum_{a \in \mathbb{C} \cup \{\infty\}} \delta_1(a;f) \leq 4 \). Then

\[
\max \left\{ \lambda_f, \lambda_g \right\} \leq \limsup_{r \to \infty} \frac{\log T(r,f \circ g)}{T(r,M[g])} \leq \frac{\rho_f \min \left\{ \frac{\sigma_g}{\sigma_g + \bar{r}_g}, \frac{\bar{r}_g}{\sigma_g + \bar{r}_g} \right\}}{\Gamma_{M - (\Gamma_M - \gamma_M)}(\infty;g)}.
\]

The proof is omitted because it can be carried out in the line of Theorem 3 and with the help of Lemma 12.

Remark 5. Under the same conditions of Theorem 4 if \( f \) be a meromorphic function with order zero and \( 0 < \lambda_f^* \leq \rho_f^* < \infty \) then with the help of Definition 2 and similar process of Theorem 4 one can easily verify that

\[
\frac{\lambda_f^{**}}{\Gamma_{M - (\Gamma_M - \gamma_M)}(\infty;g)} \leq \limsup_{r \to \infty} \frac{\log T(r,f \circ g)}{T(r,M[g])} \leq \frac{1 + o(1)}{\rho_f^*} \min \left\{ \frac{\sigma_g}{\sigma_g + \bar{r}_g}, \frac{\bar{r}_g}{\sigma_g + \bar{r}_g} \right\}.
\]

Theorem 5. Let \( f \) be a meromorphic function and \( g \) be an entire function such that (i) \( 0 < \lambda_f \leq \rho_f < \infty \), (ii) \( \rho_f = \rho_g \), (iii) \( \sigma_g < \infty \), (iv) \( \sigma_f > 0 \) and \( \Theta(\infty;f) = \sum_{a \neq \infty} \delta_p(a;f) = 1 \). Then

\[
\liminf_{r \to \infty} \frac{\log T(r,f \circ g)}{T(r,P_0[f])} \leq \frac{1}{\rho_f^{*}} \min \left\{ \frac{\rho_f}{\rho_f + \sigma_g}, \frac{\rho_f}{\sigma_f + \bar{r}_g}, \frac{\sigma_g}{\rho_f + \bar{r}_g}, \frac{\bar{r}_g}{\rho_f + \bar{r}_g} \right\}.
\]

Proof. As \( T(r,g) \leq \log^+ M(r,g) \), we have from Lemma 1 for a sequence of values of \( r \) tending to infinity that

\[
T(r,f \circ g) \leq (1 + o(1)) T(M(r,g),f)
\]

i.e., \( \log T(r,f \circ g) \leq (\lambda_f + \varepsilon) \log M(r,g) + O(1) \)

\[
\text{i.e., } \liminf_{r \to \infty} \frac{\log T(r,f \circ g)}{T(r,P_0[f])} \leq (\lambda_f + \varepsilon) \liminf_{r \to \infty} \frac{\log M(r,g)}{T(r,P_0[f])}.
\]

Now from the definition of type it follows for sufficiently large values of \( r \)

\[
\log M(r,g) \leq (\sigma_g + \varepsilon) r^\rho_g.
\]
Also from the definition of lower type we obtain for a sequence of values of $r$ tending to infinity that
\[ \log M(r, g) \leq (\bar{\sigma}_g + \epsilon) r^{\rho_g}. \] (12)

Again by Lemma 9 and Lemma 10, we have for all sufficiently large values of $r$ that
\[ T(r, P_0[f]) \geq (\bar{\alpha}_{P_0[f]} - \epsilon) r^{\rho_{P_0[f]}} \]

i.e.,
\[ T(r, P_0[f]) \geq (\gamma_{P_0[f]} \bar{\sigma}_f - \epsilon) r^{\rho_f}. \] (13)

Similarly with the help of Lemma 9 and Lemma 10 we obtain for a sequence of values of $r$ tending to infinity that
\[ T(r, P_0[f]) \leq (\sigma_{P_0[f]} - \epsilon) r^{\rho_{P_0[f]}} \]

i.e.,
\[ T(r, P_0[f]) \leq (\gamma_{P_0[f]} \sigma_f - \epsilon) r^{\rho_f}. \] (14)

Since $\rho_f = \rho_g$ we get from (11) and (14) for a sequence of values of $r$ tending to infinity that
\[ \liminf_{r \to \infty} \frac{\log M(r, g)}{T(r, P_0[f])} \leq \frac{\sigma_g}{\gamma_{P_0[f]} \sigma_f}. \] (15)

Similarly from (12) and (13) it follows for a sequence of values of $r$ tending to infinity that
\[ \liminf_{r \to \infty} \frac{\log M(r, g)}{T(r, P_0[f])} \leq \frac{\bar{\sigma}_g}{\gamma_{P_0[f]} \bar{\sigma}_f}. \] (16)

Also we obtain from (11) and (13) for all sufficiently large values of $r$,
\[ \liminf_{r \to \infty} \frac{\log M(r, g)}{T(r, P_0[f])} \leq \frac{\sigma_g}{\gamma_{P_0[f]} \bar{\sigma}_f}. \] (17)

Since $\epsilon (> 0)$ is arbitrary, from (5) and (15) we obtain that
\[ \liminf_{r \to \infty} \frac{\log T(r, f; g)}{T(r, P_0[f])} \leq \rho_f \frac{\sigma_g}{\gamma_{P_0[f]} \sigma_f}. \] (18)

Similarly from (5) and (16) it follows that
\[ \liminf_{r \to \infty} \frac{\log T(r, f; g)}{T(r, P_0[f])} \leq \rho_f \frac{\bar{\sigma}_g}{\gamma_{P_0[f]} \bar{\sigma}_f}. \] (19)

Also we get from (10) and (17) that
\[ \liminf_{r \to \infty} \frac{\log T(r, f; g)}{T(r, P_0[f])} \leq \lambda_f \frac{\sigma_g}{\gamma_{P_0[f]} \sigma_f}. \] (20)

Thus the theorem follows from (18), (19) and (20).

**Remark 6** Theorem 5 remains true with $\Gamma_{P_0[f]}$ instead of $\gamma_{P_0[f]}$ if we replace the condition
\[ \theta(\alpha; f) = \sum_{a \neq \infty} \delta_p(a; f) = 1 \] or $\delta(\infty; f) = \sum_{a \neq \infty} \delta(a; f) = 1$ by $\sum_{a \neq \infty} \theta(a; f) = 2$ and the other conditions remain the same.
**Theorem 6** Let $f$ be a transcendental meromorphic function and $g$ be an entire function such that (i) $0 < \lambda_f < \rho_f < \infty$, (ii) $\rho_f = \rho_g$, (iii) $\sigma_g < \infty$, (iv) $\sigma_f > 0$, and $\sum_{a \in \mathbb{C} \cup \{\infty\}} \delta_1(a; f) \leq 4$. Then

$$
\liminf_{r \to \infty} \frac{\log T(r, f \circ g)}{T(r, M[f])} \leq \frac{1}{\Gamma_{M-(\Gamma_M-\gamma_M)}(\Theta(\infty; f))} \min \left\{ \frac{\sigma_g}{\rho_f \sigma_f}, \frac{\sigma_g}{\rho_f \bar{\sigma_f}}, \frac{\sigma_g}{\lambda_f \bar{\sigma_f}} \right\}.
$$

The proof of the theorem can be established in the line of Theorem 6 and with the help of Lemma 12 and therefore is omitted.

In the line of Theorem 5 we may state the following theorem without proof.

**Theorem 7** Let $f$ be a meromorphic function and $g$ be an entire function such that (i) $0 < \lambda_f < \infty$, (ii) $\sigma_g < \infty$, (iii) $\bar{\sigma_f} > 0$, and (iv) $\sum_{a \neq \infty} \delta_1(a; g) = 2$. Then

$$
\liminf_{r \to \infty} \frac{\log T(r, f \circ g)}{T(r, P_0[\{g\}])} \leq \frac{\lambda_f}{\Gamma_{P_0[\{g\}]}(\sigma_g)}.
$$

**Remark 7** In addition to the conditions of Theorem 7 if $f$ be a meromorphic function with $0 < \lambda_f < \infty$ then one can easily verify that

$$
\liminf_{r \to \infty} \frac{T(r, f \circ g)}{T(r, P_0[\{g\}])} \leq \frac{1+o(1)}{\Gamma_{P_0[\{g\}]}(\sigma_g)} \cdot \frac{\sigma_g}{\lambda_f}.
$$

**Remark 8** Theorem 7 and Remark 7 remain true with $\gamma_{P_0[\{g\}]}$ instead of $\Gamma_{P_0[\{g\}]}$ if we replace the condition $\sum_{a \neq \infty} \delta_1(a; g) = 2$ by $\sum_{a \neq \infty} \delta_1(a; g) = 1$ and the other conditions are same.

In the line of Theorem 7 and in view of Lemma 12 we may state the following theorem without proof.

**Theorem 8** Let $f$ be a meromorphic function and $g$ be a transcendental entire function such that (i) $0 < \lambda_f < \infty$, (ii) $\sigma_g < \infty$, (iii) $\bar{\sigma_f} > 0$, and $\sum_{a \in \mathbb{C} \cup \{\infty\}} \delta_1(a; f) \leq 4$. Then

$$
\liminf_{r \to \infty} \frac{\log T(r, f \circ g)}{T(r, M[g])} \leq \frac{\lambda_f}{\Gamma_{M-(\Gamma_M-\gamma_M)}(\Theta(\infty; f))} \cdot \frac{\sigma_g}{\bar{\sigma_f}}.
$$

**Remark 9** In addition the conditions of Theorem 8 if $f$ be a meromorphic function with $0 < \lambda_f < \infty$ then one can easily verify that

$$
\liminf_{r \to \infty} \frac{T(r, f \circ g)}{T(r, M[g])} \leq \frac{\{1+o(1)\} \lambda_f}{\Gamma_{M-(\Gamma_M-\gamma_M)}(\Theta(\infty; f))} \cdot \frac{\sigma_g}{\bar{\sigma_f}}.
$$
**Theorem 9** Let \( f \) be a meromorphic function and \( g \) be an entire function such that \( 0 < \lambda_g < \rho_f \), \( 0 < \lambda_f \leq \rho_f < \infty, \overline{\sigma}_f > 0 \) and \( \sum \Theta(a;f) = 2 \). Then

\[
\liminf_{r \to \infty} \frac{\log T(r,fog)}{T(r,P_0[f])} \leq \min \left\{ \frac{1}{\Gamma_{P_0[f]}} \left\{ \frac{\rho_f}{\overline{\sigma}_f}, \frac{\rho_g}{\overline{\sigma}_g} \right\} \right\}.
\]

**Proof.** Since \( \lambda_g < \rho_f \), in view of Lemma 4 we obtain for a sequence of values of \( r \) tending to infinity that

\[
\log T(r,fog) < \log T(\exp(r^{\rho_f}),f)
\]

i.e., \( \log T(r,fog) < (\rho_f + \varepsilon) \log \exp (r^{\rho_f}) \)

i.e., \( \log T(r,fog) < (\rho_f + \varepsilon) r^{\rho_f} \).

(21)

Again by Lemma 8, we have for all sufficiently large values of \( r \),

\[
T(r,P_0[f]) \geq (\Gamma_{P_0[f]} - \varepsilon) r^{\rho_{P_0[f]}}
\]

i.e., \( T(r,P_0[f]) \geq (\Gamma_{P_0[f]} - \varepsilon) r^{\rho_f} \).

(22)

Therefore from (21) and (22) it follows for a sequence of values of \( r \) tending to infinity

\[
\frac{\log T(r,fog)}{T(r,P_0[f])} \leq \frac{(\rho_f + \varepsilon) r^{\rho_f}}{(\Gamma_{P_0[f]} - \varepsilon) r^{\rho_f}}
\]

i.e., \( \limsup_{r \to \infty} \frac{\log T(r,fog)}{T(r,P_0[f])} \leq \frac{\rho_f}{\Gamma_{P_0[f]} - \varepsilon} \frac{\rho_g}{\overline{\sigma}_g} \).

(23)

Similarly in view of Lemma 5 we get that

\[
\limsup_{r \to \infty} \frac{\log T(r,fog)}{T(r,P_0[f])} \leq \frac{\rho_g}{\Gamma_{P_0[f]} - \varepsilon} \frac{\rho_f}{\overline{\sigma}_f}.
\]

(24)

Thus the theorem follows from (23) and (24).

**Remark 10** Theorem 9 remains true with \( \gamma_{P_0[f]} \) instead of \( \Gamma_{P_0[f]} \) if we replace the condition

\[
\sum_{a \neq \infty} \Theta(a;f) = 2 \text{ by } \Theta(\infty;f) = \sum_{a \neq \infty} \delta_p(a;f) = 1 \text{ or } \delta(\infty;f) = \sum_{a \neq \infty} \delta(a;f) = 1 \text{ and }
\]

the other conditions remain the same.

**Theorem 10** Let \( f \) be a meromorphic function and \( g \) be an entire function such that \( 0 < \lambda_g < \rho_f \), \( 0 < \lambda_f \leq \rho_f < \infty, \overline{\sigma}_g > 0 \) and \( \Theta(\infty;g) = \sum_{a \neq \infty} \delta_p(a;g) = 1 \) or \( \delta(\infty;g) = \sum_{a \neq \infty} \delta(a;g) = 1 \). Then

\[
\liminf_{r \to \infty} \frac{\log T(r,fog)}{T(r,P_0[g])} \leq \min \left\{ \frac{1}{\gamma_{P_0[g]}} \left\{ \frac{\rho_f}{\overline{\sigma}_g}, \frac{\rho_g}{\overline{\sigma}_g} \right\} \right\}.
\]
Theorem 10 can be carried out in the line of Theorem 9 and therefore its proof is omitted.

**Remark 11** if we take \( \sum_{a \neq \infty} \Theta(a;g) = 2 \) instead of \( \Theta(\infty;g) = \sum_{a \neq \infty} \delta_p(a;g) = 1 \) or \( \delta(\infty;g) = \sum_{a \neq \infty} \delta(a;g) = 1 \) in Theorem 10 and the other conditions remain the same then

Theorem 10 remains valid with \( \Gamma_{p,[g]} \) instead of \( \gamma_{p,[g]} \).

The following two theorems can be carried out in view of Lemma 14 and in the same way of Theorem 9 and Theorem 10 respectively. Hence the proof is omitted.

**Theorem 11** Let \( f \) be a meromorphic function and \( g \) be an entire function such that \( 0 < \lambda_g < \rho_f \), \( 0 < \lambda_f \leq \rho_f < \infty \), \( \sigma_g > 0 \) and \( \sum_{a \in \mathbb{C} \cup \{\infty\}} \delta(a;f) = 4 \). Then

\[
\liminf_{r \to \infty} \frac{\log T(r,fog)}{T(r,M[f])} \leq \min \left\{ \frac{1}{(\Gamma_M - (\Gamma_M - \gamma_M) \Theta(\infty;g))} \left( \frac{\rho_f}{\sigma_f}, \frac{\rho_g}{\sigma_g} \right) \right\}.
\]

**Theorem 12** Let \( f \) be a meromorphic function and \( g \) be a transcendental entire function such that \( 0 < \lambda_g < \rho_f \), \( 0 < \lambda_f \leq \rho_f < \infty \), \( \sigma_g > 0 \) and \( \sum_{a \in \mathbb{C} \cup \{\infty\}} \delta(a;g) = 4 \). Then

\[
\liminf_{r \to \infty} \frac{\log T(r,fog)}{T(r,M[g])} \leq \min \left\{ \frac{1}{(\Gamma_M - (\Gamma_M - \gamma_M) \Theta(\infty;g))} \left( \frac{\rho_f}{\sigma_f}, \frac{\rho_g}{\sigma_g} \right) \right\}.
\]

Using the notion of weak type, we may state the following theorem without proof:

**Theorem 13** Let \( f \) be a meromorphic function and \( g \) be an entire function such that (i) \( 0 < \lambda_f \leq \rho_f < \infty \), (ii) \( \lambda_f = \lambda_g \), (iii) \( \sigma_g < \infty \), (iv) \( \tau_f > 0 \) and \( \Theta(\infty;f) = \sum_{a \neq \infty} \delta_p(a;f) = 1 \) or \( \delta(\infty;f) = \sum_{a \neq \infty} \delta(a;f) = 1 \). Then

\[
\liminf_{r \to \infty} \frac{\log T(r,fog)}{T(r,M[f])} \leq \frac{1}{\gamma_{P_0[f]}} \min \left\{ \frac{\tau_g}{\rho_f \tau_f}, \frac{\tau_g}{\rho_f \tau_f}, \frac{\lambda_f}{\tau_f} \right\}.
\]

**Remark 12** if we take \( \sum_{a \neq \infty} \Theta(a;f) = 2 \) instead of \( \Theta(\infty;f) = \sum_{a \neq \infty} \delta_p(a;f) = 1 \) or \( \delta(\infty;f) = \sum_{a \neq \infty} \delta(a;f) = 1 \) in Theorem 13 and the other conditions remain the same then

Theorem 13 remains valid with \( \Gamma_{p,[f]} \) instead of \( \gamma_{p,[f]} \).

**Theorem 14** Let \( f \) be a transcendental meromorphic function and \( g \) be an entire function such that (i) \( 0 < \lambda_f \leq \rho_f < \infty \), (ii) \( \lambda_f = \lambda_g \), (iii) \( \sigma_g < \infty \), (iv) \( \tau_f > 0 \) and \( \sum_{a \in \mathbb{C} \cup \{\infty\}} \delta(a;f) = 4 \). Then

\[
\liminf_{r \to \infty} \frac{\log T(r,fog)}{T(r,M[f])} \leq \frac{1}{(\Gamma_M - (\Gamma_M - \gamma_M) \Theta(\infty;f))} \min \left\{ \frac{\tau_g}{\rho_f \tau_f}, \frac{\tau_g}{\rho_f \tau_f}, \frac{\lambda_f}{\tau_f} \right\}.
\]
The proof is omitted as it can be carried out in the line of Theorem 13 and in view of Lemma 12.

In the line of Theorem 7 we may state the following theorem without proof.

**Theorem 15** Let \( f \) be a meromorphic function and \( g \) be an entire function such that (i) \( 0 < \lambda_f < \infty \), (ii) \( \tau_g < \infty \), (iii) \( \tau_f > 0 \) and \( \Theta(\infty; g) = \sum_{a \neq \infty} \delta(a; g) = 1 \) or \( \delta(\infty; g) = \sum_{a \neq \infty} \delta(a; g) = 1 \). Then

\[
\liminf_{r \to \infty} \frac{\log T(r, f \circ g)}{T(r, P_0[g])} \leq \frac{\lambda_f}{\gamma_{P_0[g]}} \cdot \frac{\tau_g}{\tau_g}.
\]

**Remark 13** If we take \( \sum_{a \neq \infty} \Theta(a, g) = 2 \) instead of \( \Theta(\infty; g) = \sum_{a \neq \infty} \delta_p(a; g) = 1 \) or \( \delta(\infty; g) = \sum_{a \neq \infty} \delta(a; g) = 1 \) in Theorem 15 and the other conditions remain the same then Theorem 15 is still valid with \( \Gamma_{P_0[g]} \) instead of \( \gamma_{P_0[g]} \).

**Remark 14** In addition to the conditions of Theorem 15 if \( f \) be a meromorphic function with \( 0 < \lambda_f^{**} < \infty \) then one can easily verify that

\[
\liminf_{r \to \infty} \frac{T(r, f \circ g)}{T(r, P_0[g])} \leq \frac{\{1 + o(1)\} \lambda_f^{**}}{\gamma_{P_0[g]}} \cdot \frac{\tau_g}{\tau_g}.
\]

The following theorem can be carried out in the line of Theorem 15 and in view of Lemma 12:

**Theorem 16** Let \( f \) be a meromorphic function and \( g \) be a transcendental entire function with (i) \( 0 < \lambda_f < \infty \), (ii) \( \tau_g < \infty \), (iii) \( \tau_f > 0 \) and (iv) \( \sum_{a \in \mathbb{C} \cup \{\infty\}} \delta_1(a; g) \leq 4 \). Then

\[
\liminf_{r \to \infty} \frac{\log T(r, f \circ g)}{T(r, M[g])} \leq \frac{\lambda_f}{\{\Gamma_M - (\Gamma_M - \gamma_M) \cdot \Theta(\infty; g)\}} \cdot \frac{\tau_g}{\tau_g}.
\]

**Remark 15** In addition to the conditions of Theorem 16 if \( f \) be a meromorphic function with \( 0 < \lambda_f^{**} < \infty \) then one can easily verify that

\[
\liminf_{r \to \infty} \frac{T(r, f \circ g)}{T(r, M[g])} \leq \frac{\{1 + o(1)\} \lambda_f^{**}}{\{\Gamma_M - (\Gamma_M - \gamma_M) \cdot \Theta(\infty; g)\}} \cdot \frac{\tau_g}{\tau_g}.
\]

**Theorem 17** Let \( f \) be a meromorphic function and \( g \) be an entire function such that (i) \( 0 < \tau_g < \lambda_f \), (ii) \( 0 < \lambda_f \leq \rho_f < \infty \), (iii) \( \tau_f > 0 \) and \( \Theta(\infty; f) = \sum_{a \neq \infty} \delta_p(a; f) = 1 \) or \( \delta(\infty; f) = \sum_{a \neq \infty} \delta(a; f) = 1 \). Then
\[
\liminf_{r \to \infty} \frac{\log T(r, f \circ g)}{T(r, P_{0}(f))} \leq \frac{1}{\gamma_{P_{0}(f)}} \min \left\{ \frac{\rho_{f}}{\tau_{f}}, \frac{\rho_{g}}{\tau_{f}} \right\}.
\]

The proof of the Theorem is omitted because it can be carried out in the line of Theorem 9 and using the notion of weak type.

**Remark 16** If we take \( \sum_{a \neq \infty} \delta(a; f) = 2 \) instead of \( \Theta(\infty; f) = \sum_{a \neq \infty} \delta_{p}(a; f) = 1 \) or \( \delta(\infty; f) = \sum_{a \neq \infty} \delta(a; f) = 1 \) in Theorem 17 and the other conditions remain the same then Theorem 17 is also valid with \( \Gamma_{P_{0}(f)} \) instead of \( \gamma_{P_{0}(f)} \).

**Theorem 18** Let \( f \) be a meromorphic function and \( g \) be an entire function such that \( 0 < \lambda_{g} < \lambda_{f} \), \( 0 < \lambda_{f} \leq \rho_{f} < \infty \), \( \tau_{g} > 0 \) and \( \sum_{a \in \mathbb{C} \cup \{\infty\}} \delta_{1}(a; f) = 4 \). Then

\[
\liminf_{r \to \infty} \frac{\log T(r, f \circ g)}{T(r, M(f))} \leq \frac{\min \left\{ \frac{\rho_{f}}{\tau_{f}}, \frac{\rho_{g}}{\tau_{f}} \right\}}{\{\Gamma_{M} - (\Gamma_{M} - \gamma_{M}) \Theta(\infty; f)\}}.
\]

We omit the proof of Theorem 18 because it can be carried out in the line of Theorem 17.

**Theorem 19** Let \( f \) be a meromorphic function and \( g \) be an entire function such that \( 0 < \lambda_{g} < \lambda_{f} \), \( 0 < \lambda_{f} \leq \rho_{f} < \infty \), \( \tau_{g} > 0 \) and \( \Theta(\infty; g) = \sum_{a \neq \infty} \delta_{p}(a; g) = 1 \) or \( \delta(\infty; g) = \sum_{a \neq \infty} \delta(a; g) = 1 \). Then

\[
\liminf_{r \to \infty} \frac{\log T(r, f \circ g)}{T(r, P_{0}(g))} \leq \frac{1}{\gamma_{P_{0}(g)}} \min \left\{ \frac{\rho_{f}}{\tau_{g}}, \frac{\rho_{g}}{\tau_{g}} \right\}.
\]

The proof of Theorem 19 is omitted because it can be carried out in the line of Theorem 17.

**Remark 17** If we take \( \sum_{a \neq \infty} \delta(a; g) = 2 \) instead of \( \Theta(\infty; g) = \sum_{a \neq \infty} \delta_{p}(a; g) = 1 \) or \( \delta(\infty; g) = \sum_{a \neq \infty} \delta(a; g) = 1 \) in Theorem 19 and the other conditions remain the same then Theorem 19 remain valid with \( \Gamma_{P_{0}(g)} \) instead of \( \gamma_{P_{0}(g)} \).

**Theorem 20** Let \( f \) be a meromorphic function and \( g \) be an entire function with \( 0 < \lambda_{g} < \lambda_{f} \), \( 0 < \lambda_{f} \leq \rho_{f} < \infty \), \( \tau_{g} > 0 \) and \( \sum_{a \in \mathbb{C} \cup \{\infty\}} \delta_{1}(a; g) \leq 4 \). Then

\[
\liminf_{r \to \infty} \frac{\log T(r, f \circ g)}{T(r, M(g))} \leq \frac{\min \left\{ \frac{\rho_{f}}{\tau_{g}}, \frac{\rho_{g}}{\tau_{g}} \right\}}{\{\Gamma_{M} - (\Gamma_{M} - \gamma_{M}) \Theta(\infty; g)\}}.
\]

The proof is omitted.
References


Composite entire and meromorphic functions and their growth analysis in the light of order and weak type

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Abstract: In this paper we study the growth properties of composite entire and meromorphic functions which improve some earlier results.

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I. Introduction, Definitions and Notations

We denote by \( \mathbb{C} \) the set of all finite complex numbers. Let \( f \) be a meromorphic function and \( g \) be an entire function defined on \( \mathbb{C} \). We use the standard notations and definitions in the theory of entire and meromorphic functions which are available in [4] and [10]. In the sequel we use the following notations:

\[ \log_{\tau} f = \log ( \log_{\tau-1} f ) \] for \( \tau = 1, 2, 3, \ldots \)

and

\[ \exp_{\tau} f = \exp ( \exp_{\tau-1} f ) \] for \( \tau = 1, 2, 3, \ldots \).

Definition 1 The order \( \rho_f \) and lower order \( \lambda_f \) of an entire function \( f \) are defined as

\[ \rho_f = \limsup_{r \to \infty} \frac{\log |f(r)|}{\log r} \quad \text{and} \quad \lambda_f = \liminf_{r \to \infty} \frac{\log |f(r)|}{\log r}. \]

If \( f \) is meromorphic then

\[ \rho_f = \limsup_{r \to \infty} \frac{\log T(r, f)}{\log r} \quad \text{and} \quad \lambda_f = \liminf_{r \to \infty} \frac{\log T(r, f)}{\log r}. \]

The following definition is also well known:

Definition 2 [3] The weak type \( \tau_f \) of an meromorphic function \( f \) of finite positive lower order \( \lambda_f \) is defined by

\[ \tau_f = \liminf_{r \to \infty} \frac{T(r, f)}{r^{\lambda_f}}. \]

For entire \( f \),

\[ \tau_f = \liminf_{r \to \infty} \frac{\log M(r, f)}{r^{\lambda_f}}, \quad 0 < \lambda_f < \infty. \]

Similarly one can define the growth indicator \( \bar{\tau}_f \) of a meromorphic function \( f \) of finite positive lower order \( \lambda_f \) as

\[ \bar{\tau}_f = \limsup_{r \to \infty} \frac{T(r, f)}{r^{\lambda_f}}. \]

When \( f \) is entire, it can be easily verified that

\[ \bar{\tau}_f = \limsup_{r \to \infty} \frac{\log M(r, f)}{r^{\lambda_f}}, \quad 0 < \lambda_f < \infty. \]
Definition 3 [9] A function \( \rho(r) \) is called a proximate order of \( f \) relative to \( T(r, f) \) if

(i) \( \rho(r) \) is non-negative and continuous for \( r \geq r_0 \), say.

(ii) \( \rho(r) \) is differentiable for \( r \geq r_0 \) except possibly at isolated points at which \( \rho'(r) \) and \( \rho'(r + 0) \) exist.

(iii) \( \lim_{r \to \infty} \rho(r) = \rho_f < \infty \),

(iv) \( \lim_{r \to \infty} r \rho'(r) \log r = 0 \) and

(v) \( \limsup_{r \to \infty} \frac{T(r, f)}{r^{\rho_f}} = 1. \)

In the line of Definition 3 the following definition may be given:

Definition 4 A function \( \lambda_f(r) \) is called a lower proximate order of \( f \) relative to \( T(r, f) \) if

(i) \( \lambda_f(r) \) is non-negative and continuous for \( r \geq r_0 \), say.

(ii) \( \lambda_f(r) \) is differentiable for \( r \geq r_0 \) except possibly at isolated points at which \( \lambda_f'(r) \) and \( \lambda_f'(r + 0) \) exist.

(iii) \( \lim_{r \to \infty} \lambda_f(r) = \lambda_f < \infty \),

(iv) \( \lim_{r \to \infty} r \lambda_f'(r) \log r = 0 \) and

(v) \( \liminf_{r \to \infty} \frac{T(r, f)}{r^{\lambda_f}} = 1. \)

In the paper we establish some newly developed results based on the comparative growth properties of composite entire or meromorphic functions.

II. Lemmas.

In this section we present some lemmas which will be needed in the sequel.

Lemma 1 [1] Let \( f \) be meromorphic and \( g \) be entire. Then for all sufficiently large values of \( r \),

\[
T(r, f g) \leq \left(1 + o(1)\right) \frac{T(r, g)}{\log M(r, g)} T(M(r, g), f).
\]

Lemma 2 [2] Let \( f \) be meromorphic and \( g \) be entire and suppose that \( 0 < \mu < \rho_g \leq \infty \). Then for a sequence of values of \( r \) tending to infinity,

\[
T(r, f g) \geq T(\exp(r^\mu), f).
\]

Lemma 3 [6] Let \( f \) be meromorphic and \( g \) be entire such that \( 0 < \mu < \rho_g \leq \infty \) and \( \lambda_f > 0 \). Then for a sequence of values of \( r \) tending to infinity,

\[
T(r, f g) \geq T(\exp(r^\mu), g).
\]

Lemma 4 [5] If \( f \) be an entire function then for \( \delta > 0 \) the function \( r^{\lambda_f + \delta - \rho_f(r)} \) is ultimately an increasing function of \( r \).

Lemma 5 [7] Let \( f \) be an entire function. Then for \( \delta > 0 \) the function \( r^{\lambda_f + \delta - \lambda_f(r)} \) is ultimately an increasing function of \( r \).

III. Theorems.

In this section we present the main results of the paper.

Theorem 1 Let \( f, h \) be any two meromorphic functions and \( g, k \) be any two entire functions such that \( \rho_h < \infty, \rho_k < \rho_g \) and \( \lambda_f > 0 \). Then

\[
\liminf_{r \to \infty} \frac{\log \left( T(r, h f g) \log M(r, k) \right)}{\log T(r, f g)} = 0.
\]
Proof. As \( \rho_\delta < \rho_k \), we can choose \( \varepsilon (> 0) \) in such a way that
\[
\rho_k + \varepsilon < \rho_\delta - \varepsilon < \rho_\delta. \tag{1}
\]

Now from (1) and Lemma 2 it follows that for a sequence of values of \( \rho \) tending to infinity that
\[
\log T(r, f \circ g) \geq \log T(\exp^{(\rho_\delta - \varepsilon)}, h)
\]
\[= \log T(r, f \circ g) \geq (\lambda_f - \varepsilon) \log \exp^{(\rho_\delta - \varepsilon)} \]
\[i.e., \log T(r, f \circ g) \geq (\lambda_f - \varepsilon) r^{(\rho_\delta - \varepsilon)}. \tag{2}\]

Again we have from Lemma 1 for all sufficiently large values of \( \rho \),
\[
\log T(r, h \circ k) \leq (1 + o(1))T(r, k)T(M(r, k), h)
\]
\[i.e., \log \{T(r, h \circ k) \log M(r, k)\} \leq (\rho_k + \varepsilon) \log r + (\rho_k + \varepsilon) \log M(r, k) + O(1)
\]
\[i.e., \log \{T(r, h \circ k) \log M(r, k)\} \leq (\rho_k + \varepsilon) \log r + (\rho_k + \varepsilon) r^{(\rho_\delta + \varepsilon)} + O(1). \tag{3}\]

Therefore from (2) and (3) we obtain for a sequence of values of \( \rho \) tending to infinity that
\[
\lim_{r \to \infty} \frac{\log T(r, h \circ k) \log M(r, k)}{\log T(r, f \circ g)} \leq \frac{(\rho_\delta + \varepsilon) \log r + (\rho_\delta + \varepsilon) r^{(\rho_\delta + \varepsilon)} + O(1)}{(\lambda_f - \varepsilon) r^{(\rho_\delta - \varepsilon)}}. \tag{4}
\]

Now in view of (1) it follows from (4) that
\[
\lim_{r \to \infty} \frac{\log T(r, h \circ k) \log M(r, k)}{\log T(r, f \circ g)} = 0 .
\]

This proves the theorem .

Remark 1. For the validity of Theorem 1, the conditions \( \rho_\delta < \infty, \rho_k < \rho_\delta \) and \( \lambda_f > 0 \) are necessary but for meromorphic \( h \) with order zero Theorem 1 also holds for \( \rho \leq \rho_k \) which are evident from the following examples :

Example 1. Let \( f = k = \exp z, g = \exp(z^2) \) and \( h = \exp(2z) \).

Then \( \lambda_f = 1 > 0, \rho_h = \infty \) and \( \lambda_k = \rho_k = 1 < 2 = \rho_\delta \).

Now
\[
T(r, f \circ g) \leq \log M(r, f \circ g) = \exp(r^2)
\]
and \( 3T(2r, h \circ k) \geq \log M(r, h \circ k) = \exp(2r) \).

So
\[
\frac{\log T(r, h \circ k) \log M(r, k)}{\log T(r, f \circ g)} = \frac{\log T(r, h \circ k) + \log(2)M(r, k)}{\log T(r, f \circ g)} \geq \frac{\exp r^2 + \log r + O(1)}{r^2}
\]
\[i.e., \lim_{r \to \infty} \frac{\log T(r, h \circ k) \log M(r, k)}{\log T(r, f \circ g)} = \infty .
\]

Example 2. Suppose \( f = h = g = \exp(z) \) and \( k = \exp(z^2) \).

Then \( \lambda_f = \lambda_h = \lambda_g = \rho_\delta = 1 \) and \( \rho_k = 2 \).

Now
\[
3T(2r, h \circ k) \geq \log M(r, h \circ k) = \exp(r^2) = r^2
\]
\[ \log T(r, h, k) \geq \frac{r^2}{4} + O(1) \]  

Also
\[ T(r, f, g) \sim \frac{\exp}{(2e^1)^2} \]

Therefore
\[ \frac{\log \{T(r, h, k) \log M(r, k)\}}{\log T(r, f, g)} = \frac{\log T(r, h, k) + \log^2 M(r, k)}{\log T(r, f, g)} \]

\[ \geq \frac{\frac{r^2}{4} + O(1) + 2 \log r}{r - \frac{1}{2} \log r + O(1)} \]

\[ \text{i.e., } \liminf_{r \to \infty} \frac{\log^2 T(r, h, k) \log M(r, k)}{\log T(r, f, g)} = \infty. \]

**Example 3** Suppose \( f = z, g = \exp(z^2) \) and \( h = k = \exp(z) \).

Then \( \lambda_f = \rho_f = 0 < \infty, \lambda_h = \rho_h = \lambda_k = \rho_k = 1 < 2 = \rho_g \).

Therefore
\[ T(r, f, g) \leq \log M(r, f, g) = r^2 \]

\[ \text{i.e., } \log T(r, f, g) \leq 2 \log r. \]

Also
\[ T(r, h, k) \sim \frac{\exp}{(2e^1)^2} \]

Thus
\[ \frac{\log \{T(r, h, k) \log M(r, k)\}}{\log T(r, f, g)} = \frac{\log T(r, h, k) + \log^2 M(r, k)}{\log T(r, f, g)} \]

\[ \geq \frac{r - \frac{1}{2} \log r + \log r + O(1)}{2 \log r} \]

\[ \text{i.e., } \liminf_{r \to \infty} \frac{\log T(r, h, k) \log M(r, k)}{\log T(r, f, g)} = \infty. \]

**Example 4** Let \( f = g = \exp(z), h = z \) and \( k = \exp(z^2) \).

Then \( \rho_f = \rho_g = 1, \lambda_h = \rho_h = 0 \) and \( \lambda_k = \rho_k = 2 \).

Now
\[ T(r, h, k) \leq \log M(r, h, k) = \log \exp(r^2) = r^2 \]

and \( T(r, f, g) \sim \frac{\exp}{(2e^1)^2} \).

So
\[ \frac{\log \{T(r, h, k) \log M(r, k)\}}{\log T(r, f, g)} = \frac{\log T(r, h, k) + \log^2 M(r, k)}{\log T(r, f, g)} \]

\[ \leq \frac{4 \log r}{r - \frac{1}{2} \log r + O(1)} \]

\[ \text{i.e., } \liminf_{r \to \infty} \frac{\log T(r, h, k) \log M(r, k)}{\log T(r, f, g)} = 0. \]

**Example 5** Let \( g = k = \exp(z), h = z \).

Then \( \rho_f = \rho_g = 1, \lambda_h = \rho_h = 0 \) and \( \lambda_k = \rho_k = 2 \).

Now
\[ T(r, h, k) \leq \log M(r, h, k) = r \]

and \( T(r, f, g) \sim \frac{\exp}{(2e^1)^2} \).

So
\[ \frac{\log \{T(r, h, k) \log M(r, k)\}}{\log T(r, f, g)} = \frac{\log T(r, h, k) + \log^2 M(r, k)}{\log T(r, f, g)} \]

\[ \leq \frac{2 \log r}{r - \frac{1}{2} \log r + O(1)} \]
Since \( g_1 \lambda_1 \tau_1 \]

Now from (5) and (6) we obtain for a sequence of values of 
where \( g_1 \lambda_1 \).

Similarly in view of Lemma 3 we have for a sequence of values of \( r \) tending to infinity 

\[ T(r, fog) \log M(r, g) \leq \left( \rho_g + \varepsilon \right) \log r + \left( \rho_f + \varepsilon \right) \log M(r, g) + O(1) \]

Then 

\[ \log \frac{[T(r, hok) \log M(r, k)]}{T(r, fog)} \leq \frac{\rho_k}{\rho_g} \]

The proof is omitted.

In the line of Theorem 2 the following corollary may be deduced:

Corollary 1 Let \( f, h \) be meromorphic and \( g, k \) be entire such that \( \rho_f < \infty, \rho_k < \rho_g \) and \( \lambda_f > 0 \). Then

\[ \log \frac{[T(r, hok) \log M(r, k)]}{T(r, fog)} \leq 1. \]

Proof. By Lemma 1 we obtain for all sufficiently large values of \( r \),

\[ T(r, fog) \log M(r, g) \leq (1 + o(1)) T(r, g) T(M(r, g), f) \]

\[ \log \{ T(r, fog) \log M(r, g) \} \leq \left( \rho_f + \varepsilon \right) \log r + \left( \rho_f + \varepsilon \right) \log M(r, g) + O(1) \]

Since \( \lambda_g < \rho_k \), in view of Lemma 2 it follows for a sequence of values of \( r \) tending to infinity that

\[ \log T(r, hok) \geq \log T(\exp (r^q), h) \]

\[ \log T(r, hok) \geq (\lambda_k - \varepsilon) \log \exp (r^q) \]

Similarly in view of Lemma 3 we have for a sequence of values of \( r \) tending to infinity

\[ \log T(r, hok) \geq \log T(\exp (r^q), k) \]

\[ \log T(r, hok) \geq (\lambda_k - \varepsilon) \log \exp (r^q) \]

where \( 0 < \varepsilon < \min \{ \lambda_h, \lambda_k \} \).

Now from (5) and (6) we obtain for a sequence of values of \( r \) tending to infinity that

\[ \log \frac{[T(r, hok) \log M(r, g)]}{T(r, hok)} \leq \frac{\rho_g + \varepsilon + (\rho_f + \varepsilon)(\tau_g + \varepsilon) r^q + O(1)}{(\lambda_g - \varepsilon) r^q} \]
\[ \liminf_{r \to \infty} \frac{\log(T(r, hog) \log M(r, g))}{\log T(r, hog)} \leq \frac{\rho_f}{\lambda_h}. \]  

(8)

Analogously from (5) and (7) it follows for a sequence of values of \( r \) tending to infinity that

\[ \log(E) \log M(r, g))] \leq (\rho_g + \varepsilon) \log r + (\rho_f + \varepsilon)(\tau_0 + \varepsilon) r^{\lambda_g} + O(1) \]

(i.e., \( \liminf_{r \to \infty} \frac{\log T(r, hog) \log M(r, g))}{\log T(r, hog)} \leq \frac{\rho_f}{\lambda_h} \)).

(9)

Thus the theorem follows from (8) and (9).

In the line of Theorem 3 one can easily prove the following theorem:

**Theorem 4** Let \( f, h \) be meromorphic and \( g, k \) be entire such that (i) \( \rho_h < \infty \), (ii) \( \lambda_f > 0 \), (iii) \( \lambda_g > 0 \), (iv) \( \lambda_k > \rho_g \) and (v) \( 0 < \lambda_k < \infty \), \( \tau_0 < \infty \). Then

\[ \limsup_{r \to \infty} \frac{\log(T(r, f, g))}{\log T(r, h, g)} \geq (\rho_h \tau_0)^{-1} \cdot \max\{ \lambda_f, \lambda_g \}. \]

The proof is omitted.

**Theorem 5** Let \( f \) be a meromorphic function and \( g, h \) be two entire functions such that \( \rho_g < \infty \), \( \rho_f < \infty \) and \( \lambda_h > 0 \). Then for any \( a > 1 \)

\[ \liminf_{r \to \infty} \frac{\log T(r, f, g)}{\log T(r, h, g)} \leq \left( \frac{\rho_f}{\lambda_h} \right) \cdot (4a)^{\rho_g}. \]

**Proof.** Since \( T(r, g) \leq \log^+ M(r, g) \), we obtain by Lemma 1 for \( \varepsilon > 0 \) and for all sufficiently large values of \( r \),

\[ T(r, f, g) \leq (1 + \varepsilon)T(M(r, g), f) \]

i.e., \( \log T(r, f, g) \leq (\rho_f + \varepsilon) \log M(r, g) + O(1) \). (10)

For all sufficiently large values of \( r \) we know that

\[ T(r, hog) \geq \frac{1}{2} \log M \left( \frac{r}{2}, g \right) + o(1), \]  \( \{ \text{cf. [8]} \} \)

For \( \varepsilon(0 < \varepsilon < \min\{ \lambda_h, \lambda_k \}) \) we get for all sufficiently large values of \( r \),

\[ \log T(r, hog) \geq (\lambda_h - \varepsilon) \log M \left( \frac{r}{4}, g \right) + o(1) \]

i.e., \( \log T(r, hog) \geq (\lambda_h - \varepsilon) \log M \left( \frac{r}{4}, g \right) + O(1) \)

i.e., \( \log T(r, hog) \geq (\lambda_h - \varepsilon) T \left( \frac{r}{4}, g \right) + O(1) \)

i.e., \( \log T(r, hog) \geq (\lambda_h - \varepsilon) T \left( \frac{r}{4}, g \right) + O(1) \).

(11)

Since \( \varepsilon(0 > 0) \) is arbitrary, it follows from (10) and (11) for all sufficiently large values of \( r \),

\[ \liminf_{r \to \infty} \frac{\log T(r, f, g)}{\log T(r, h, g)} \leq \frac{\rho_f}{\lambda_h} \liminf_{r \to \infty} \frac{\log M(r, g)}{\log T(r, g)}. \]

(12)

Since \( \limsup_{r \to \infty} \frac{T(r, g)}{T \left( \frac{r}{4}, g \right)} = 1 \), for given \( \varepsilon(0 \leq \varepsilon < 1) \) we get for all sufficiently large values of \( r \),

\[ T(r, g) < (1 + \varepsilon)T^{\rho_g(r)} \]

and for a sequence of values of \( r \) tending to infinity

\[ T(r, g) > (1 - \varepsilon)T^{\rho_h(r)}. \]

(14)
Since for any \( a > 1 \), \( \log M(r, g) \leq \frac{a+1}{a-1} T(\alpha r, g) \), in view of (13), (14) and for any \( \delta(>0) \) we get for a sequence of values of \( r \) tending to infinity that

\[
\frac{\log M(r, g)}{T(\alpha r, g)} \leq \frac{a+1}{a-1} \left( \frac{1+\epsilon}{(1-\epsilon)} \right) \cdot \frac{(ar)^{\rho_g+\delta}}{(ar)^{\rho_g+\delta-\rho_g(ar)}} \cdot \frac{1}{\zeta^{\rho_g(\frac{a}{a-1})}} \\
\leq \frac{(\alpha + 1)}{\alpha - 1} \left( \frac{1+\epsilon}{(1-\epsilon)} \right) \cdot \frac{(4\alpha)^{\rho_g+\delta}}{(4\alpha)^{\rho_g}} \\
\leq \frac{(\alpha + 1)}{\alpha - 1} \left( \frac{1+\epsilon}{(1-\epsilon)} + 1 \right) \cdot (4\alpha)^{\rho_g+\delta}
\]

because \((\rho_g+\delta)\) is ultimately an increasing function of \( r \). Since \( \epsilon(>0) \) and \( \delta(>0) \) are arbitrary, we obtain that

\[
\liminf_{r \to \infty} \frac{\log M(r, g)}{T(\alpha r, g)} \leq \frac{(a+1)}{(a-1)} \cdot (4\alpha)^{\rho_g}.
\]

Thus from (12) and (15) it follows that

\[
\liminf_{r \to \infty} \frac{\log T(\alpha f, g)}{\log T(\alpha h, g)} \leq \frac{(a+1)}{(a-1)} \cdot \frac{\rho_f}{\lambda_k} \cdot (4\alpha)^{\rho_g}.
\]

In the line of Theorem 5 one can easily prove the following theorem using the definition of lower proximate order:

**Theorem 6** Let \( f \) be a meromorphic function and \( g \), \( h \), \( k \) be any three entire functions such that \( \rho_g < \lambda_k < \infty \) and \( \lambda_h < \infty \). Then for any \( a > 1 \)

\[
\liminf_{r \to \infty} \frac{\log T(\alpha f, g)}{\log T(\alpha h, g)} \leq \frac{(a+1)}{(a-1)} \cdot \frac{\rho_f}{\lambda_k} \cdot (4\alpha)^{\rho_g}.
\]

The proof is omitted.

References